

## Exercise Sheet #4

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- P1.** Let  $(X, \mathcal{B}, \mu)$  be a measure space, and let  $f : X \rightarrow \mathbb{C}$  be a measurable function. Then prove that  $f = 0$  a.e. if and only if  $\int_E f d\mu = 0$  for every measurable set  $E \in \mathcal{B}$ .
- P2.** Prove that a space is complete if and only if the following property holds: for functions  $f, g; X \rightarrow \mathbb{C}$ , if  $f$  is measurable and  $f = g$  a.e., then  $g$  is also measurable.
- P3.** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Let  $\mathcal{N} = \{N \in \mathcal{B} : \mu(N) = 0\}$  be the  $\sigma$ -ideal of  $\mu$ -null sets. Show that the family  $\overline{\mathcal{B}} = \{E \cup F : E \in \mathcal{B}, F \subseteq N \in \mathcal{N}\}$  is a  $\sigma$ -algebra, and there is a unique extension  $\bar{\mu}$  of  $\mu$  to  $\overline{\mathcal{B}}$ .
- P4.** Let  $(X, \mathcal{B}, \mu)$  be a measure space. Prove that simple functions are dense in  $L^1(X)$ .
- P5.** Let  $(X, \mathcal{B}, \mu)$  be a measure space and  $E \in \mathcal{B}$  a set of positive measure. If  $\{E_n\}_{n \in \mathbb{N}}$  is an increasing sequence of measurable sets and  $E = \bigcup_{n \in \mathbb{N}} E_n$  prove that for every function  $f \in L^1(X)$

$$\lim_{n \rightarrow \infty} \int_{E_n} f d\mu = \int_E f d\mu$$

and state and prove an analogous result for decreasing sequences.

- P6. (Problem 3.2.)** Let  $(X, \mathcal{B}, \mu)$  be a probability space. Let  $(A_n)_{n \in \mathbb{N}}$  be a family of measurable sets with  $a = \inf_{n \in \mathbb{N}} \mu(A_n) > 0$ . We aim to show that there is a set  $E \subseteq \mathbb{N}$  such that  $\bar{d}(E) := \limsup_{N \rightarrow \infty} \frac{|E \cap \{1, \dots, N\}|}{N} \geq a$ , and for any finite set  $F \subseteq E, F \neq \emptyset$ , one has  $\mu(\bigcap_{n \in F} A_n) > 0$ .
- (a) Justify that we can assume, without loss of generality, that  $\bigcap_{n \in F} A_n \neq \emptyset$  if and only if  $\mu(\bigcap_{n \in F} A_n) > 0$ . To do this, it may help to define the countable set

$$\mathcal{F} = \{F \subseteq \mathbb{N} \mid |F| < \infty, \bigcap_{n \in F} A_n \neq \emptyset, \mu\left(\bigcap_{n \in F} A_n\right) = 0\}.$$

- (b) Prove that  $\int \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{A_n}(x) d\mu(x) \geq a$ .
- (c) Define  $E = \{n \in \mathbb{N} \mid x \in A_n\}$  for some suitable  $x \in X$ . Conclude the proof.