Exercise Sheet #12

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P1. Let (X, μ, T) a measure space. We say that $(f_n)_n$ *converges in measure* to f if given $\epsilon > 0$ we have $\lim_{n\to\infty} \mu(\{x \mid |f(x) - f_n(x)| > \epsilon\}) = 0$.

(a) Assume that $\mu(X) < \infty$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges in measure to f if and only if every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges a.e. to f.

(b) What happens if $\mu(X) = \infty$?

P2. Let $(X, \mathcal{F})$ be a measurable space, and let ν and μ be two measures such that $\nu \ll \mu$ and $g = \frac{\partial \nu}{\partial \mu}$. Prove that if $g \in L^p(\mu)$ and $A \in \mathcal{F}$, then

$$\nu(A) \leq \|g\|_{L^p(\mu)} \mu(A)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

P3. Let $(X, \mathcal{B})$ be a measurable space, and let $\mu : \mathcal{B} \to [-\infty, \infty]$ be a signed measure with total variation $|\mu|$ (see Def. 9.9 in the notes). Then show that for any $E \in \mathcal{B}$,

$$\begin{aligned} |\mu|(E) &= \inf\{\nu(E) : \nu \text{ is a measure and } |\mu(F)| \leq \nu(F) \text{ for all } F \in \mathcal{B}\} \\ &= \sup\left\{\sum_{n=1}^{\infty} |\mu(E_n)| : E = \bigsqcup_{n\in\mathbb{N}} E_n\right\}. \end{aligned}$$