Exercise Sheet #3

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P1. **(Problem 3.3.)**(Borel–Cantelli lemma) If $(A_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of a probability space $(X, \mathcal{B}, \mu)$ and $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 0.$$

P2. Let $(X, \mathcal{B}, \mu)$ be a probability space, and $(A_n)_{n\in\mathbb{N}}$ a family of *independent* measurable subsets such that $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$. By independence we mean that for any finite index set $\mathcal{I}$, we have $\mu(\bigcap_{n\in\mathcal{I}} A_n) = \Pi_{n\in\mathcal{I}}\mu(A_n)$. Show that

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 1.$$

Hint: Note the useful inequality $1 - a \leq e^{-a}$ *for* $a \in [0, 1]$.

P3. Let $(X, \mathcal{F}, \mu)$ be a measure space and f a measurable function. Prove the Markov-Chebyshev inequality:

$$\forall \alpha > 0, \mu(\{|f| > \alpha\}) \leq \frac{1}{\alpha} \int_{\{|f|>\alpha\}} |f| \, d\mu \leq \frac{1}{\alpha} \int |f| \, d\mu,$$

where we denote $\{|f| > \alpha\} = \{x \in X \mid |f(x)| > \alpha\}$.

P4. Let $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \nu)$ be probability spaces, and let $T : X \to Y$ be a measurable function. Define $T\mu(A) := \mu(T^{-1}(A))$ for each $A \in \mathcal{F}_Y$. Prove that $\nu = T\mu$ if and only if for all integrable function f:

$$\int_Y f \, d\nu = \int_X f \circ T \, d\mu.$$