MATH 303 – Measures and Integration
Final Exam Problems

Part I: Main definitions and theorems
Solve problems 1 and 2. Make sure that each solution starts on a new page and that each page is clearly labeled with your name and the relevant problem number.

Problem 1. Give a full statement of the following theorems related to integration of functions:

- Monotone convergence theorem
- Fatou's lemma

Prove that the monotone convergence theorem and Fatou's lemma are equivalent. That is, give a proof of the following two implications:

- (monotone convergence theorem) $\implies$ (Fatou's lemma)
- (Fatou's lemma) $\implies$ (monotone convergence theorem)

Problem 2. In this course, we defined an outer measure to be a function $\mu^* : \mathscr{P}(X) \to [0, \infty]$ such that $\mu^*(\emptyset) = 0$ and μ^* is monotone and countably subadditive.

(a) What does it mean for μ^* to be monotone?

(b) What does it mean for μ^* to be countably subadditive?

(c) Show that a function $\mu^* : \mathscr{P}(X) \to [0, \infty]$ is an outer measure if and only if $\mu^*(\emptyset) = 0$ and μ^* satisfies the following property: if $A \subseteq X$, $(B_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, and $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$, then $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$.

Part II: Problem Solving
Solve 4 of the 6 problems below. Make sure that each solution starts on a new page and that each page is clearly labeled with your name and the relevant problem number.

Problem 3. Let X be an uncountable set.

(a) Prove that the collection $\mathcal{B} = \{E \subseteq X : E \text{ is countable or } X \setminus E \text{ is countable}\}$ is a σ-algebra on X.

(b) Define a function $\mu : \mathcal{B} \to \{0, 1\}$ by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if $X \setminus E$ is countable. Prove that μ is a measure.

(c) Describe the collection of measurable functions from X to $\mathbb{R}$ and compute their integrals with respect to μ.

Problem 4. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a measurable function. Suppose f is integrable (with respect to μ). Prove that for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\int_{\{|f|>M\}} |f| \, d\mu < \varepsilon.$$

Problem 5. Let $(X, \mathcal{B})$ be a measurable space, let $\mu : \mathcal{B} \to [0, \infty]$ be a measure, and let $\nu : \mathcal{B} \to [0, \infty)$ be a finite measure. Prove that the following are equivalent:

(i) for any $E \in \mathcal{B}$, if $\mu(E) = 0$, then $\nu(E) = 0$;

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \in \mathcal{B}$ and $\mu(E) < \delta$, then $\nu(E) < \varepsilon$.

Show that the implication (i) $\implies$ (ii) may fail if ν is an infinite measure.

Problem 6. Consider the set of integers $\mathbb{Z}$ as a discrete topological space.

(a) Describe the space $C_c(\mathbb{Z})$ of compactly supported continuous functions on $\mathbb{Z}$.

(b) Describe the positive linear functionals on $C_c(\mathbb{Z})$.

(c) Let $\varphi : C_c(\mathbb{Z}) \to \mathbb{C}$ be a positive linear functional, and let μ be the Radon measure representing φ via the Riesz representation theorem. When is μ a finite measure? (Give a characterization in terms of properties of φ.)

Problem 7. Let X be an LCH space and $\mu : \mathrm{Borel}(X) \to [0, \infty]$ a Radon measure on X. Prove that if $K \subseteq X$ is compact, then

$$\mu(K) = \inf \left\{ \int_X f \, d\mu : f \in C_c(X), K \prec f \right\}.$$

Problem 8. Let $F, G : \mathbb{R} \to \mathbb{R}$ be increasing, right-continuous functions with $F(0) = G(0) = 0$. Let μ_F and μ_G be the Lebesgue–Stieltjes measures with distribution functions F and G respectively. Show that if either F is continuous or G is continuous, then

$$\int_{(a,b]} F \, d\mu_G + \int_{(a,b]} G \, d\mu_F = F(b)G(b) - F(a)G(a).$$