Exercise Sheet Solutions #12

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P1. Let (X, μ, T) a measure space. We say that $(f_n)_n$ *converges in measure* to f if given $\epsilon > 0$ we have $\lim_{n\to\infty} \mu(\{x : |f(x) - f_n(x)| > \epsilon\}) = 0$.

(a) Assume that $\mu(X) < \infty$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges in measure to f if and only if every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges a.e. to f.

Solution: ($\Longleftarrow$) First, we assume that every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges a.e. to f. We notice that $(f_n)_{n\in\mathbb{N}}$ converge in measure to f if and only if every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges to f in measure (by characterization of convergent sequences in $\mathbb{R}$). Thus, we can assume without loss of generality that $(f_n)_{n\in\mathbb{N}}$ converges a.e. to f. Let $A = \{x \in X \mid f_n(x) \to f(x)\}$, and fix an $\epsilon > 0$. Then $\mu(X\backslash A) = 0$. Since $\mu(X) < \infty, \mu(A) < \infty$ and we may apply Egoroff's theorem. Thus, fixing a $\delta > 0$, there exists a set E such that $\mu(A\backslash E) < \delta$ and $f_n \to f$ uniformly on E. Given $\epsilon > 0$, there exists N_δ such that $|f(x) - f_n(x)| < \epsilon$ for all $n > N_\delta$ and all $x \in E$. So, for $n > N_\delta, |f(x) - f_n(x)|$ can be greater than ϵ only on $(A\backslash E)\cup(X\backslash A)$. This means that

$$\begin{aligned}\mu\left(\{x : |f_n(x) - f(x)| > \epsilon\}\right) &\leq \mu(A\backslash E) + \mu(X\backslash A) \\ &< \delta + 0 = \delta\end{aligned}$$

for all $n > N_\delta$. This exactly shows that $\lim_{n\to\infty} \mu(\{x : |f(x) - f_n(x)| > \epsilon\}) = 0$.

($\Longrightarrow$) For the converse, we assume that $f_n \to f$ in measure. Let $\epsilon = 2^{-k}$. Given k, there exists $N(k) \geq k \vee N(k-1)$ such that

$$\mu\left(\left\{x : \left|f(x) - f_{N(k)}(x)\right| > 2^{-k}\right\}\right) < 2^{-k}.$$

Let $E_k = \left\{x : |f_{N(k)}(x) - f(x)\ | > \ 2^{-k}\right\}$. Then $\mu\left(E_k\right) < 2^{-k}$. If $x \notin \cup_{i=k}^\infty E_i$, then $x \in \left(\cup_{i=k}^\infty E_i\right)^c = \cap_{i=k}^\infty E_i^c$. For such x we have

$$\left|f_{N(i)}(x) - f(x)\right| < 2^{-i} \quad \text{for every } i \geq k$$

which implies $f_{N(i)}(x) \to f(x)$.
Let

$$A = \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty E_i.$$

So if $x \notin A$, then $f_{N(i)}(x) \to f(x)$. For any k,

$$\mu(A) \leq \mu\left(\cup_{i=k}^\infty E_i\right) \leq \sum_{i=k}^\infty 2^{-i} = 2^{-k+1},$$

so $\mu(A) = 0$, concluding.
Alternatively, we notice that $\sum_{k\geq1} \mu(E_k) < \infty$, and thus by the Borel-Cantelli lemma (P1 on sheet 3), we have that $\mu(E_k \text{ infinitely often}) = 0$, concluding.

(b) What happens if $\mu(X) = \infty$?

Consider $X = \mathbb{R}$ with the Lebesgue measure, and the functions $f_n = \mathbb{1}_{[n,n+1]}$. Then, $f_n \to 0$ poinwise, but $\{x \in X \mid |f_n(x)| = 1\}$ has measure 1 for each $n \in \mathbb{N}$. This shows that ($\Longleftarrow$) becomes false if we do not assume that $\mu(X) < \infty$. On the other hand, we never used the hypothesis $\mu(X) < \infty$ in ($\Longrightarrow$), so this implication is still true.

P2. Let $(X, \mathcal{F})$ be a measurable space, and let ν and μ be two measures such that $\nu \ll \mu$ and $g = \frac{\partial \nu}{\partial \mu}$. Prove that if $g \in L^p(\mu)$ and $A \in \mathcal{F}$, then

$$\nu(A) \leq \|g\|_{L^p(\mu)} \mu(A)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Solution: We have by Holder's inequality that

$$\begin{aligned} \nu(A) &= \int_A g d\mu \\ &= \int \mathbb{1}_A \cdot g d\mu \\ &\leq ||\mathbb{1}_A||_q \cdot ||g||_p = \mu(A)^{1/q} ||g||_p, \end{aligned}$$

concluding.

P3. Let $(X, \mathcal{B})$ be a measurable space, and let $\mu : \mathcal{B} \to [-\infty, \infty]$ be a signed measure with total variation $|\mu|$ (see Def. 9.9 in the notes). Then show that for any $E \in \mathcal{B}$,

$$\begin{aligned} |\mu|(E) &= \inf\{\nu(E) : \nu \text{ is a measure and } |\mu(F)| \leq \nu(F) \text{ for all } F \in \mathcal{B}\} \\ &= \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : E = \bigsqcup_{n \in \mathbb{N}} E_n \right\}. \end{aligned}$$

Solution: We prove the first equality. As $|\mu_+ - \mu_-| \leq \mu_+ + \mu_-$ we have that $|\mu| = \mu_+ + \mu_-$ participates in the infimum, and thus

$$|\mu|(E) \geq \inf\{\nu(E) : \nu \text{ is a measure and } |\mu(F)| \leq \nu(F) \text{ for all } F \in \mathcal{B}\}.$$

On the other hand, for ν a measure such that $|\mu(F)| \leq \nu(F)$ for each $F \in \mathcal{B}$. By Hahn's theorem, there is a partition $X = P \sqcup N$ such that μ_+ is supported on P and μ_- is supported on N. Thus, for each $A \in \mathcal{B}$

$$|\mu|(A) = \mu_+(A) + \mu_-(A) = |\mu(A \cap P)| + |\mu(A \cap N)| \leq \nu(A \cap P) + \nu(A \cap N) = \nu(A).$$

As ν was arbitrary, we conclude

$$|\mu|(E) \leq \inf\{\nu(E) : \nu \text{ is a measure and } |\mu(F)| \leq \nu(F) \text{ for all } F \in \mathcal{B}\}.$$

Now, we show the second equality. Let $E = \bigsqcup_{n \in \mathbb{N}} E_n \in \mathcal{B}$. Then

$$|\mu|(E) = \sum_{n \in \mathbb{N}} |\mu|(E_n) \geq \sum_{n \in \mathbb{N}} |\mu(E_n)|.$$

Taking the supremum gives

$$|\mu|(E) \geq \sup\left\{\sum_{n=1}^{\infty}|\mu\left(E_n\right)| : E = \bigsqcup_{n\in\mathbb{N}} E_n\right\}.$$

On the other hand, we have that

$$|\mu|(E) = \mu(E\cap P) - \mu(E\cap N) = |\mu(E\cap P)| + |\mu(E\cap N)| \leq \sup\left\{\sum_{n=1}^{\infty}|\mu\left(E_n\right)| : E = \bigsqcup_{n\in\mathbb{N}} E_n\right\},$$

concluding.