Exercise Sheet Solutions #2

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P1. (Problem 2.2.) Prove that the extended real line $[-\infty, \infty]$ is homeomorphic to the closed unit interval $[0, 1]$.

> ***Solution:*** As $[-1, 1]$ is homeomorphic with $[0, 1]$, we prove the statement for the interval $[-1, 1]$ rather than $[0, 1]$ (just for the sake of notation). We define the function $\varphi : [-\infty, \infty] \to [-1, 1]$ as
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> $$\varphi(t) = \begin{cases} 1 - \frac{1}{2t} & \text{if } t \in [1, \infty] \\ t/2 & \text{if } t \in (-1, 1) \\ -1 + \frac{1}{2t} & \text{if } t \in [-\infty, -1] \end{cases} . \tag{1}$$
>
> **Remark:** Another option is to define the function $\psi : [-\infty, \infty] \to [-\pi/2, \pi/2]$ given by $\psi(t) = \arctan(t)$.
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> Notice that $\varphi \mid_{(-\infty,\infty)}: (-\infty, \infty) \to (-1, 1)$ is clearly an homemorphism. For the extremal points, it is enough to notice that φ is bijective with $\varphi(\infty) = 1$ and $\varphi(-\infty) = -1$, therefore (by the definition of the topology in $[-\infty, \infty]$) we have that φ is continuous. Moreover, is an homeomorphism (this can be check manually justifying analogously for the inverse, or noticing that φ is a continuous bijective map between compact and Hausdorff spaces).

P2. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Show that $\mathcal{B}$ is generated by each of the following families:

(i) the collection of all open intervals,

(ii) the collection of all closed intervals.

> ***Solution:***
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> (i) Let $\mathcal{E} = \{(a, b) : a, b \in \mathbb{R}\}$. As $\mathcal{E} \subset \mathcal{B}$, then also $\sigma(\mathcal{E}) \subset \mathcal{B}$. Note also, that by P2 on Sheet 1, we have that any open set in $\mathbb{R}$ can be written as a countable (disjoint) union of open intervals, and so is contained in $\sigma(\mathcal{E})$. As $\mathcal{B}$ is the smallest σ-algebra containing all open sets, we conclude that $\mathcal{B} \subset \sigma(\mathcal{E})$. The two inclusions show that $\mathcal{B} = \sigma(\mathcal{E})$.
>
> (ii) Let $\mathcal{F} = \{[a, b] : a, b \in \mathbb{R}\}$. As any $[a, b]$ can be written as $[a, b] = \bigcap_n (a - \frac{1}{n}, b + \frac{1}{n})$, we get that $\mathcal{F} \subset \sigma(\mathcal{E})$, and so $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$. For the reverse inclusion, note that any open interval (a, b) can be written as $(a, b) = \bigcup_n [a + \frac{1}{n}, b - \frac{1}{n}]$, and so $\mathcal{E} \subset \sigma(\mathcal{F})$, which gives $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$. The two inclusions show that $\sigma(\mathcal{E}) = \sigma(\mathcal{F})$.

P3. (Problem 2.3.) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[-\infty, \infty]$, and let $c \in \mathbb{R}$. If $(x_n)_{n\in\mathbb{N}}$ converges to an extended real number, then the sequence $(cx_n)_{n\in\mathbb{N}}$ also converges, and

$$\lim_{n\to\infty} (cx_n) = c \cdot \lim_{n\to\infty} x_n. \tag{2}$$

> ***Solution:*** If $(x_n)_{n\in\mathbb{N}}$ is bounded, then the convergence as a sequence in $[-\infty, \infty]$ reduces to convergence in $(-\infty, \infty)$, in which we already know that Eq. (2) holds. Otherwise, as $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, we have either $x_n \to \infty$ as $n \to \infty$ or $x_n \to -\infty$ as $n \to \infty$. Assume without loss of generality that $x_n \to \infty$ as $n \to \infty$. If $c = 0$ then $cx_n = 0$ for all $n \in \mathbb{N}$, and $c \cdot \lim_{n\to\infty} x_n = 0$, so Eq. (2) holds. If $c \neq 0$, we can assume without loss of generality that $c > 0$. So, we have that
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> $$c \cdot \lim_{n\to\infty} x_n = c \cdot \infty = \infty.$$

> For the left-hand side expression, it is enough to justify that $(cx_n)_{n\in\mathbb{N}}$ is a sequence that goes to ∞. Indeed, for each $M \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n \in [M, \infty]$. Hence, taking M/c instead of M, the same goes for the sequence $(cx_n)_{n\in\mathbb{N}}$, and therefore $\lim_{n\to\infty} cx_n = \infty$, concluding.

P4. Let X, Y be two sets and $f : X \to Y$ a function.

(a) Prove that if $\mathcal{F}_Y \subseteq P(Y)$ is a σ-algebra, then $\mathcal{F}_X := \{f^{-1}(A) \mid A \in \mathcal{F}_Y\}$ is a σ-algebra.

> ***Solution:*** First, $X = f^{-1}(Y)$, so we have that $X \in \mathcal{F}_X$. Second, for $A \in \mathcal{F}_Y$, we have that $f^{-1}(A)^c = f^{-1}(A^c) \in \mathcal{F}_X$ because $A^c \in \mathcal{F}_Y$. Finally, for $(B_n)_{n\in\mathbb{N}}$ a countable family of elements in $\mathcal{F}_X$, we write $B_n = f^{-1}(A_n)$ for $A_n \in \mathcal{F}_Y$ for each $n \in \mathbb{N}$. Then, we write
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> $$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) = f^{-1}(\bigcup_{n\in\mathbb{N}} A_n), \tag{3}$$
>
> and as $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}_Y$, we have that $f^{-1}(\bigcup_{n\in\mathbb{N}} A_n) \in \mathcal{F}_X$. In consequence, $\mathcal{F}_X$ is a sigma algebra.

