

Exercise Sheet Solutions #9

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P1. (Area interpretation of the integral) Let (X, \mathcal{B}, μ) a s -finite probability space and $f : X \rightarrow \mathbb{R}$ a measurable function. Let $R_f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq f(x)\}$. Prove that

(a) If $f : X \rightarrow [0, \infty]$ is measurable then $R_f \in \mathcal{B} \otimes \text{Borel}(\mathbb{R})$ and

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_f).$$

where λ is the Lebesgue measure.

Solution: We start proving that R_f is $\mathcal{B} \otimes \text{Borel}(\mathbb{R})$ measurable. Indeed, notice that R_f is the intersection of $X \times [0, \infty)$ (which is measurable for the product sigma algebra) with the inverse image of $[0, \infty)$ through the map taking $(x, t) \in X \times \mathbb{R}$ to $f(x) - t \in \mathbb{R}$. This last map is measurable by being the composition of the map $(y, t) \in \mathbb{R}^2 \rightarrow y - t$ with the map $(x, t) \in X \times \mathbb{R} \rightarrow (f(x), t)$, where both maps are measurable by definition of the product sigma algebra and by hypothesis of f being measurable.

As $\mathbb{1}_{R_f}$ is measurable and the spaces (X, \mathcal{B}, μ) and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ are s -finite, by Fubini's theorem we have that

$$\begin{aligned} (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_f) &= \int_X \int_{\mathbb{R}} \mathbb{1}_{[0, f(x)]}(t) d\lambda(t) d\mu(x) \\ &= \int_X f(x) d\mu(x). \end{aligned}$$

(b) If $f : X \rightarrow \mathbb{R}$ integrable then

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f+}) - (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f-}).$$

Solution: We have that $f = f_+ - f_-$ with f_+ and f_- integrable positive functions and

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu.$$

By the previous part, this implies that

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f+}) - (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f-}),$$

concluding.

P2. Denote by λ the Lebesgue measure on \mathbb{R} . Show that if $A, B \subseteq \mathbb{R}$ are Lebesgue measurable sets such that $\lambda(A), \lambda(B) > 0$ then there exists a $t \in \mathbb{R}$ satisfying $\lambda(A \cap (B - t)) > 0$.

Solution: As λ is s -finite and the function $(x, t) \in \mathbb{R}^2 \rightarrow \mathbb{1}_A(x)\mathbb{1}_B(x+t) \in \mathbb{R}$ is measurable for the product sigma algebra (as it can be obtained using measurable operations from the functions $(x, t) \rightarrow \mathbb{1}_A(x)$, $(x, t) \rightarrow \mathbb{1}_B(t)$, and $(x, t) \rightarrow (x, x+t)$ which are measurable), Fubini's theorem

yields

$$\begin{aligned}\int \lambda(A \cap (B - t)) d\lambda(t) &= \int \int \mathbf{1}_A(x) \mathbf{1}_B(x+t) d\lambda(x) d\lambda(t) \\ &= \int \int \mathbf{1}_A(x) \mathbf{1}_B(x+t) d\lambda(t) d\lambda(x) \\ &= \int \int \mathbf{1}_A(x) \mathbf{1}_B(t) d\lambda(t) d\lambda(x) = \lambda(B) \lambda(A) > 0.\end{aligned}$$

Thus, there must be $t \in \mathbb{R}$ such that $\lambda(A \cap (B - t)) > 0$.

P3. Consider $X = [0, 1]$ equipped with the standard topology and $Y = [0, 1]$ equipped with the discrete topology. Define $\varphi : C_c(X \times Y) \rightarrow \mathbb{C}$ by $\varphi(f) = \sum_y \int_0^1 f(x, y) dx$ where the integral corresponds to the Riemman integral. Let μ be the Radon measure corresponding to φ . Is μ a product of $\lambda|_{\text{Borel}([0,1])}$ and the counting measure?

Hint: Consider the rectangle $\{0\} \times [0, 1]$.

Solution: The answer is no because the product $\rho = \lambda|_{\text{Borel}([0,1])} \otimes \nu$ where ν is the counting measure on $[0, 1]$, cannot be regular: If $U \times [0, 1]$ is an open set covering $\{0\} \times [0, 1]$ then we have that $\rho(U \times [0, 1]) = \lambda(U) \times \nu([0, 1]) = \lambda(U) \cdot \infty = \infty$, where we used that $\lambda(U) > 0$. Nevertheless, the measure of $\{0\} \times [0, 1]$ by definition of product measure is $\rho(\{0\} \times [0, 1]) = \lambda(\{0\}) \times \nu([0, 1]) = 0 \cdot \infty = 0$.