

## Exercise Sheet #8

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**P1.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and let  $(Y, \mathcal{B})$  be a measurable space. Further, let  $f : X \rightarrow Y$  be a measurable function. We define a *pushforward* measure  $f_*\mu$  on  $(Y, \mathcal{B})$  by

$$f_*\mu(A) := \mu(f^{-1}(A)), \quad A \in \mathcal{B}.$$

Show that  $f_*\mu$  is well-defined, and is a probability measure if  $\mu$  is a probability measure.

**P2.** Let  $(X, \mathcal{F}, \mathbb{P})$  be probability space, and let  $X_n$  be a sequence of real valued random variables on  $X$  i.e. a sequence of Borel measurable maps  $X_n : X \rightarrow \mathbb{R}$ . Define the *distribution* of  $X_n$  as the pushforward measure on  $\mathbb{R}$  given by

$$\mu_n(A) := \mathbb{P}(X_n^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}).$$

(a) Show that if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous bijection, with continuous inverse, then if  $\mu_n \rightarrow \mu$  *vaguely*<sup>1</sup>, then also the laws of  $g(X_n)$  converge vaguely to  $g_*\mu$ .

(b) Now show that if  $\mu_n \rightarrow \mu$  *weakly*<sup>2</sup>, and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $g_*\mu_n \rightarrow g_*\mu$  weakly.

**P3.** Let  $X$  be a compact metric space, and  $T : X \rightarrow X$  be a continuous map. Show that there exists a  $T$ -invariant, Borel probability measure on  $X$ , i.e. a Borel probability measure  $\mu$  such that  $T_*\mu = \mu$ .<sup>3</sup>

**P4.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $X, Y$  be two real valued random variables on  $\Omega$  with laws  $\mu$  and  $\nu$ . Define the *Fourier transform* of  $\mu$  (equivalently the *Characteristic function* of  $X$ ) as

$$\hat{\mu}(\xi) := \int_X e^{i\xi x} d\mu(x), \quad \xi \in \mathbb{R}.$$

Show that if  $\hat{\mu} = \hat{\nu}$  pointwise, then  $\mu = \nu$ .

**P5.** Consider  $\mathbb{Z}$  as a discrete topological space.

(a) Describe the space  $C_c(\mathbb{Z})$ .

(b) Describe the positive linear functionals on  $C_c(\mathbb{Z})$ .

(c) Let  $\phi : C_c(\mathbb{Z}) \rightarrow \mathbb{C}$  be a positive linear functional, and let  $\mu$  be the Radon measure representing  $\phi$  via the Riesz representation theorem. When is  $\mu$  a finite measure? (Give a characterisation in terms of properties of  $\phi$ ).

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<sup>1</sup>see def. 5.12 in the notes

<sup>2</sup>Weak convergence is defined analogously to vague convergence but replacing  $C_c$  by  $C_b$ .

<sup>3</sup>*Hint:* consider the measures  $\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} (T^k)_*\delta_x$ , where  $\delta_x$  is the Dirac measure at some  $x \in X$ .