Exercise Sheet Solutions #3

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Szymon Sobczak

P1. **(Problem 3.3.)**(Borel–Cantelli lemma) If $(A_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of a probability space $(X, \mathcal{B}, \mu)$ and $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 0.$$

> ***Solution:*** Notice that what we want to prove is $\mu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_j) = 0$. As μ is finite and $(\bigcup_{n\geq N} A_n)_N$ is a nested sequence of sets, we have that by continuity of the measure and subadditivity:
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> $$\mu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_j) = \lim_{N\to\infty} \mu(\bigcup_{n\geq N} A_j) \leq \lim_{N\to\infty} \sum_{j\geq N} \mu(A_j) = 0, \tag{1}$$
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> in where the last equality comes from the fact that $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$.

P2. Let $(X, \mathcal{B}, \mu)$ be a probability space, and $(A_n)_{n\in\mathbb{N}}$ a family of *independent* measurable subsets such that $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$. By independence we mean that for any finite index set $\mathcal{I}$, we have $\mu(\bigcap_{n\in\mathcal{I}} A_n) = \Pi_{n\in\mathcal{I}} \mu(A_n)$. Show that

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 1.$$

> ***Solution:*** Let us start by showing that the independence property holds for countable index sets. Let I_n be a sequence of finite nested sets such that $I = \bigcup_{n\in\mathbb{N}} I_n$. Then, by continuity of the measure and independence of finite sets, we have that
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> $$\mu(\bigcap_{i\in I} A_i) = \lim_{n\to\infty} \mu(\bigcap_{i\in I_n} A_i) = \lim_{n\to\infty} \prod_{i\in I_n} \mu(A_i) = \prod_{i\in I} \mu(A_i). \tag{2}$$
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> Now, notice that what we want to prove is $\mu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_j) = 1$, which by de Morgan's laws is equivalent to proving that $\mu(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} A_j^c) = 0$. By continuity of measure and the hint we get
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> $$\mu(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} A_j^c) = \lim_{N\to\infty} \mu(\bigcap_{n\geq N} A_j^c) = \lim_{N\to\infty} \prod_{n\geq N} (1 - \mu(A_n)) \leq \lim_{N\to\infty} e^{-\sum_{n\geq N} \mu(A_n)} = 0.$$

P3. Let $(X, \mathcal{F}, \mu)$ be a measure space and f a measurable function. Prove the Markov-Chebyshev inequality:

$$\forall \alpha > 0, \mu(\{|f| > \alpha\}) \leq \frac{1}{\alpha} \int_{\{|f|>\alpha\}} |f| \, d\mu \leq \frac{1}{\alpha} \int |f| \, d\mu,$$

where we denote $\{|f| > \alpha\} = \{x \in X \mid |f(x)| > \alpha\}$.

> ***Solution:*** Observe that
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> $$\mu(\{|f| > \alpha\}) = \int 1 \cdot \mathbb{1}_{\{|f|>\alpha\}}(x) d\mu(x) \leq \int \frac{|f(x)|}{\alpha} \mathbb{1}_{\{|f|>\alpha\}}(x) d\mu(x) = \frac{1}{\alpha} \int_{\{|f|>\alpha\}} |f| \, d\mu \leq \frac{1}{\alpha} \int |f| \, d\mu. \tag{3}$$

P4. Let $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \nu)$ be probability spaces, and let $T : X \to Y$ be a measurable function. Define $T\mu(A) := \mu(T^{-1}(A))$ for each $A \in \mathcal{F}_Y$. Prove that $\nu = T\mu$ if and only if for all integrable function f:

$$\int_Y f \, d\nu = \int_X f \circ T \, d\mu. \tag{4}$$

Solution: $(\Leftarrow)$ For any set $A \in \mathcal{F}_Y$, take $f = \mathbb{1}_A$ in eq. (8) to get $\nu(A) = \mu(T^{-1}A)$.
$(\Rightarrow)$ Let $f = \sum_{i \in I} c_i \mathbb{1}_{A_i}$ a simple function. Then

$$\int_X f \circ T d\mu = \int_X \sum_{i \in I} c_i \mathbb{1}_{A_i} \circ T d\mu = \sum_{i \in I} c_i \int \mathbb{1}_{T^{-1}A_i} d\mu = \sum_{i \in I} c_i \mu(T^{-1}A_i) = \sum_{i \in I} c_i \nu(A_i) = \int \sum_{i \in I} c_i \mathbb{1}_{A_i} d\nu. \tag{5}$$

Hence, the statement is true for simple functions. Let f be a positive measurable function. Take $(f_n)_n$ a sequence of positive simple functions such that $f_n \nearrow f$ as $n \to \infty$. Then $f_n \circ T \nearrow f \circ T$ as $n \to \infty$ as well, so by monotone convergence theorem, we have that having for each $n \in \mathbb{N}$

$$\int_Y f_n \, d\nu = \int_X f_n \circ T \, d\mu, \tag{6}$$

implies that

$$\int_Y f \, d\nu = \lim_{n \to \infty} \int_Y f_n \, d\nu = \lim_{n \to \infty} \int_X f_n \circ T \, d\mu = \int_X f \circ T \, d\mu. \tag{7}$$

Consequently, we have the statement for positive functions. To conclude, notice that for every measurable function f, we can write $f = f_+ - f_-$ where f_+ and f_- are positive integrable functions. Thus,

$$\int_Y f \, d\nu = \int_Y f_+ \, d\nu - \int_Y f_- \, d\nu = \int_X f_+ \circ T \, d\mu - \int_Y f_- \, d\nu = \int_X f_- \circ T \, d\mu = \int_X (f_+ - f_-) \circ T \, d\mu = \int_X f \circ T \, d\mu, \tag{8}$$

concluding.