Exercise Sheet Solutions #1

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P1. Show that $J^*(\mathbb{Q} \cap [0,1]) = J^*([0,1] \setminus \mathbb{Q}) = 1$, and $J_*(\mathbb{Q} \cap [0,1]) = J_*([0,1] \setminus \mathbb{Q}) = 0$.

> ***Solution:*** We will prove for just $\mathbb{Q} \cap [0,1]$ given that the proofs for $[0,1] \setminus \mathbb{Q}$ are analogous.
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> Let $S \subseteq \mathbb{Q} \cap [0,1]$ be a simple set (disjoint union of boxes). If $B = [b_1, b_2] \subseteq S$ is a box that participates in S, then $B \subseteq \mathbb{Q}$, but as $\mathbb{Q}$ has no interior that implies that $b_1 = b_2$ and B is a point. In particular $Vol(B) = 0$. Thus $Vol(S) = 0$ and we conclude that $J_*(\mathbb{Q} \cap [0,1]) = 0$.
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> Secondly, take a simple set $S \supseteq \mathbb{Q} \cap [0,1]$. We will show that $S \supseteq [0,1]$. Let $r \in [0,1] \setminus \mathbb{Q}$. If $r \notin S$ then there is an open interval I containing R such that $I \cap S = \emptyset$ (given that S is closed and $r \in S^c$). On the other hand I clearly contains an element from $\mathbb{Q} \cap [0,1]$ which is a contradiction. Thus $S \supseteq [0,1]$ and therefore $J^*(\mathbb{Q} \cap [0,1]) \geq 1$ by definition of infimum. Nevertheless, this infimum is reached with $S = [0,1]$, so we are done.

P2. Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.

> ***Solution:*** Let $(u_n)_{n \in \mathbb{N}}$ be a countable dense set on U. We define I_1 as the largest open interval on U that contains u_1. Assume that we have defined $I_1, \ldots, I_m$ in this way. If $I_1, \ldots, I_m$ covers $(u_n)_n$ then we are done. If not, there is $n_m \geq m$ such that u_{n_m} is the first element that is not covered by these intervals. We define I_{m+1} as the largest interval that covers u_{n_m} inside U. By definition, we have that $\bigcup_m I_m \subseteq U$. On the other hand, if $u \in U$, then there is an interval $I \subseteq U$ that contains u, which also contains an element of $(u_n)_{n \in \mathbb{N}}$, and thus there is n such that $I \subseteq I_n$. We conclude that
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> $$\bigcup_m I_m = U.$$

P3. Let $U = \{(x,y) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.

> ***Solution:*** We present two solutions. First, if by contradiction we express U as a disjoint union of countably many open boxes $(B_i)_{i=1}^{\infty}$ then we have that
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> $$U = B_1 \cup \bigcup_{i=2}^{\infty} B_i, \tag{1}$$
>
> which is union of two non-empty open sets, which contradicts the connectedness of U.
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> For the second solution, we take a box $B = (a,b) \times (c,d)$ inside U. We notice that any point in the boundary of B cannot be cover with a box without overlapping the box B, which makes impossible to have the desirable expression.

P4. Give an example to show that the statement

$$\lambda^*(E) = \sup_{U \subset E, U \text{ open}} \lambda^*(U)$$

is false.

Solution: Take $E = \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$. Then, the right-hand side is going to be 0 (given that the only open set contained in E is the empty set). Meanwhile, the left-hand side $\lambda^*(E)$, is going to be 1 given that any open set that contains the irrational numbers must contain the whole interval $[0,1]$.

P5. (Area interpretation of the Riemann integral). Let $[a,b]$ be an interval, and let $f : [a,b] \to \mathbf{R}_+ := [0,\infty)$ be a bounded function. Show that f is Riemann integrable if and only if the set $E_+ := \{(x,t) : x \in [a,b]; 0 \leq t \leq f(x)\}$ is Jordan measurable in $\mathbf{R}^2$, in which case one has

$$\int_a^b f(x)dx = m^2(E_+).$$

where m^2 denotes two-dimensional Jordan measure.

