Transhipment

Problem definition and properties

Michel Bierlaire

Introduction to optimization and operations research

The transhipment problem

Motivation

- The transhipment problem is probably the most typical network optimization problem.
- It is also known as the min-cost flow problem.
- We start by formulating it.

Motivation

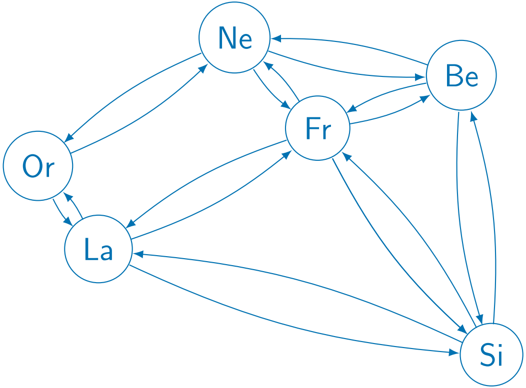
Logistics

- Goods have to be transported
  - from the warehouses,
  - to the clients.
- They can be transshipped anywhere on the network.
- Transportation costs must be minimum.

Data

Nodes

- Supply (warehouses).
- Demand (customers).
- Transhipment.



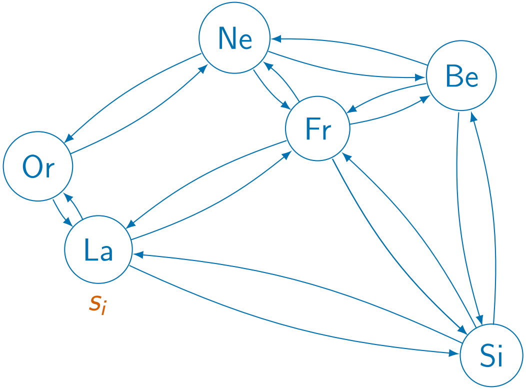
Data

Supply/demand

- $s_i > 0$ if supply node.
- $s_i < 0$ if demand node.
- $s_i = 0$ is transhipment node.

Validity

$$\sum_{i \in \mathcal{N}} s_i = 0.$$



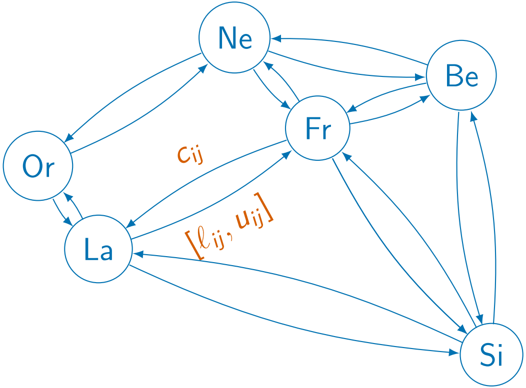
Data

Links

- Costs.
- Capacities.

