**Warmup sheet MATH303** 8/9/2025

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**Exercise 1.**

Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable. We denote by $L_{[0,1]}(f, P)$ the lower Riemann sum of $f$ with respect to the partition $P$ of $[0,1]$, and by $U_{[0,1]}(f, P)$ the upper Riemann sum. We then also denote:

$$L_{[0,1]}(f) := \sup_P L_{[0,1]}(f, P); \qquad U_{[0,1]}(f) := \inf_P U_{[0,1]}(f, P).$$

Show that:

(i) $L_{[0,1]}(f) = \lim_{n\to\infty} L_{[0,1]}(f, P_n)$,

(ii) $U_{[0,1]}(f) = \lim_{n\to\infty} U_{[0,1]}(f, P_n)$,

where $P_n$ is the partition of $[0,1]$ into $2^n$ intervals of equal length.

**Exercise 2.**

Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable, and let $\tilde{f}$ be the functioned obtained by changing the value of $f$ at finitely many points. Show that $\tilde{f}$ is Riemann integrable and

$$\int_0^1 f(x)dx = \int_0^1 \tilde{f}(x)dx.$$

Is this statement still true if we change the values of $f$ on a countably infinite set?

**Exercise 3.**

Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = n^2 x^n (1-x)$. Show that:

(i) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$,

(ii) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

**Exercise 4.**

Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as $f_n(x) := n\chi_{[-\frac{1}{2n}, \frac{1}{2n}]}$. Let $g : \mathbb{R} \to \mathbb{R}$ be of class $C_c^\infty(\mathbb{R})$, meaning it is infinitely differentiable and compactly supported. Show that as $n \to \infty$

$$\int_{\mathbb{R}} f_n(x)g(x)dx \to g(0).$$

**Bonus:** Replace the $f_n$ by a Gaussian kernel with variance $\frac{1}{n}$ $f_n(x) = \frac{1}{\sqrt{2\pi\frac{1}{n}}} e^{-\frac{x^2}{2\frac{1}{n}}}$ and show the same result. (Now the integrals make sense as improper Riemann integrals.)

*In both cases we can say that functions $f_n$ converge to the Dirac delta distribution $\delta_0$ in the sense of distributions. $\delta_0$ is a purely atomic measure with all its mass at 0, which is why it 'picks out' the value of $g$ at the origin.*