Exercise Sheet #8

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Szymon Sobczak

P1. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(Y, \mathcal{B})$ be a measurable space. Further, let $f : X \to Y$ be a measurable function. We define a *pushforward* measure $f_*\mu$ on $(Y, \mathcal{B})$ by

$$f_*\mu(A) := \mu(f^{-1}(A)),\ A \in \mathcal{B}.$$

Show that $f_*\mu$ is well-defined, and is a probability measure if μ is a probability measure.

> See Homework 1, Problem 1.

P2. Let $(X, \mathcal{F}, \mathbb{P})$ be probability space, and let X_n be a sequence of real valued random variables on X i.e. a sequence of Borel measurable maps $X_n : X \to \mathbb{R}$. Define the *distribution* of X_n as the pushforward measure on $\mathbb{R}$ given by

$$\mu_n(A) := \mathbb{P}(X_n^{-1}(A)),\ A \in \mathcal{B}(\mathbb{R}).$$

(a) Show that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous bijection, with continuous inverse, then if $\mu_n \to \mu$ vaguely, then also the laws of $g(X_n)$ converge vaguely to $g_*\mu$.

(b) Now show that if $\mu_n \to \mu$ *weakly*[1], and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g_*\mu_n \to g_\mu$ weakly.

> **(a)** First we note that the law of $g(X_n)$ is given by the pushforward measure $g_*\mu_n$. Now let $f \in C_c(\mathbb{R})$. Then we see that
>
> $$\int_{\mathbb{R}} f d(g_*\mu_n) = \int_{\mathbb{R}} f(g(x)) d\mu_n(x).$$
>
> As $f \circ g \in C_c(\mathbb{R})$, by the vague convergence of μ_n to μ, we have that
>
> $$\lim_{n\to\infty} \int_{\mathbb{R}} f d(g_*\mu_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f(g(x)) d\mu_n(x) = \int_{\mathbb{R}} f(g(x)) d\mu(x) = \int_{\mathbb{R}} f d(g_*\mu).$$
>
> Thus, $g_*\mu_n \to g_*\mu$ vaguely.
>
> **(b)** Now we take $f \in C_b(\mathbb{R})$, and note that $f \circ g \in C_b(\mathbb{R})$. Thus, by the weak convergence of μ_n to μ, we have that
>
> $$\lim_{n\to\infty} \int_{\mathbb{R}} f d(g_*\mu_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f(g(x)) d\mu_n(x) = \int_{\mathbb{R}} f(g(x)) d\mu(x) = \int_{\mathbb{R}} f d(g_*\mu).$$

P3. Let X be a compact metric spacew, and $T : X \to X$ be a continuous map. Show that there exists a T-invariant, Borel probability measure on X, i.e. a Borel probability measure μ such that $T_*\mu = \mu$.

Hint: consider the measures $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1}(T^k)_*\delta_x$*, where* δ_x *is the Dirac measure at some* $x \in X$*.*

---

[1] Weak convergence is defined analogously to vague convergence but replacing C_c by C_b.

Following the hint, define $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1}(T^k)_*\delta_x$. Each μ_n is a Borel probability measure, as it is a convex combination of Borel probability measures. Since X is compact, the family $\{\mu_n\}$ is precompact (see corolary in the lecture notes), so there exists a subsequence μ_{n_j} converging weakly to some Borel probability measure μ. Let us show that μ is T-invariant. For this, let $f \in C_b(X)$, then we have

$$\int_X f d(T_*\mu) = \int_X f \circ T d\mu = \lim_{j\to\infty}\int_X f \circ T d\mu_{n_j} = \lim_{j\to\infty}\int_X f dT_*\mu_{n_j}.$$

Note

$$T_*\mu_{n_j} = \frac{1}{n_j}\sum_{k=0}^{n_j-1}(T^{k+1})_*\delta_x = \mu_{n_j} + \frac{1}{n_j}\left((T^{n_j})_*\delta_x - \delta_x\right).$$

We also note the bounds

$$\left|\int_X f d((T^{n_j})_*\delta_x)\right| \le \|f\|_\infty, \quad \left|\int_X f d\delta_x\right| \le \|f\|_\infty,$$

which imply that

$$\left|\int_X f dT_*\mu_{n_j} - \int_X f d\mu_{n_j}\right| \le \frac{2}{n_j}\|f\|_\infty \to 0, \text{ as } j \to \infty.$$

Thus, we have

$$\int_X f d(T_*\mu) = \lim_{j\to\infty}\int_X f dT_*\mu_{n_j} = \lim_{j\to\infty}\int_X f d\mu_{n_j} = \int_X f d\mu,$$

which shows that $T_*\mu = \mu$.

