# MATH 303 – Measure Theory
# Riesz Representation Theorem Jigsaw
# Part A

**Riesz Representation Theorem.** Let $X$ be a locally compact Hausdorff space. Given a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu$ such that

$$\varphi(f) = \int_X f \; d\mu$$

for every $f \in C_c(X)$.

**Instructions:** For this "jigsaw" activity, you will be (re)constructing the proof of the Riesz representation theorem in groups. Your task, as someone working on Part A, is to prove the uniqueness part of the theorem, which can be broken into the following two problems. Discuss the problem with your Part A group and be prepared to present a solution to peers who handled different parts of the theorem.

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**Problem 1.** Let $X$ be a locally compact Hausdorff space, and suppose $\mu$ is a Radon measure on $X$. Show that for any every open set $U \subseteq X$, one has the following identity:

$$\mu(U) = \sup \left\{ \int_X f \; d\mu : f \in C_c(X), 0 \leq f \prec U \right\}.$$

**Problem 2.** Using Problem 1, show that if $\mu$ and $\nu$ are two Radon measures on $X$ and $\int_X f \; d\mu = \int_X f \; d\nu$ for every $f \in C_c(X)$, then $\mu = \nu$.

# MATH 303 – Measure Theory
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# Part B

> **Riesz Representation Theorem.** Let $X$ be a locally compact Hausdorff space. Given a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu$ such that
>
> $$\varphi(f) = \int_X f \, d\mu$$
>
> for every $f \in C_c(X)$.

**Instructions:** For this "jigsaw" activity, you will be (re)constructing the proof of the Riesz representation theorem in groups. Your task, as someone working on Part B, is to construct an outer measure corresponding to a given positive linear functional. The problems below will guide you through the construction. Discuss the problem with your Part B group and be prepared to present a solution to peers who handled different parts of the theorem.

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For each open set $U \subseteq X$, define

$$m(U) = \sup \{\varphi(f) : f \in C_c(X), 0 \leq f \prec U\}.$$

Then define

$$\mu^*(E) = \inf \{m(U) : U \text{ is open and } E \subseteq U\}.$$

**Problem 1.** Check that if $U$ is an open set, then $\mu^*(U) = m(U)$.

**Problem 2.** Check that $\mu^*(\emptyset) = 0$ and $\mu^*$ is monotone.

**Problem 3.** Prove that $\mu^*$ is countably subadditive.[1]

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[1] After you have given some thought to this problem and made an initial attempt, the following hint may be useful: If $f \in C_c(X)$ and $f \prec \bigcup_{n\in\mathbb{N}} U_n$, then by compactness, there is a finite subcollection $U_{n_1}, \ldots, U_{n_k}$ such that $f \prec \bigcup_{j=1}^k U_{n_j}$. You may then apply partition of unity with $K = \text{supp}(f) \subseteq \bigcup_{j=1}^k U_{n_j}$ to obtain a useful decomposition of $f$.

# MATH 303 – Measure Theory
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# Part C

**Riesz Representation Theorem.** Let $X$ be a locally compact Hausdorff space. Given a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu$ such that

$$\varphi(f) = \int_X f \, d\mu$$

for every $f \in C_c(X)$.

**Instructions:** For this "jigsaw" activity, you will be (re)constructing the proof of the Riesz representation theorem in groups. Your task, as someone working on Part C, is to prove that Borel sets are measurable for the outer measure that was constructed in Part B. The problems below will guide you through the proof. Discuss the problem with your Part C group and be prepared to present a solution to peers who handled different parts of the theorem.

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For each open set $U \subseteq X$, define

$$m(U) = \sup \{\varphi(f) : f \in C_c(X), 0 \leq f \prec U\}.$$

Then define

$$\mu^*(E) = \inf \{m(U) : U \text{ is open and } E \subseteq U\}.$$

In Part B, it was shown that $\mu^*$ is an outer measure. You will now show that Borel sets are measurable by solving the following two problems.

**Problem 1.** Justify that it suffices to prove the following: if $U$ is an open set and $E$ is an arbitrary subset of $X$ with $\mu^*(E) < \infty$, then

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U). \tag{1}$$

**Problem 2.** Prove that (1) holds for every open set $U$ and every subset $E$ with $\mu^*(E) < \infty$.

# MATH 303 – Measure Theory
# Riesz Representation Theorem Jigsaw
# Part D

**Riesz Representation Theorem.** Let $X$ be a locally compact Hausdorff space. Given a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu$ such that

$$\varphi(f) = \int_X f \, d\mu$$

for every $f \in C_c(X)$.