Solution: We prove it first for piecewise constant functions. Let $f = \sum_{i=1}^n c_i I_n$, where $(I_i)_{i=1}^n$ is a partition of intervals of $[a,b]$ and c_i are positive coefficients. This function is Riemann integrable with integral

$$\int_a^b f = \sum_{i=1}^n c_i m(I_i). \tag{2}$$

On the other hand, we have that

$$\begin{aligned} E &= \{(x,t) : x \in [a,b], t \in [0, f(x)]\} \\ &= \bigcup_{i=1}^n \{(x,t) : x_1 \in I_i, t \in [0, f(x)]\} \\ &= \bigcup_{i=1}^n I_i \times [0, c_i], \end{aligned}$$

which implies that E is simple.

By Theorem 1.6 from the lecture notes, we know that E_+ is Jordan measurable if and only if $J_*(E_+) = J^*(E_+)$, so for concluding, it is enough to show that

$$J_*(E_+) = \sup\{\int_a^b h : h \text{ is a piecewise constant function with } h \leq f\}, \tag{3}$$

and

$$J^*(E_+) = \inf\{\int_a^b h : h \text{ is a piecewise constant function with } h \geq f\}. \tag{4}$$

We will just prove the former one, given that the latter is analogous. By the previous calculation it follows that if $h \leq f$ is a piecewise function, then

$$\int_a^b h = \sum_{i=1}^n c_i m(I_i) = m^2(\bigcup_{i=1}^n I_i \times [0, c_i]) \leq J_*(E_+). \tag{5}$$

On the other hand, let $S = \bigcup_{i=1}^n I_i \times C_i$ be a simple set contained in E_+. By refining and enlarging this boxes, we can assume that the sets $(I_i)_i$ are disjoint and that C_i is of the form $[0, c_i]$, where the quantity $\sum_{i=1}^n m(I_i)m(C_i)$ may only get bigger. Then, defining $h(x) = \sum_{i=1}^n c_i \mathbb{1}_{I_i}(x)$

we get

$$Vol(S) \leq \sum_{i=1}^{n} m(I_i)m(C_i) \leq \sum_{i=1} m(I_i)c_i = \int h \leq \sup\{\int_a^b h : h \text{ is a piecewise constant function with } h \leq f\}. \quad (6)$$

As S was arbitrary, we conclude that

$$J_*(E_+) \leq \sup\{\int_a^b h : h \text{ is a piecewise constant function with } h \leq f\}, \quad (7)$$

concluding.

P6. Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^{d} [a_i, b_i)\Big)$.

Solution: Let $(u_n)_{n\in\mathbb{N}}$ be a countable dense set on U. Then since U is open, for each n we can find an open box containing u_n which is contained in U. Making it slightly smaller if necessary, we can find a half-open box B_n containing u_n which is contained in U. Then $U = \bigcup_n B_n$ is a countable union of half-open boxes. We now describe how to construct a disjoint union using these. We start by taking $S = B_1$. Now, if $B_2 \cap B_1 = \emptyset$, we add it to the disjoint union $S = B_1 \cup B_2$. If not, let us describe how to partition $B_2 \setminus B_1$ into disjoint half-open boxes. Let $B_1 = \prod_{i=1}^{d}[a_i, b_i)$ and $B_2 = \prod_{i=1}^{d}[c_i, d_i)$. Then denote $I_i = [a_i, b_i) \cap [c_i, d_i)$, which is itself a half-open interval. Then $[c_i, d_i) = I_i \cup \bigcup_k J_k^i$, where J_k^i are disjoint half-open intervals, and k runs over a finite set (at most two elements). Then we can write $B_2 = \prod_{i=1}^{d} I_i \cup \prod_{i=1}^{d} \bigcup_k J_k^i$. Expanding the second term, we get a finite union of half-open boxes, which we add to S. For the inductive step, let S be a finite disjoint union of half-open intervals such that S covers $\bigcup_{i=1}^{n} B_i$. We take B_{n+1}, and repeat the above procedure looking at each box in S and partitioning B_{n+1} accordingly. After this finite number of steps, we add a finite number of boxes to S, which now covers $\bigcup_{i=1}^{n+1} B_i$. Continuing this way, we get a countable disjoint union of half-open boxes, which covers $\bigcup_{i=1}^{\infty} B_i = U$.