(b) Prove that for $\mathcal{C} \subseteq P(Y)$, we have that $\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma(\mathcal{C}))$.

> ***Solution:*** First, as $\mathcal{C} \subseteq \sigma(\mathcal{C})$, we have that $f^{-1}(\mathcal{C}) \subseteq f^{-1}(\sigma(\mathcal{C}))$. By part (a), we know that $f^{-1}(\sigma(\mathcal{C}))$ is a sigma algebra, so we have
>
> $$\sigma(f^{-1}(\mathcal{C})) \subseteq f^{-1}(\sigma(\mathcal{C})).$$
>
> For showing the reverse inclusion, we define the set
>
> $$\mathcal{D} = \{D \subseteq Y \mid f^{-1}(D) \in \sigma(f^{-1}(\mathcal{C}))\}.$$
>
> Observe that $\sigma(\mathcal{C}) \subseteq \mathcal{D}$ is equivalent to $f^{-1}(\sigma(\mathcal{C})) \subseteq \sigma(f^{-1}(\mathcal{C}))$. In consequence, as clearly $\mathcal{C} \subseteq \mathcal{D}$, it is enough to show that $\mathcal{D}$ is a sigma algebra. For this, first observe that $f^{-1}(Y) = X \in \sigma(f^{-1}(\mathcal{C}))$ by definition of sigma algebra. Second, for $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{D}$ we have that
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> $$f^{-1}(\bigcup_{i\in\mathbb{N}} A_i) = \bigcup_{i\in\mathbb{N}} f^{-1}(A_i) \in \sigma(f^{-1}(\mathcal{C})),$$
>
> because of the definition of sigma algebra and $A_i \in \mathcal{D} \iff f^{-1}(A_i) \in \sigma(f^{-1}(\mathcal{C}))$ for each $i \in \mathbb{N}$. Finally, for $A \in \mathcal{D}$, we have that
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> $$f^{-1}(A_i^c) = f^{-1}(A_i)^c \in \sigma(f^{-1}(\mathcal{C})),$$
>
> given that the complement of an element of a sigma algebra is in the sigma algebra. This proves that $\mathcal{D}$ is a sigma algebra, concluding.

P5. Let $(X, \mathcal{T}, \mu)$ be a finite measure space, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Show that if $\mathcal{A}$ generates $\mathcal{T}$, then for every $B \in \mathcal{T}$ and for every $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A\Delta B) \leq \epsilon$.

> ***Solution:*** Notice that what we want to prove can be reformulated as
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> $$\sigma(\mathcal{A}) \subseteq \{B \subseteq X \mid \forall\epsilon > 0, \exists A \in \mathcal{A}, \mu(A\Delta B) \leq \epsilon\}. \tag{4}$$

Let us call $\mathcal{B}$ the right-hand side set. It is straightforward to see that $\mathcal{A} \subseteq \mathcal{B}$, so it is enough to show that $\mathcal{B}$ is a sigma algebra to conclude. First, $X \in \mathcal{A} \subseteq \mathcal{B}$. For $(B_i)_{i\in\mathbb{N}} \subseteq \mathcal{B}$, let $\epsilon > 0$. We have that for each $\epsilon_i > 0$, there is $A_i \in \mathcal{A}$ such that $\mu(A_i \Delta B_i) \leq \epsilon_i$. Given that $\bigcup_{i=1}^N B_i \nearrow \bigcup_{i=1}^\infty B_i$ and the fact that the measure is finite, we have that for given $\epsilon_0 > 0$ there is N such that

$$\mu(\bigcup_{i=1}^{\infty} B_i \Delta \bigcup_{i=1}^{N} B_i) = \mu(\bigcup_{i=1}^{\infty} B_i) - \mu(\bigcup_{i=1}^{N} B_i) \leq \epsilon_0. \tag{5}$$

Thus, we have the approximation

$$\begin{aligned}\mu(\bigcup_{i\in N} A_i \Delta \bigcup_{i\in\mathbb{N}} B_i) &\leq \mu(\bigcup_{i\in\mathbb{N}} B_i \Delta \bigcup_{i\in N} B_i) + \mu(\bigcup_{i\in N} A_i \Delta \bigcup_{i\in N} B_i)\\ &\leq \epsilon_0 + \mu(\bigcup_{i\in N} A_i \Delta B_i) \leq \epsilon_0 + \sum_{i\in N} \mu(A_i \Delta B_i)\\ &\leq \sum_{i=0}^{N} \epsilon_i,\end{aligned}$$

so, taking $\sum_{i\in\mathbb{N}_0} \epsilon_i \leq \epsilon$ (for example $\epsilon_i = \epsilon/2^{i+1}$) and the fact that $\bigcup_{n=1}^N A_i \in \mathcal{A}$ by definition of algebra,we conclude that $\bigcup_{i\in\mathbb{N}} B_i \in \mathcal{B}$. Finally, for $B \in \mathcal{B}$, we need to see that $B^c \in \mathcal{B}$. Indeed, for $\epsilon > 0$, we take $A \in \mathcal{A}$ such that $\mu(A \Delta B) \leq \epsilon$. Then

$$\mu(A^c \Delta B^c) = \mu((A^c \cup B^c) \cap (A^c \cap B^c)^c) = \mu((A \cup B) \cap (A \cup B)^c) = \mu(A \Delta B) \leq \epsilon. \tag{6}$$

Thus, we conclude that $\mathcal{B}$ is a sigma algebra.