Exercise Sheet Solutions #6

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P1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a random variable, i.e., $\forall A \in \mathcal{B}(\mathbb{R}), X^{-1}(A) \in \mathcal{F}$. Prove that the distribution function of X, $F_X : \mathbb{R} \to [0,1]$ defined by $F(x) = \mathbb{P}(X \leq x)$, determines the measure induced by X, $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A))$.

> ***Solution:*** Notice that $F(x) = \mathbb{P}(X \leq x) = \mathbb{P}_X((-\infty, x])$. In addition, for $y \leq x$, we have $F(x) - F(y) = \mathbb{P}_X((y,x])$. Thus, by this and the continuity of the measure, we conclude that F determines the values of $\mathbb{P}_X$ in the semi-algebra:
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> $$S = \{(a,b] \mid a,b \in \overline{\mathbb{R}}\}. \tag{1}$$
>
> As this semialgebra generates $\mathcal{B}(\mathbb{R})$, by Caratheodory's theorem - and the fact that $\mathbb{P}_X$ is sigma finite by being a probability measure- we conclude that there is a unique extension to $\mathcal{B}(\mathbb{R})$.

P2. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue measure. Let μ be a measure on $\mathcal{B}(\mathbb{R})$ that satisfies the following conditions:

i) For all $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$: $\mu(A) = \mu(A + x)$.
ii) $0 < \mu((0,1]) < \infty$.

Show that there exists $\alpha > 0$ such that $\mu = \alpha\lambda$.

> ***Solution:*** Notice that the natural candidate for α is $\mu((0,1])$.
>
> Let us first prove that for $a,b \in \overline{\mathbb{R}}$, $\mu((a,b]) = \alpha\lambda((a,b])$. Indeed, we start taking $a,b \in \mathbb{Z}$ with $a < b$. In this case, using that $(i,i+1] - i = (0,1]$ for $i \in \mathbb{Z}$ and the fact that μ is invariant under translation
>
> $$\begin{aligned} \mu((a,b]) &= \mu\left(\bigsqcup_{i=a}^{b-1}(i,i+1]\right) \\ &= \sum_{i=a}^{b-1} \mu((i,i+1]) \\ &= \sum_{i=a}^{b-1} \mu((0,1]) \\ &= (b-a) \cdot \alpha \\ &= \alpha \cdot \lambda((a,b]). \end{aligned}$$
>
> For extending this for $a,b \in \mathbb{Q}$, we write $a = p_a/q$ and $b = p_b/q$ with $q \in \mathbb{N}$ and $p_a, p_b \in \mathbb{Z}$. Calling $p = p_b - p_a$ and using the translation invariance once again
>
> $$\mu((a,b]) = \mu\left(\left(0, \frac{p_b - p_a}{q}\right]\right) = \mu\left(\left(0, \frac{p}{q}\right]\right). \tag{2}$$

On the ther hand, we observe that

$$\begin{aligned}\mu\left((0,p]\right) &= \mu\left(\bigsqcup_{i=0}^{q-1}\left(\frac{p}{q}i,\frac{p}{q}(i+1)\right]\right)\\ &= \sum_{i=0}^{q-1}\mu\left(\left(\frac{p}{q}i,\frac{p}{q}(i+1)\right]\right)\\ &= \sum_{i=0}^{q-1}\mu\left(\left(0,\frac{p}{q}\right]\right)\\ &= q\cdot\mu\left(\left(0,\frac{p}{q}\right]\right).\end{aligned}$$

In consequence

$$\mu((a,b]) = \frac{1}{q}\mu\left((0,p]\right) = \frac{p}{q}\cdot\mu((0,1]) = \alpha\cdot\lambda((a,b]). \tag{3}$$

By continuity of the measures, we get that

$$\mu((a,b]) = \alpha\lambda((a,b]) \tag{4}$$

for each $a,b\in\overline{\mathbb{R}}$. Hence, we conclude that the measure μ and $\alpha\lambda$ coincide in the generating semialgebra

$$\mathcal{S} = \{(a,b] \mid a,b\in\overline{\mathbb{R}}\}, \tag{5}$$

and by Carathéodory extension theorem, we conclude that $\mu=\alpha\lambda$ by the fact that both measures are sigma finite.

P3. In this exercise, we prove that there exists a Lebesgue non-measurable subset of $\mathbb{R}$. For this, we define an equivalence relation on $[0,1)$ by $x\mathcal{R}y$ if $y-x\in\mathbb{Q}$. By axion of choice, let $E\subseteq[0,1)$ be a set containing exactly one representative of each equivalence class, and for each $t\in\mathbb{Q}\cap[0,1]$ let $E_t=\{x+t \bmod 1 \mid x\in E\}\subseteq[0,1)$.

- **(a)** Show that the sets $(E_t)_{t\in\mathbb{Q}\cap[0,1]}$ are pairwise disjoint.
- **(b)** Show that

$$\bigsqcup_{t\in\mathbb{Q}\cap[0,1]} E_t = [0,1).$$

- **(c)** Assume by contradiction that E is Lebesgue measurable. Show that for every $t\in\mathbb{Q}\cap[0,1)$, E_t is Lebesgue measurable and $\lambda(E_t)=\lambda(E)$.
- **(d)** Conclude by arriving to a contradiction.

Solution: See Theorem 5.30 in the lecture notes.

P4. Consider λ as the Lebesgue measure on $\mathbb{R}$, and let A be a Lebesgue measurable set. Prove that if $\lambda(A)>0$, then A contains a non-measurable set.

Solution: Let $V\subseteq[0,1)$ be a Vitali's set (this is a set E as in the proof of theorem 5.30 in the lecture notes, or in the exercise above). We first prove that every measurable subset C of V must have measure 0. Indeed, notice that $\{C+q\}_{q\in\mathbb{Q}\cap[0,1]}$ are all disjoint and measurable. Hence

$$\sum_{q\in\mathbb{Q}\cap[0,1]}\lambda(C+q) = \lambda(\bigcup^{\cdot}_{q\in\mathbb{Q}\cap[0,1]} C+q)) \le \lambda([-1,2]) = 3,$$

as all terms in the sum are equal, it is imperative that $\lambda(C) = 0$.

Now, let A be a measurable set with $\lambda(A) > 0$. Without loss of generality, we can assume $A \subseteq [0,1]$, because we can always fine $n \in \mathbb{Z}$ with $\mu(A \cap [n, n+1]) > 0$, and then $(A - n) \cap [0,1]$ must have positive measure, so by replacing A by $A - n$ we can assume this. Notice that

$$A = \bigsqcup_{q \in \mathbb{Q} \cap [0,1]} A \cap (V + q), \tag{6}$$

and if there is $q \in \mathbb{Q} \cap [0,1]$ such that $A \cap (V + q)$ is not measurable, we are done. In consequence, we can assume by contradiction that $A \cap (V + q)$ are all measurable, and then they must have measure 0, given that $A \cap (V + q) = (A - q) \cap V$. Consequently

$$0 \leq \lambda(A) = \lambda(\dot{\bigcup_{q \in \mathbb{Q} \cap [0,1]}} A \cap (V + q)) \leq \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda(A \cap (V + q)) = 0,$$

which is a contradiction.