Validity

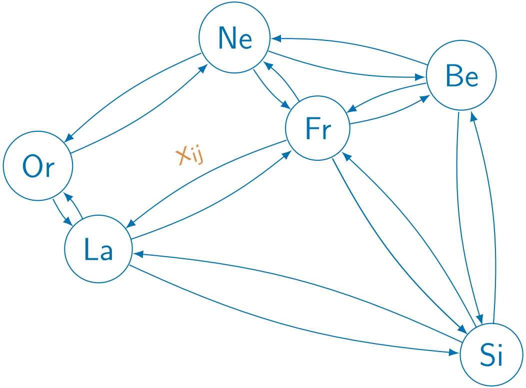
$$\ell_{ij} \leq u_{ij}.$$



Decision variables

Flow vector

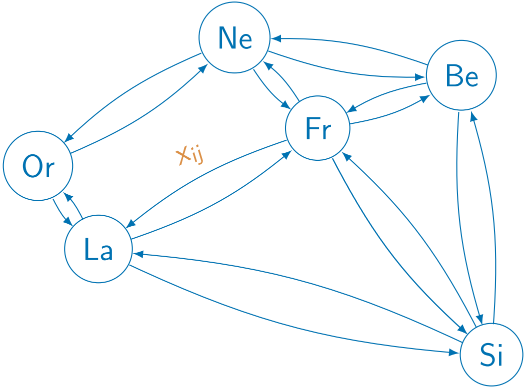
$$x \in \mathbb{R}^n.$$



Objective function

Total costs

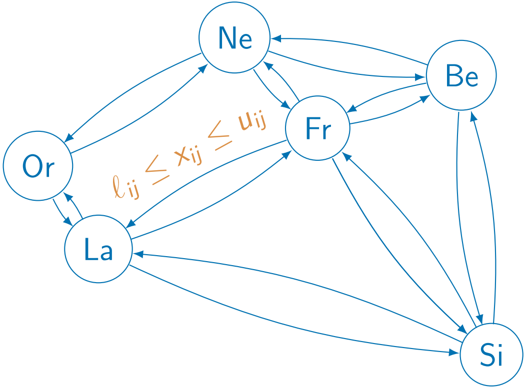
$$\sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}.$$



Constraints

Capacity

$$\ell_{ij} \leq x_{ij} \leq u_{ij},\ \forall (i,j) \in \mathcal{A}.$$

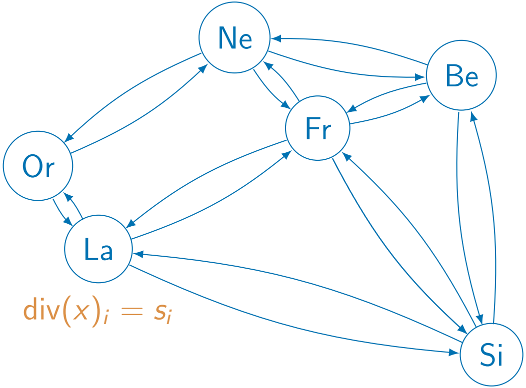


Constraints

Flow conservation

$$\mathrm{div}(x)_i = s_i, \ \forall i \in \mathcal{N}.$$

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = s_i, \ \forall i \in \mathcal{N}.$$



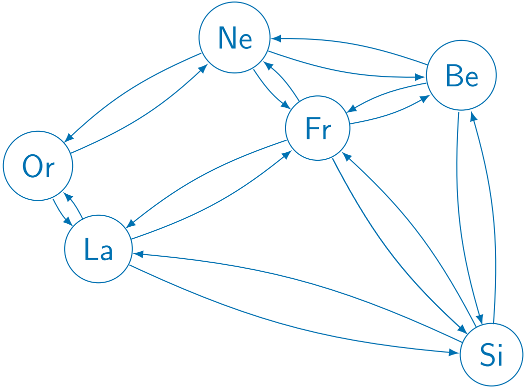
Linear optimization problem

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N}.$$

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \forall (i,j) \in \mathcal{A}.$$



Standard form

Motivation

- The transhipment problem is a linear optimization problem.
- The simplex algorithm that we have seen requires the problem to be written in standard form.
- We would like the problem in standard form to have an underlying network structure.

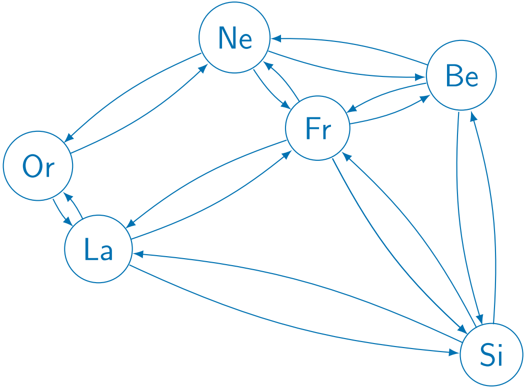
Linear optimization problem

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N}.$$

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \forall (i,j) \in \mathcal{A}.$$