P4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let X, Y be two real valued random variables on Ω with laws μ and ν. Define the *Fourier transform* of μ (equivalently the *Characteristic function* of X) as

$$\hat{\mu}(\xi) := \int_X e^{i\xi x} d\mu(x),\ \xi \in \mathbb{R}.$$

Show that if $\hat{\mu} = \hat{\nu}$ pointwise, then $\mu = \nu$.

Let us first note, that the characteristic function is always well defined for probability measures, as $x \mapsto e^{i\xi x}$ is bounded. Now, let us assume that $\hat{\mu} = \hat{\nu}$ pointwise. We want to show that $\mu = \nu$. For this let us consider a fumction $f \in C_c^\infty(\mathbb{R})$. By the Fourier inversion formula and Fubini's theorem, we have that

$$\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}}\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)e^{i\xi x}d\xi d\mu(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\hat{\mu}(\xi)d\xi.$$

Now since $\hat{\mu} = \hat{\nu}$, we have that

$$\int_{\mathbb{R}} f(x)d\mu(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\hat{\mu}(\xi)d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\hat{\nu}(\xi)d\xi = \int_{\mathbb{R}} f(x)d\nu(x).$$

Since we can approximate any bounded continuous function by functions in $C_c^\infty(\mathbb{R})$ with respect to pointwise convergence, we conclude by an application of dominated convergence theorem that $\mu = \nu$.

P5. Consider $\mathbb{Z}$ as a discrete topological space.

(a) Describe the space $C_c(\mathbb{Z})$.

(b) Describe the positive linear funcitonals on $C_c(\mathbb{Z})$.

(c) Let $\phi : C_c(\mathbb{Z}) \to \mathbb{C}$ be a positive linear functional, and let μ be the Radon measure representing ϕ via the Riesz representation theorem. When is μ a finite measure? (Give a characterisation in terms of properties of ϕ).

(a) Every function on a discrete space is continuous. Moreover, a subset $K \subset \mathbb{Z}$ is compact if and only if it is finite. Therefore, $C_c(\mathbb{Z})$ is the space of functions $f : \mathbb{Z} \to \mathbb{C}$ that vanish outside of a finite set. This can be identified with the direct sum $\bigoplus_{n\in\mathbb{Z}} \mathbb{C}$ by taking a basis of functions $e_n = 1_{\{n\}}$ for each $n \in \mathbb{Z}$.

(b) A linear functional is determined by its values on a basis. Given a positive linear functional $\phi : C_c(\mathbb{Z}) \to \mathbb{C}$, define $a_\phi : \mathbb{Z} \to \mathbb{C}$ as $a_\phi(n) := \phi(e_n)$. Since ϕ is positive, we have that $a_\phi(n) \geq 0$ for all $n \in \mathbb{Z}$. Conversely, given any function $a : \mathbb{Z} \to [0, \infty)$, we can define a positive linear functional $\phi_a : C_c(\mathbb{Z}) \to \mathbb{C}$ as

$$\phi_a(f) := \sum_{n\in\mathbb{Z}} a(n)f(n).$$

Thus, the positive linear functionals on $C_c(\mathbb{Z})$ are in one-to-one correspondence with functions $a : \mathbb{Z} \to [0, \infty)$.

(c) From the description above, let $a : \mathbb{Z} \to [0, \infty)$ be the function corresponding to ϕ. If $E \subset \mathbb{Z}$ is a finite set, then $1_E \in C_c(\mathbb{Z})$, so $\mu(E) = \phi(1_E) = \sum_{n\in E} a(n)$. Hence, by continuity from below we have

$$\mu(\mathbb{Z}) = \lim_{N\to\infty} \mu([-N, N] \cap \mathbb{Z}) = \lim_{N\to\infty} \sum_{n=-N}^{N} a(n) = \sum_{n\in\mathbb{Z}} a(n).$$

It follows that μ is a finite measure if and only if $a \in \ell^1(\mathbb{Z})$.