Exercise Sheet Solutions #7

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P1. Throughout this problem we denote λ as the Lebesgue measure on $\mathbb{R}$.

(a) Let A be a Lebesgue-measurable set on the real line such that $\lambda(A) > 0$. Show that the difference set $A - A = \{x - y \mid x, y \in A\}$ contains an open neighborhood of 0 in $\mathbb{R}$.
Hint: Prove that for each $r \in (1/2, 1)$, there is an interval $(a, b) \subseteq \mathbb{R}$ such that $\lambda(A \cap (a, b))/(b - a) \geq r$.

> ***Solution:*** By regularity, there is an open set $U \supseteq A$ such that $\mu(U) \leq \mu(A) + \epsilon$. As U is countable union of disjoint open intervals, we can write $U = \bigsqcup_i (a_i, b_i)$, and thus
>
> $$\sum_i \mu((a_i, b_i)) = \mu(U) \leq \sum_i \mu(A \cap (a_i, b_i)) + \epsilon.$$
>
> If the statement in the hint is false, then $\mu(A \cap (a_i, b_i) < r\mu((a_i, b_i))$ and thus
>
> $$\sum_i \mu((a_i, b_i)) = \mu(U) \leq \sum_i r\mu((a_i, b_i)) + \epsilon, \iff \mu(U)(1 - r) \leq \epsilon.$$
>
> So taking $\epsilon = \mu(A)(1 - r)/2$ we arrive to a contradiction, given that
>
> $$\mu(A) \leq \mu(U) \leq \epsilon/(1 - r) = \mu(A)/2 \Longrightarrow \mu(A) = 0.$$
>
> In particular, there is an interval (a, b) such that $r \leq \mu((a, b) \cap A)/\mu((a, b))$. Call $I = (a, b) \cap A$.
> Let $\delta \in (0, b - a)$. We assume by contradiction that $(-\delta, \delta) \cap (A - A)^c \neq \emptyset$. In other words, there is $x \in (-\delta, \delta)$ such that $(x + A) \cap A = \emptyset$. In particular $(x + ((a, b) \cap A)) \cap ((a, b) \cap A) = \emptyset$. This implies that
>
> $$\mu((I + x) \cup I) = 2\mu(I). \tag{1}$$
>
> On the other hand
>
> $$\mu((I + x) \cup I) \leq \mu((a, b) + x \cup (a, b)) < b - a + \delta.$$
>
> Thus
>
> $$b - a + \delta \geq 2\mu(I) \geq 2(b - a)r$$
>
> or equivalently $\delta > (b - a)(2r - 1)$. Taking δ small enough (less than $(b - a)(2r - 1)$), this leads to a contradiction.

(b) Let $(H, +)$ be a Lebegue measurable proper subgroup of $(\mathbb{R}, +)$. Show that $\lambda(H) = 0$.

> ***Solution:*** If by contradiction $\lambda(H) > 0$ then by the previous part, there is $\delta > 0$ such that $(-\delta, \delta) \subseteq H - H = H$. However, the previous inclusion implies $H = \mathbb{R}$ which is a contradiction.

P2. **(a)** Show that the Dirac functional $\delta_0 \in \mathcal{M}[0, 1]$ defined by $\delta_0(f) := f(0)$ is not of the form

$$\delta_0(f) = \int_0^1 f(t)g(t)\mathrm{d}t \quad (f \in C[0, 1])$$

for any $g \in C[0, 1]$.

Solution: If we assume by contradiction that there is a $g \in C([0,1])$ such that for all $f \in C([0,1])$ $f(0) = \int_0^1 f(t)g(t)dt$, then for each $\epsilon > 0$, take $\delta > 0$ and for each $f \in C([\epsilon, 1])$, extend f continuously to a function f' such that for $x \in [0,1] \setminus (\epsilon - \delta, \epsilon)$:

$$f'(x) = \begin{cases} 0 & \text{if } x \in [0, \epsilon - \delta] \\ f(x) & \text{if } x \in [\epsilon, 1] \end{cases}$$

and f' is a line that connects 0 and $f(\epsilon)$ in $[\epsilon - \delta, \epsilon]$. Then, we will have

$$0 = f'(0) = \int_0^1 f'(t)g(t)dt = \int_{\epsilon-\delta}^{\epsilon} f'(t)g(t)dt + \int_{\epsilon}^1 f(t)g(t)dt. \tag{2}$$

Noticing that

$$|\int_{\epsilon-\delta}^{\epsilon} f'(t)g(t)dt| \leq |f(\epsilon)| \cdot ||g||_\infty \delta,$$

and making $\delta \to 0$ in equation (2) we get

$$0 = \int_{\epsilon}^1 f(t)g(t)dt.$$

Given that $f \in C([\epsilon, 1])$ was arbitrary, we get $g(t) = 0$ for all $t \geq \epsilon$. As $\epsilon > 0$ was arbitrary and g is continuous, this implies $g = 0$, which is a contradiction with the hypothesis (by taking $f \equiv 1$ for example).

(b) Define $\psi : C[0,1] \to \mathbb{R}$ by

$$\psi(f) = \frac{f(0) + f(1)}{2} + \int_0^1 tf(t)\mathrm{d}t.$$

Determine the measure from the Riesz-Markov-Kakutani theorem corresponding to ψ, i.e. a regular Borel measure μ on $[0,1]$ such that $\psi(f) = \int_{[0,1]} f \, \mathrm{d}\mu$ for $f \in C[0,1]$. Calculate $\mu([0,1])$.