Set the lower bounds to zero

$$x'_{ij} = x_{ij} - \ell_{ij} \ , \ x_{ij} = x'_{ij} + \ell_{ij}$$

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij} + \sum_{(i,j) \in \mathcal{A}} c_{ij} \ell_{ij} = \min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x'_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x'_{ki} = s_i + \sum_{k|(k,i) \in \mathcal{A}} \ell_{ki} - \sum_{j|(i,j) \in \mathcal{A}} \ell_{ij} = s'_i, \ \forall i \in \mathcal{C},$$

$$0 \leq x'_{ij} \leq u_{ij} - \ell_{ij} = u'_{ij}, \ \forall (i,j) \in \mathcal{A}.$$

Slack variables

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x'_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x'_{ki} = s'_i \ \forall i \in \mathcal{C},$$

$$0 \leq x'_{ij} \leq u'_{ij}, \ \forall (i,j) \in \mathcal{A}.$$

Slack variables

$$0 \leq x'_{ij} \leq u'_{ij}, \ \forall (i,j) \in \mathcal{A}.$$

Slack variables: y_{ij}

$$x'_{ij} + y_{ij} = u'_{ij},$$
$$x'_{ij} \geq 0, \ y_{ij} \geq 0.$$

Interpretation

$$x'_{ij} + y_{ij} = u'_{ij}$$

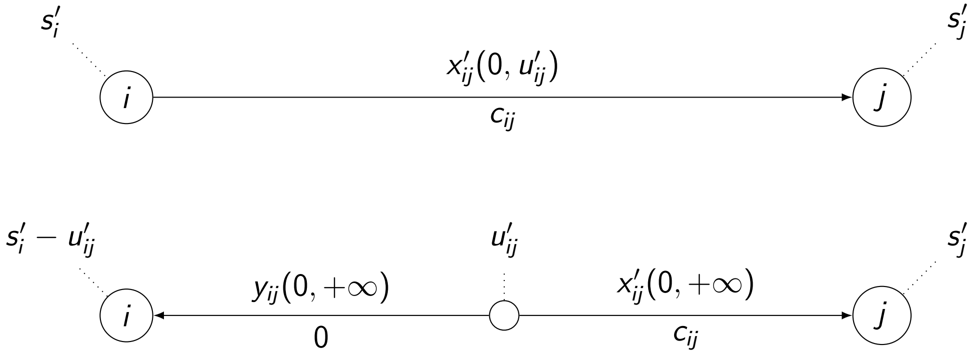
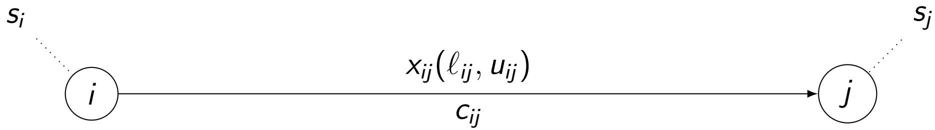
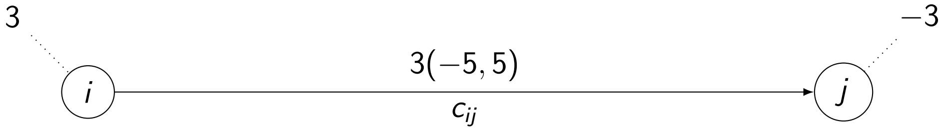


