Exercise Sheet #11

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P1. Prove Young's convolution inequality: Suppose that $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Then $f * g$ is defined a.e. and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Hint: It may be useful to notice that if s, t are such that $\frac{1}{s} = 1 - \frac{1}{q}$ and $\frac{1}{t} = 1 - \frac{1}{p}$ then for each $a, b \geq 0$ one has

$$ab = (a^p b^q)^{1/r} (a^p)^{1/s} (b^q)^{1/t}.$$

P2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function with

$$f(x + y) = f(x) + f(y), \ \forall x, y \in \mathbb{R}.$$

(a) Using Lusin's and Steinhaus' Theorems, prove that f is continuous at $x = 0$.

(b) Conclude that $f(x) = x f(1)$ for each $x \in \mathbb{R}$.

P3. Let (X, τ) be a locally compact Hausdorff space. Let μ be a Radon measure that is inner regular on sets with finite measure. We will show that for each function $f \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there exist functions $g, h : X \to \mathbb{R}$ such that g is upper semicontinuous[1] and bounded above, h is lower semicontinuous[2] and bounded below,

$$g \leq f \leq h, \text{ and } \int_X (h - g) \mathrm{d}\mu < \epsilon.$$

For this:

(a) Justify that one can assume without loss of generality that f is positive.
From now on, we assume that $f \geq 0$.

(b) Show that there are measurable sets $(E_n)_{n \in \mathbb{N}}$ and constants $(c_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$ such that $f = \sum_{n=1}^{\infty} c_n \mathbb{1}_{E_n}$.

(c) Find appropriate compact sets $(K_n)_{n \in \mathbb{N}}$ and open sets $(U_n)_{n \in \mathbb{N}}$ to define $g = \sum_{n=1}^{N} c_n \mathbb{1}_{K_n}$ for some carefully chosen $N \in \mathbb{N}$ and $h = \sum_{n \in \mathbb{N}} c_n \mathbb{1}_{U_n}$. Conclude.

[1] A function $f : X \to \overline{\mathbb{R}}$ is upper semicontinuous if for each $x \in X$, $\limsup_{y \to x} f(y) \leq f(x)$.

[2] A function $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous if $-f$ is upper semicontinuous.