**Instructions:** For this "jigsaw" activity, you will be (re)constructing the proof of the Riesz representation theorem in groups. Your task, as someone working on Part D, is to prove regularity properties of the measure constructed in Parts B and C. The problems below will guide you through the proof. Discuss the problem with your Part D group and be prepared to present a solution to peers who handled different parts of the theorem.

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For each open set $U \subseteq X$, define

$$m(U) = \sup \{\varphi(f) : f \in C_c(X), 0 \leq f \prec U\}.$$

Then define

$$\mu^*(E) = \inf \{m(U) : U \text{ is open and } E \subseteq U\}.$$

In Part B, it was shown that $\mu^*$ is an outer measure, and in Part C, it was shown that Borel sets are $\mu^*$-measurable. Let $\mu$ be the measure obtained by restricting $\mu^*$ to the Borel subsets of $X$. You will now show that $\mu$ is a Radon measure. You may freely use the fact that $\mu(U) = m(U)$ for open sets $U$ (this was proved in Part B).

**Problem 1.** Check that the definition of $\mu$ ensures that $\mu$ is outer regular.

**Problem 2.** Show that $\mu$ is inner regular on open sets.

**Problem 3.** Prove that $\mu$ is locally finite.

# MATH 303 – Measure Theory
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# Part E

> **Riesz Representation Theorem.** Let $X$ be a locally compact Hausdorff space. Given a positive linear functional $\varphi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu$ such that
>
> $$\varphi(f) = \int_X f \, d\mu \tag{1}$$
>
> for every $f \in C_c(X)$.

**Instructions:** For this "jigsaw" activity, you will be (re)constructing the proof of the Riesz representation theorem in groups. Your task, as someone working on Part E, is to prove that the Radon measure constructed in Parts B–D satisfies (1). The problems below will guide you through the proof. Discuss the problem with your Part E group and be prepared to present a solution to peers who handled different parts of the theorem.

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For each open set $U \subseteq X$, define

$$m(U) = \sup \{\varphi(f) : f \in C_c(X), 0 \le f \prec U\}.$$

Then define

$$\mu(E) = \inf \{m(U) : U \text{ is open and } E \subseteq U\}$$

for Borel sets $E \subseteq X$. The combined efforts of Parts B, C, and D show that $\mu$ is a Radon measure on $X$. You will prove that it satisfies (1) by completing the following problems.

**Problem 1.** Justify that it suffices to prove (1) for $f \in C_c(X)$ satisfying $0 \le f \le 1$.

Let $f \in C_c(X)$ with $0 \le f \le 1$, and let $K = \operatorname{supp}(f)$. Let $N \in \mathbb{N}$ be a large number, and decompose $K$ as a disjoint union $K = \bigsqcup_{n=0}^{N} K_n$ with

$$K_0 = \{x \in K : f(x) = 0\}$$

and

$$K_n = \left\{x \in K : f(x) \in \left(\frac{n-1}{N}, \frac{n}{N}\right]\right\}$$

for $n \in \{1, \ldots, N\}$. Then for $n \in \{1, \ldots, N\}$, define

$$f_n(x) = \begin{cases} 0, & \text{if } x \in K_m, m < n; \\ f(x) - \frac{n-1}{N}, & \text{if } x \in K_n; \\ \frac{1}{N}, & \text{if } x \in K_m, m > n. \end{cases}$$

**Problem 2.** Show that $f_n \in C_c(X)$ for each $n \in \{1, \ldots, N\}$, and $f = \sum_{n=1}^{N} f_n$.

**Problem 3.** Check that $\bigsqcup_{m>n} \overline{K}_m \prec N f_n \prec \bigsqcup_{m \ge n} K_m$ for each $n \in \{1, \ldots, N\}$.

**Problem 4.** Deduce that

$$\frac{1}{N}\mu\left(\bigsqcup_{m>n} K_m\right) \leq \varphi(f_n) \leq \frac{1}{N}\mu\left(\bigsqcup_{m\geq n} K_m\right)$$

and

$$\frac{1}{N}\mu\left(\bigsqcup_{m>n} K_m\right) \leq \int_X f_n \ d\mu \leq \frac{1}{N}\mu\left(\bigsqcup_{m\geq n} K_m\right)$$

for each $n \in \{1, \dots, N\}$.

**Problem 5.** Finish the proof of (1) by summing over $n \in \{1, \dots, N\}$ and applying the squeeze theorem.