Illustration: initial formulation

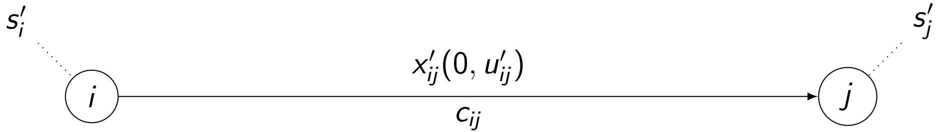


$$x_{ij} = s_i, \ -x_{ij} = s_j, \ \ell_{ij} \leq x_{ij} \leq u_{ij}.$$

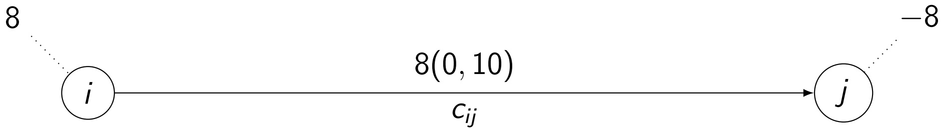


$$3 = 3, \ -3 = -3, \ -5 \leq 3 \leq 5.$$

Illustration: change of variables

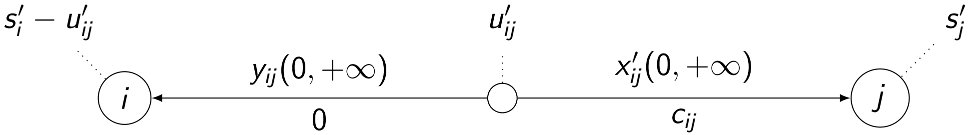


$$x_{ij} - \ell_{ij} = s_i - \ell_{ij}, \ -x_{ij} + \ell_{ij} = s_j + \ell_{ij}, \ 0 \le x_{ij} - \ell_{ij} \le u_{ij} - \ell_{ij}.$$

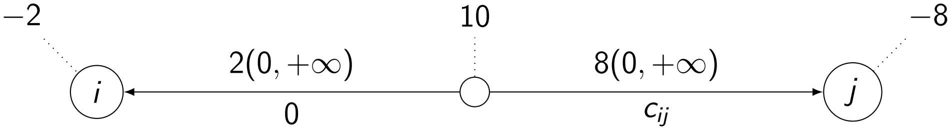


$$3 - (-5) = 3 - (-5), \ -3 - 5 = -3 - 5, \ 0 \le 3 - (-5) \le 5 - (-5).$$

Illustration: slack variables

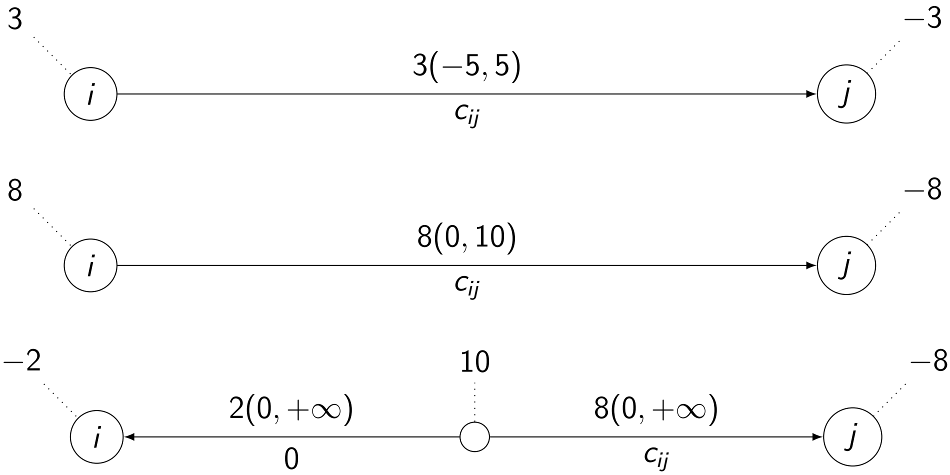


$$s'_i - u_{ij} = -y_{ij},\ x'_{ij} + y_{ij} = u'_{ij},\ s'_j = -x'_{ij}.$$



$$8 - 10 = -2,\ 8 + 2 = 10,\ -8 = -8.$$

Illustration: summary



Standard form

Comments

- Transforming the problem in standard form is equivalent to transforming the network.
- Main motivation: keep the structure of the matrix A.
- This can be done for any transhipment problem as pre-processing.
- Without loss of generality, we can therefore assume that the transhipment problem is written in standard form.