Solution: By Riesz representation theorem, there is a Radon measure ν such that $\int_0^1 tf(t)dt = \int f\nu$ (actually this equation defines the measure ν).
We will have that $\psi(f) = \int fd(\frac{\delta_0+\delta_1}{2} + \nu)$. Call $\mu = (\frac{\delta_0+\delta_1}{2} + \nu)$. Let us see that μ is a Radon measure on $[0,1]$. First of all, as μ is clearly positive, if we compute the measure of $[0,1]$ then we will prove that is finite in compact sets (by being finite). Notice that $\mu([0,1]) = \int_{[0,1]} 1 d\mu = \frac{1+1}{2} + \int_0^1 t \cdot 1dt = 1 + (\frac{t^2}{2}) \mid_0^1 = \frac{3}{2}$.
For the outer regularity, if $E \subseteq [0,1]$ then we have 3 cases: If $E \subseteq (0,1)$ then $\mu(E) = \nu(E)$, in which the outer regularity follows from the regularity of ν. If for example $0 \in E$ and $1 \notin E$ then

$$\begin{aligned} inf\{\mu(U) : U \text{ is open and } E \subseteq U\} &= \frac{1}{2} + inf\{\nu(U) : U \text{ is open and } E \subseteq U\} \\ &= \frac{\delta_0(E) + \delta_1(E)}{2} + \nu(E) \\ &= \mu(E), \end{aligned}$$

which gives the regularity in this case, where we used that the infimum is reach with open sets excluding 1, i.e. contained in $[0, 1)$ given that E is contained in this set. A similar strategy gives the inner regularity, which concludes that μ is a Radon measure.

P3. In this exercise, we will construct a Haar measure[1] on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. For this, recall that one can identify functions $f : \mathbb{T}^n \to \mathbb{C}$ with $\mathbb{Z}^n$-invariant functions $F : \mathbb{R}^n \to \mathbb{C}$ on $\mathbb{R}^n$ (i.e. we require $F(x + m) = F(x)$ for all $m \in \mathbb{Z}^n$). Furthermore, f is continuous (measurable) if and only if F is continuous (measurable). We define a measure m on $\mathbb{T}^n$ by requiring that

$$\int_{\mathbb{T}^n} f \,\mathrm{d}m = \int_{[0,1]^n} F \,\mathrm{d}m_{\mathbb{R}^n}$$

where $m_{\mathbb{R}^n}$ is the Lebesgue measure on $\mathbb{R}^n$ and f, F are measurable and correspond to each other. Justify that m is well defined and show that m is a Haar measure on $\mathbb{T}^n$.

Define $\Phi : C(\mathbb{T}^r) \to C(\mathbb{R}^r)$ as $\Phi(f) = F$ where F is constructed as in the statement of the question (i.e. $F(x) = f(x \bmod 1)$). This operation is well defined given that for each $f \in C(\mathbb{T}^r)$ results in a continuous function $\Phi(f)$ (because is isometric, doting $\mathbb{T}^r$ of the distance $d(x, y) = ||x-y||_{\mathbb{T}^r}$ where $||x||_{\mathbb{T}^r}$ is the minimum distance from x to $\mathbb{Z}^r$). Therefore, the operator $\psi : C(\mathbb{T}^r) \to \mathbb{C}$

$$\psi(f) = \int_{[0,1]^r} \Phi(f) dm_{\mathbb{R}^r}$$

is well defined.

This functional is clearly linear and positive (given that Φ is) thus, m is well defined. Now, for proving that is a Haar measure on $\mathbb{T}^n$, what is left to prove is that is left-invariant. Let $t \in \mathbb{T}^r$ and define $f_t(x) = f(x + t)$. We want to show that

$$\int_{\mathbb{T}^r} f_t dm = \int_{\mathbb{T}^r} f dm. \tag{3}$$

Without loss of generality, assume that $t = (0, \ldots, 0, t_i, 0, \ldots, 0)$ (if we prove the invariance for

[1] A Haar measure is a Radon measure on a locally compact topological group $(G, +)$ that is left-invariant, meaning that for any Borel set S and $g \in G$, $\mu(g + S) = \mu(S)$.

each coordinate, the global invariance will follow). Then, denoting $F = \Phi(f)$, and $F_t = \Phi(f_t)$:

$$
\begin{aligned}
\int_{\mathbb{T}^r} f_t dm &= \int_{[0,1]^n} \Phi(f_t) dm_{\mathbb{R}^n} \\
&= \int_{[0,1]^n} F_t(x) dm_{\mathbb{R}^n}(x) \\
&= \int_{[0,1]^{i-1}\times[t_i,1+t_i]\times[0,1]^{n-i}} F dm_{\mathbb{R}^n} \\
&= \int_{[0,1]^{i-1}\times[t_i,1]\times[0,1]^{n-i}} F dm_{\mathbb{R}^n} + \int_{[0,1]^{i-1}\times[1,1+t_i]\times[0,1]^{n-i}} F dm_{\mathbb{R}^n} \\
&= \int_{[0,1]^{i-1}\times[t_i,1]\times[0,1]^{n-i}} F dm_{\mathbb{R}^n} + \int_{[0,1]^{i-1}\times[0,t_i]\times[0,1]^{n-i}} F(x + (0,...,0,1,0,...,0)) dm_{\mathbb{R}^n}(x) \\
&= \int_{[0,1]^{i-1}\times[t_i,1]\times[0,1]^{n-i}} F dm_{\mathbb{R}^n} + \int_{[0,1]^{i-1}\times[0,t_i]\times[0,1]^{n-i}} F(x) dm_{\mathbb{R}^n}(x) \\
&= \int_{[0,1]^n} F dm_{\mathbb{R}^n} \\
&= \int_{\mathbb{T}^r} f dm
\end{aligned}
$$

concluding that m is invariant.