Exercise Sheet Solutions #4

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Szymon Sobczak

P1. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a measurable function. Then prove that $f = 0$ a.e. if and only if $\int_E f d\mu = 0$ for every measurable set $E \in \mathcal{B}$.

Solution: If $f = 0$ a.e. it follows that $f\mathbb{1}_E = 0$ a.e. and thus

$$\int_E f\mu = \int_X \mathbb{1}_E f d\mu = 0.$$

Let us assume now that $\int_E f d\mu = 0$ for every measurable set $E \in \mathcal{B}$. As

$$0 = \int_E f d\mu = \int_E Re(f) d\mu + i \int_E Im(f) d\mu \Rightarrow 0 = \int_E Re(f) d\mu = \int_E Im(f) d\mu,$$

whence it follows that we can assume without loss of generality that $f : X \to \mathbb{R}$. Take $E = \{x \in X \mid f(x) > 0\}$. We have that $f\mathbb{1}_E \geq 0$ a.e.. Besides

$$0 = \int_E f d\mu = \int \mathbb{1}_E f d\mu,$$

which implies that $\mathbb{1}_E f = 0$ a.e.. By a similar argument with the set E^c we get that $\mathbb{1}_{E^c} f = 0$ a.e., which concludes that $f = 0$ a.e..

P2. Prove that a space is complete if and only if the following property holds: for functions $f, g; X \to \mathbb{C}$, if f is measurable and $f = g$ a.e., then g is also measurable.

Solution: Assume that the space is complete. Let $B \subseteq \mathbb{C}$ an open set. Then, we notice that $g^{-1}(B) \setminus f^{-1}(B), f^{-1}(B) \setminus g^{-1}(B) \subseteq \{x \in X \mid f(x) \neq g(x)\}$ which has measure 0. As the space is complete, we have that $g^{-1}(B) \setminus f^{-1}(B)$ and $f^{-1}(B) \setminus g^{-1}(B)$ are measurable. The latter set being measurable implies that $(g^{-1}(B) \cap f^{-1}(B))$ is measurable due to

$$(g^{-1}(B) \cap f^{-1}(B)) = f^{-1}(B) \setminus (f^{-1}(B) \setminus g^{-1}(B)).$$

Thus we have that the set $g^{-1}(B) = (g^{-1}(B) \setminus f^{-1}(B)) \cup (g^{-1}(B) \cap f^{-1}(B))$ is measurable, and thus g is measurable by the fact that B was arbitrary. For the other direction, let $A \subseteq X$ be a measurable set of measure 0. Let $B \subseteq A$. Then $\mathbb{1}_A = \mathbb{1}_B$ a.e., which implies that $\mathbb{1}_B$ is measurable, and thus B is in the sigma algebra. In this way, we conclude the completeness.

P3. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{B} : \mu(N) = 0\}$ be the σ-ideal of μ-null sets. Show that the family $\overline{\mathcal{B}} = \{E \cup F : E \in \mathcal{B}, F \subseteq N \in \mathcal{N}\}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to $\overline{\mathcal{B}}$.

Solution: First, we see that $\overline{\mathcal{B}}$ is a sigma algebra. Indeed, we have that $X \in \mathcal{B} \subseteq \overline{\mathcal{B}}$. On the other hand, if $E \cup F \in \overline{\mathcal{B}}$ with $E \in \mathcal{B}$ and $F \subseteq N \in \mathcal{N}$, we have that

$$(E \cup F)^c = E^c \cap F^c = E^c \cap N^c \cup (E^c \cap F^c \setminus N^c),$$

in where the first set is measurable and the second is contained in the set N which has measure 0. Hence, $(E \cup F)^c \in \mathcal{B}$.

Now, we take $E_i \cup F_i \in \overline{\mathcal{B}}$ with $E_i \in \mathcal{B}$ and $F_i \subseteq N_i \in \mathcal{N}$, for $i \in \{1, 2, \dots\}$. Then, given that

$\bigcup_i E_i \in \mathcal{B}$, $\bigcup_i F_i \subseteq \bigcup_i N_i \in \mathcal{N}$, we have that

$$\bigcup_i (E_i \cup F_i) = \bigcup_i E_i \cup \bigcup_i F_i \in \overline{\mathcal{B}},$$

concluding thus that $\overline{\mathcal{B}}$ is a sigma algebra. Finally, we define $\overline{\mu}(E \cup F) := \mu(E)$. This clearly defines a measure (check this! it should follow from elementary properties). This extension is unique by the fact that any extension of μ to $\overline{\mathcal{B}}$ must give sets $F \subseteq N \in \mathcal{N}$ measure 0, thus if ν is one of those extensions: $\nu(E \cup F) = \nu(E) = \mu(E)$.