Problem in standard form: modified network

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j | (i,j) \in \mathcal{A}} x_{ij} - \sum_{k | (k,i) \in \mathcal{A}} x_{ki} = s_i, \; \forall i \in \mathcal{N},$$

$$x_{ij} \geq 0, \; \forall (i,j) \in \mathcal{A}.$$

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = s,$$

$$x \geq 0.$$

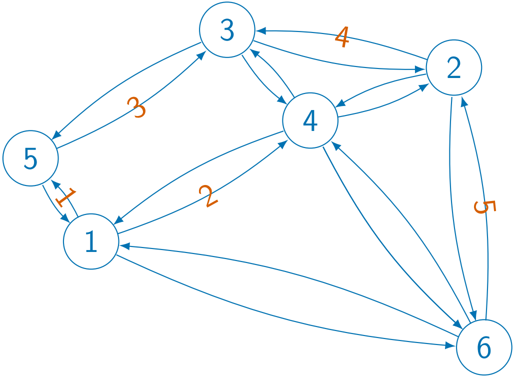
Incidence matrix

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = s_i, \ \forall i \in \mathcal{N}.$$

For each arc (i, j) of index k,

$$a_{ik} = 1 \text{ and } a_{jk} = -1.$$

Incidence matrix



	1	2	3	4	5	...
1						
2						
3						
4						
5						
6						

Optimality conditions

Motivation

- The transhipment problem is a linear optimization problem, where the constraints have a special structure.
- We analyze now the dual problem and the dual variables.
- We introduce the optimality conditions called "complementarity slackness".

Transhipment problem in standard form

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \ \forall i \in \mathcal{N},$$

$$x_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}.$$

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = s,$$

$$x \geq 0.$$

Lagrangian

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$

$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

Lagrangian: $L(x, \lambda, \mu) =$

$$\sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

$$+ \sum_{i \in \mathcal{N}} \lambda_i \left(\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} - s_i \right)$$

$$- \sum_{(i,j) \in \mathcal{A}} \mu_{ij} x_{ij}.$$

Optimality conditions

Coefficients of x_{ij} must be 0:

$$L(x, \lambda, \mu) = \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$+ \sum_{i\in\mathcal{N}} \lambda_i \left(\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} - s_i \right)$$

$$- \sum_{(i,j)\in\mathcal{A}} \mu_{ij}x_{ij},$$

$$\frac{\partial L}{\partial x_{ij}} = c_{ij} + \lambda_i - \lambda_j - \mu_{ij} = 0.$$

$$\mu_{ij} = c_{ij} + \lambda_i - \lambda_j$$

$$\mu_{ij} \geq 0 \text{ and } \mu_{ij}x_{ij} = 0$$

Complementarity slackness

For all (i, j)

$$c_{ij} + \lambda_i - \lambda_j \geq 0.$$

For all (i, j) such that $x_{ij} > 0$

$$c_{ij} + \lambda_i - \lambda_j = 0.$$

Sufficient and necessary optimality conditions.

Total unimodularity

Motivation

- The incidence matrix defining the constraints of a transhipment problem has a special structure.
- It also has a great property called "total unimodularity".
- It is great because it guarantees to generate an integer solution, if the data is integer.
- We now formally define this property.

Transhipment problem in standard form

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j \mid (i,j) \in \mathcal{A}} x_{ij} - \sum_{k \mid (k,i) \in \mathcal{A}} x_{ki} = s_i, \; \forall i \in \mathcal{N},$$

$$x_{ij} \geq 0, \; \forall (i,j) \in \mathcal{A}.$$

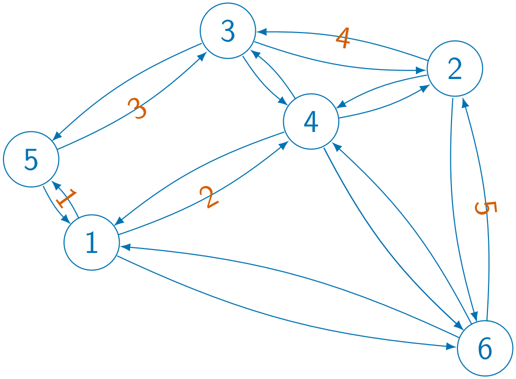
$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = s,$$

$$x \geq 0.$$

Incidence matrix



	1	2	3	4	5	...
1	1	1	0	0	0	
2	0	0	0	1	-1	
3	0	0	-1	-1	0	
4	0	-1	0	0	0	
5	-1	0	1	0	0	
6	0	0	0	0	1	

