

Exercise Sheet #1

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- P1.** Show that $J^*(\mathbb{Q} \cap [0, 1]) = J^*([0, 1] \setminus \mathbb{Q}) = 1$, and $J_*(\mathbb{Q} \cap [0, 1]) = J_*([0, 1] \setminus \mathbb{Q}) = 0$.
- P2.** Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.
- P3.** Let $U = \{(x, y) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.
- P4.** Give an example to show that the statement

$$\lambda^*(E) = \sup_{U \subset E, U \text{ open}} \lambda^*(U)$$

is false.

- P5.** (Area interpretation of the Riemann integral). Let $[a, b]$ be an interval, and let $f : [a, b] \rightarrow \mathbf{R}_+ := [0, \infty)$ be a bounded function. Show that f is Riemann integrable if and only if the set $E_+ := \{(x, t) : x \in [a, b]; 0 \leq t \leq f(x)\}$ is Jordan measurable in \mathbf{R}^2 , in which case one has

$$\int_a^b f(x) dx = m^2(E_+).$$

where m^2 denotes two-dimensional Jordan measure.

Remark: For $f : [a, b] \rightarrow \mathbb{R}$ not necessarily positive, one can prove that f is Riemann integrable if and only if the sets E_+ and $E_- := \{(x, t) : x \in [a, b]; f(x) \leq t \leq 0\}$ are both Jordan measurable in \mathbf{R}^2 , in which case one has $\int_a^b f(x) dx = m^2(E_+) - m^2(E_-)$.

- P6.** Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^d [a_i, b_i)$).