Exercise Sheet Solutions #11

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P1. Prove Young's inequality: Suppose that $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Then $f * g$ is defined a.e. and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Solution: If $r = \infty$, then p and q are conjugate and we obtain a pointwise bound on $f * g$ by Hölder's inequality:

$$|(f * g)(x)| = \left| \int_{\mathbb{R}} f(x-y) g(y) \, dy \right| \leq \int_{\mathbb{R}} |f(x-y)g(y)| \, dy \leq \|\tilde{f}\|_p \|g\|_q$$

where $\tilde{f}(y) = f(x-y)$. But by translation-invariance and reflection-invariance of the Lebesgue measure, $\|\tilde{f}\|_p = \|f\|_p$. Thus, $\|f * g\|_\infty \leq \sup_{x \in \mathbb{R}} |(f * g)(x)| \leq \|f\|_p \|g\|_q$.

Assume $r < \infty$. Then $\frac{1}{p} + \frac{1}{q} > 1$, so $p, q < \infty$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f * g = 0$ a.e., so $\|f * g\|_r = 0$. Assume $\|f\|_p > 0$ and $\|g\|_q > 0$. Note that $(cf) * (dg) = cd(f * g)$ for constants $c, d \in \mathbb{C}$. Using absolute homogeneity of the norms, we may therefore normalize the functions and assume $\|f\|_p = \|g\|_q = 1$. Let s, t be such that $\frac{1}{s} = 1 - \frac{1}{q}$ and $\frac{1}{t} = 1 - \frac{1}{p}$. Then for numbers $a, b \geq 0$,

$$ab = (a^p b^q)^{1/r} (a^p)^{1/s} (b^q)^{1/t}. \tag{1}$$

Moreover, $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$, so we may apply the generalized Hölder inequality:

$$\begin{aligned}
|(f * g)(x)| &= \left| \int_{\mathbb{R}} f(x-y) g(y) \, dy \right| \\
&\leq \int_{\mathbb{R}} |f(x-y)||g(y)| \, dy \\
&= \int_{\mathbb{R}} \underbrace{(|f(x-y)|^p |g(y)|^q)^{1/r}}_{h_1(y)} \underbrace{(|f(x-y)|^p)^{1/s}}_{h_2(y)} \underbrace{(|g(y)|^q)^{1/t}}_{h_3(y)} \, dy \\
&\leq \|h_1\|_r \|h_2\|_s \|h_3\|_t \\
&= \left(\int_{\mathbb{R}} |f(x-y)|^p |g(y)|^q \, dy \right)^{1/r} \left(\underbrace{\int_{\mathbb{R}} |f(x-y)|^p \, dy}_{\|f\|_p^p = 1} \right)^{1/s} \left(\underbrace{\int_{\mathbb{R}} |g(y)|^q \, dy}_{\|g\|_q^q = 1} \right)^{1/t} \\
&= \left(\int_{\mathbb{R}} |f(x-y)|^p |g(y)|^q \, dy \right)^{1/r}.
\end{aligned}$$

Therefore, by Tonelli's theorem,

$$\|f * g\|_r^r = \int_{\mathbb{R}} |(f * g)(x)|^r \, dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)|^p |g(y)|^q \, dy \, dx = \|f\|_p^p \|g\|_q^q = 1.$$

P2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function with

$$f(x+y) = f(x) + f(y), \ \forall x, y \in \mathbb{R}.$$

(a) Using Lusin's and Steinhaus' Theorems, prove that f is continuous at $x = 0$.

Solution: By Lusin's theorem, for each $\epsilon > 0$ there is a closed set $E \subseteq X$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous. Take $R > 0$ big enough such that $\tilde{E} = E \cap [-R, R]$ has positive measure. By Steinhaus theorem, there is $\xi > 0$ such that $(-\xi, \xi) \subseteq \tilde{E} - \tilde{E}$. Notice that $f|_{\tilde{E}}$ is uniformly continuous. So, for each $\eta > 0$ there is $\delta > 0$ such that if $|x - y| < \delta$ for $x, y \in E \cap [-R, R]$ then $|f(x) - f(y)| < \eta$.
For $z \in \mathbb{R}$ with $|z| < \min(\delta, \xi)$, there are $x, y \in E \cap [-R, R]$ such that $z = x - y$. By hypothesis, we have $f(z) = f(x) - f(y)$. As $|z| = |x-y| < \delta$ then $|f(z)| = |f(x)-f(y)| < \eta$, so we conclude that f is continuous at 0.