Optimal solution

$$x_B^* = B^{-1}s.$$

Cramer's rule

$$x_B^* = \frac{1}{\det(B)} C(B)^T s$$

Each entry of the matrix is 0, 1 or -1.
Each entry of $C(B)$ is an integer.
If s is integer, $C(B)^T s$ is integer.
If $\det(B)$ is 1 or -1, x_b^* is integer.

Total unimodularity

Definition

The matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular
if the determinant of each square submatrix of A is

$$0, -1 \text{ or } +1.$$

Total unimodularity

Definition

The matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular
if the determinant of each square submatrix of A is

$$0, -1 \text{ or } +1.$$

In particular,

every entry of the matrix is 0, -1 or $+1$.

Example

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Each square submatrix of size 1, that is each element of the matrix, is 0, -1 or $+1$. Each square submatrix of size 2 is unimodular:

$$\det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1,$$

$$\det \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = -1,$$

$$\det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = 1.$$

Integrality of the basic solutions

Consider the polyhedron represented in standard form

$$\{x \in \mathbb{R}^n | Ax = b,\ x \geq 0\},$$

where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.
If A is totally unimodular, then every basic solution is integer.

Proof

$$x_B = B^{-1}b.\ x_B = \frac{1}{\det(B)} C(B)^T b,$$

Total unimodularity of the incidence matrix

Motivation

- Total unimodularity of the matrix defining the constraints of a linear optimization problem guarantees that the basic solutions are integer if the data are integer.
- Therefore, the optimal solution is also integer.
- This is a great property, given the difficulty of discrete optimization.
- We show now that the incidence matrix of the transhipment problem is totally unimodular.

Transhipment problem in standard form

$$\min_{x\in\mathbb{R}^n} \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

subject to

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = s_i,\ \forall i \in \mathcal{N},$$

$$x_{ij} \geq 0,\ \forall (i,j) \in \mathcal{A}.$$

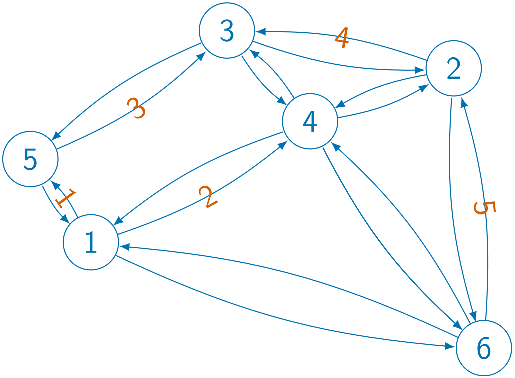
$$\min_{x\in\mathbb{R}^n} c^T x$$

subject to

$$Ax = s,$$

$$x \geq 0.$$

Incidence matrix



	1	2	3	4	5	…
1	1	1	0	0	0	
2	0	0	0	1	-1	
3	0	0	-1	-1	0	
4	0	-1	0	0	0	
5	-1	0	1	0	0	
6	0	0	0	0	1	

Theorem 22.6

The incidence matrix of a network is totally unimodular.

Proof

- By contradiction, assume that it is not the case.
- There exist square matrices with determinant $\notin \{0, 1, -1\}$.
- Among all of them, consider the one with minimum size k: B.
- Note that $k > 1$.
- Number of non-zero entries in a column of B: 0, 1 or 2.
- We treat each case separately.