P4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove that simple functions are dense in $L^1(X)$.

Solution: Let $f \in L^1(X)$ taking positive values. The, we already know that there are simple functions $(f_n)_n \subseteq L^1(X)$ such that $0 \leq f_n \leq f$ and $f_n \nearrow f$. Thus

$$\|f - f_n\|_1 = \int |f - f_n| d\mu = \int f - f_n d\mu \to 0.$$

Now, for a function f taking real values in $L^1(X)$, we divide $f = f_+ - f_-$. Let $(f_n)_n$ and $(g_n)_n$ sequences of simple functions such that $f_n \to f_+$ and $g_n \to f_-$. Then the sequence of simple functions $(f_n - g_n)_n$ is such that

$$\|f - (f_n - g_n)\|_1 = \|(f_+ - f_n) - (f_- - g_n)\|_1 \leq \|(f_+ - f_n)\|_1 + \|(f_- - g_n)\|_1 \to 0 \text{ as } n \to \infty.$$

A similar argument gives the approximation of a complex function f, by dividing $f = Re(f) + iIm(f)$, concluding that simple functions are dense in $L^1(X)$.

P5. Let $(X, \mathcal{B}, \mu)$ be a measure space and $E \in \mathcal{B}$ a set of positive measure. If $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets and $E = \bigcup_{n\in\mathbb{N}} E_n$ prove that for every function $f \in L^1(X)$

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \int_E f d\mu$$

and state and prove an analogous result for decreasing sequences.

Solution: We notice that $\mathbb{1}_{E_n} \to \mathbb{1}_E$ a.e., and thus $\mathbb{1}_{E_n} f \to \mathbb{1}_E f$. This in addition to $|\mathbb{1}_{E_n} f| \leq |f|$ and the fact that $|f|$ is integrable, implies by dominated convergence theorem, we have that

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \lim_{n\to\infty} \int \mathbb{1}_{E_n} f d\mu = \int \mathbb{1}_E f d\mu = \int_E f d\mu,$$

concluding.

P6. **(Problem 3.2.)** Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(A_n)_{n\in\mathbb{N}}$ be a family of measurable sets with $a = inf_{n\in\mathbb{N}} \mu(A_n) > 0$. We aim to show that there is a set $E \subseteq \mathbb{N}$ such that $\bar{d}(E) := \limsup_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} \geq a$, and for any finite set $F \subseteq E, F \neq \emptyset$, one has $\mu\left(\bigcap_{n\in F} A_n\right) > 0$.

Solution: First of all, notice that we can assume without loss of generality that for every $F \subseteq \mathbb{N}$ finite, we have that $\bigcap_{n\in F} A_n \neq \emptyset$ if and only if $\mu(\bigcap_{n\in F} A_n) > 0$. Indeed, we can replace $(A_n)_{n\in\mathbb{N}}$ by $(\tilde{A}_n)_{n\in\mathbb{N}}$ where

$$\tilde{A}_n = A_n \setminus \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} A_n,$$

where

$$\mathcal{F} = \{F \subseteq \mathbb{N} \mid |F| < \infty, \bigcap_{n\in F} A_n \neq \emptyset, \mu(\bigcap_{n\in F} A_n) = 0\}.$$

Notice that $\bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} A_n$ is countable union of finite intersection of measurable sets, so it is measurable. Moreover, it has measure zero because the aforementioned sets have measure zero. So, we have that for each $n \in \mathbb{N}$, $\tilde{A}_n \subseteq A_n$ is such that

$$\mu(A_n) = \mu(\tilde{A}_n),$$

and the same applies for every finite intersection, which shows that we can replace A_n by $\tilde{A}_n$ Notice that by Fatou's lemma

$$\int \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) d\mu(x) \geq \limsup_{N\to\infty} \int \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) d\mu(x) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_n) \geq a.$$

Therefore, there is $x \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) \geq a.$$

Call $E = \{n \in \mathbb{N} \mid x \in A_n\}$. Then notice that $n \in E$ if and only if $\mathbb{1}_{A_n}(x) = 1$. Hence, we have that

$$\overline{d}(E) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{E}(n) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) \geq a.$$

In addition, for every finite subset $F \subseteq E$, we have that $\forall n \in F$, $x \in A_n$, and thus $\bigcap_{n\in F} A_n \neq \emptyset$, which implies that $\mu\left(\bigcap_{n\in F} A_n\right) > 0$.