(b) Conclude that $f(x) = xf(1)$ for each $x \in \mathbb{R}$.

We notice that f is continuous everywhere. Indeed, for $x \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$, we have that $x - x_n \to 0$ as $n \to \infty$. So, by continuity at 0:

$$f(x) - f(x_n) = f(x - x_n) \to f(0) = 0 \text{ as } n \to \infty.$$

On the other hand, by an standard induction argument on $\mathbb{Z}$ and then on $\mathbb{Q}$, we have that for each $x \in \mathbb{Q}$,

$$f(x) = xf(1).$$

That being so, the conclusion follows by continuity.

P3. Let (X, τ) be a locally compact Hausdorff space. Let μ be a Radon measure that is inner regular on sets with finite measure. We will show that for each function $f \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there exist functions $g, h : X \to \mathbb{R}$ such that g is upper semicontinuous and bounded above, h is lower semicontinuous and bounded below,

$$g \leq f \leq h, \text{ and } \int_X (h - g)\mathrm{d}\mu < \epsilon.$$

For this:

(a) Justify that one can assume without loss of generality that f is positive.

Solution: We know that f can be written as $f = f_+ - f_-$ with $f_+, f_- \in L^1(\mu)$ positive functions. If the statement is true for positive functions, then there are g_-, g_+ upper semicontinuous functions bounded above and h_-, h_+ lower semicontinuous functions bounded below. Thus

$$g_+ - h_- \leq f_+ - f_- \leq h_+ - g_-,$$

where $-h_-$ is upper semicontinuous bounded by above and $-g_-$ is lower semicontinuous bounded by below. Thus, the result follows for f.

From now on, we assume that $f \geq 0$.

(b) Show that there are measurable sets $(E_n)_{n\in\mathbb{N}}$ and constants $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $f = \sum_{n=1}^{\infty} c_n \mathbb{1}_{E_n}$.

Solution: We know that there is a non decreasing sequence of positive simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \nearrow f$. We define $t_n = f_n - f_{n-1} \geq 0$ with $f_0 = 0$. Then, we have that

$$\sum_{n=1}^{N} t_n = f_N \nearrow f.$$

Thus $f = \sum_{n\in\mathbb{N}} t_n$. As $(t_n)_{n\in\mathbb{N}}$ are positive simple functions, we conclude the statement.

(c) Find appropriate compact sets $(K_n)_{n\in\mathbb{N}}$ and open sets $(U_n)_{n\in\mathbb{N}}$ to define $g = \sum_{n=1}^{N} c_n \mathbb{1}_{K_n}$ for some carefully chosen $N \in \mathbb{N}$ and $h = \sum_{n\in\mathbb{N}} c_n \mathbb{1}_{U_n}$. Conclude.

Solution: For each $n \in \mathbb{N}$, by regularity of μ and the fact that the sets $(E_n)_{n\in\mathbb{N}}$ have finite measure as $f \in \mathcal{L}^1(\mu)$, we find a compact set K_n and an open set U_n such that $K_n \subseteq E_n \subseteq U_n$ and

$$c_n\mu(U_n \setminus K_n) < 2^{-(n+1)}\epsilon. \tag{2}$$

We notice that $\mathbb{1}_{U_n}$ is lower semicontinuous for each n, and likewise $\mathbb{1}_{K_n}$ is upper semicontinuous for each n. This suggests to define $h = \sum_{n=1}^{\infty} c_n \mathbb{1}_{U_n}$ which is bounded by below. Nevertheless, the if we define and $g = \sum_{n=1}^{\infty} c_n \mathbb{1}_{K_n}$ this function is not necessarily bounded by above. So, we take $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} c_n\mu(U_n) < \frac{\epsilon}{2}$$

and set $g = \sum_{n=1}^{N} c_n \mathbb{1}_{K_n}$. Observe that this is possible due to the fact that the series $\sum_{n=1}^{\infty} c_n\mu(U_n)$ converge by equation (2) and the fact that $f \in \mathcal{L}^1(\mu)$.
Thus, we get $g \leq f \leq h$. On the other hand, notice that by monotone convergence theorem

$$\int h - g d\mu = \sum_{n=1}^{\infty} c_n\mu(U_n) - \sum_{n=1}^{N} c_n\mu(K_n) = \sum_{n=1}^{\infty} c_n\mu(U_n \setminus K_n) + \sum_{n=N+1}^{\infty} c_n\mu(U_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

concluding.