No non-zero entry

Context

- B is $k \times k$.
- $\det(B) \notin \{0, 1, -1\}$.
- $\forall\ \ell \times \ell$ submatrices B': $\det(B') \in \{0, 1, -1\}$.
- One column of B contains only zeros.

Impossible, because it would mean that $\det(B) = 0$.
Therefore, no column of B contains only 0s.

One non-zero entry

Context

- B is $k \times k$.
- $\det(B) \notin \{0, 1, -1\}$.
- $\forall\ \ell \times \ell$ submatrices B': $\det(B') \in \{0, 1, -1\}$.
- One column of B contains only one non-zero entry.

$$B = \begin{pmatrix} b_{11} & b_{12} \cdots b_{1k} \\ 0 & \\ \vdots & B' \\ 0 & \end{pmatrix}$$

$$\det(B) = b_{11} \det(B') = \det(B') \text{ or } -\det(B')$$

Impossible as B is the smallest matrix.

Two non-zero entries

Context

- B is $k \times k$.
- $\det(B) \notin \{0, 1, -1\}$.
- $\forall\ \ell \times \ell$ submatrices B': $\det(B') \in \{0, 1, -1\}$.
- Each column of B contains exactly two non-zero entries.

The two non zero entries are 1 and -1.
If you sum the elements of each column, it is equal to zero.
So, summing all the rows of B gives 0
B is singular, and its determinant is zero.
Contradiction.

Corollary

If the supply vector s of the transhipment problem is integer, that is, if $s \in \mathbb{Z}^m$, and if the problem is bounded, then it has an optimal solution which is integer.

The shortest path problem

Motivation

- The transhipment problems embeds a variety of other problems.
- These problems inherit the properties of the transhipment problem.
- In particular, the integrality of the optimal solution if the data is integer.
- We first introduce the shortest path problem as an instance of the transhipment problem.

Shortest path

Problem definition

- Consider a network with nodes and arcs.
- Each arc is associated with a cost.
- Consider an origin and a destination.
- Find a simple forward path from o to d with the smallest cost.

Shortest paths problems

- Single origin – single destination. (this lecture)
- Single origin – all destinations. (next topic)
- All origins – single destination.
- All origins – all destinations.

Cost of a path

Definition

$$C(P) = \sum_{(i,j)\in P^{\rightarrow}} c_{ij}x_{ij} - \sum_{(i,j)\in P^{\leftarrow}} c_{ij}x_{ij}.$$

Forward path

$$C(P) = \sum_{(i,j)\in P^{\rightarrow}} c_{ij}x_{ij} = \sum_{(i,j)\in P} c_{ij}x_{ij}$$

Transhipment

Flow

Send one unit of flow from o to d.
Supply at o: $s_o = 1$
Supply at d: $s_d = -1$
Supply at any other node: $s_i = 0$
Total unimodularity: if there is a solution, flow on (i, j): 0 or 1

Cost of a path

$$C(P) = \sum_{(i,j) \in P} c_{ij} x_{ij} = \sum_{(i,j) \in P} c_{ij}$$

Transhipment

- cost on each arc: c_{ij}
- supply at the origin: $s_o = 1$
- supply at the destination: $s_d = -1$
- supply at any other node: 0
- lower bound on each arc: 0
- upper bound on each arc: ∞

Transhipment

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(o,j)\in\mathcal{A}} x_{oj} - \sum_{k|(k,o)\in\mathcal{A}} x_{ko} = 1,$$

$$\sum_{j|(d,j)\in\mathcal{A}} x_{dj} - \sum_{k|(k,d)\in\mathcal{A}} x_{kd} = -1,$$

$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki} = 0, \ \forall i \in \mathcal{N}, i \neq o, i \neq d,$$

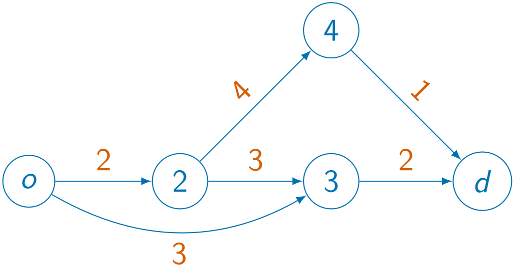
$$x_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}.$$

The maximum flow problem

Motivation

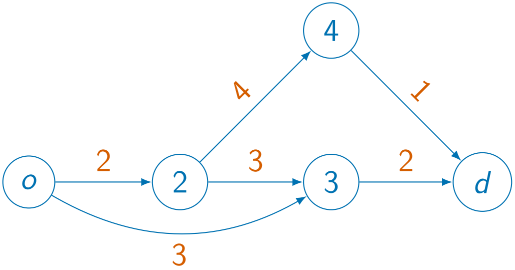
- The maximum flow problem was motivated by the analysis performed by the American army of the railway network operated by the Soviet Union across Eastern Europe during the Cold War.
- We have a network, where each arc has a capacity.
- Consider an origin o and a destination d.
- What is the maximum amount of flow that the network can carry from o to d?

The maximum flow problem

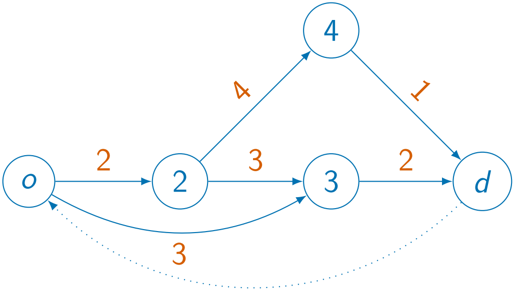


- $o \to 2 \to 4 \to d$: 1 unit of flow.
- $o \to 3 \to d$: 2 units of flow.
- $o \to 2 \to 3 \to d$: saturated.

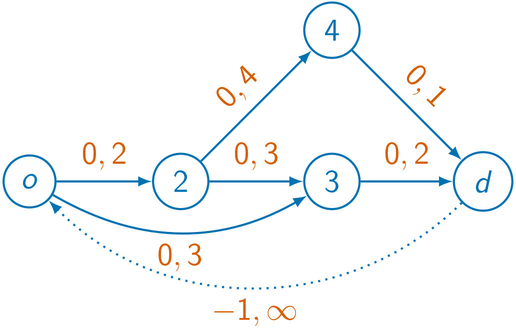
Transhipment



Transhipment



Transhipment



- cost on the artificial arc: -1
- cost on every other arc: 0
- supply/demand for each node: 0
- lower bound on each arc: 0
- upper bound on the artificial arc: ∞
- upper bound on every other arc: from the problem definition

Transhipment

$$\min_{x \in \mathbb{R}^{n+1}} -x_{do}$$

subject to

$$\sum_{j|(i,j)\in\mathcal{A}\cup(d,o)} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}\cup(d,o)} x_{ki} = 0, \ \forall i \in \mathcal{N}, \tag{1}$$

$$x_{ij} \le u_{ij}, \ \forall (i,j) \in \mathcal{A}, \tag{2}$$

$$x_{ij} \ge 0, \ \forall (i,j) \in \mathcal{A}, \tag{3}$$

$$x_{do} \ge 0. \tag{4}$$

The transportation problem

Motivation

- The transportation problem is somehow a transhipment problem where transhipment is not allowed.
- We have suppliers and customers.
- There is a cost to transport items from a supplier to a customer.
- We need to decide how much each supplier will deliver to each customer.

Electricity in Switzerland

Suppliers

- Mühleberg: 3110 GWh/year
- Beznau: 3198 GWh/year
- Leibstadt: 10205 GWh/year

Customers

- Zürich: 8961 GWh/year
- Geneva: 3777 GWh/year
- Lausanne: 2517 GWh/year
- Bern: 1258 GWh/year

Electricity in Switzerland

Unit costs

	Zürich	Geneva	Lausanne	Bern
Mühleberg	18	6	10	9
Beznau	9	16	13	7
Leibstadt	14	9	16	5

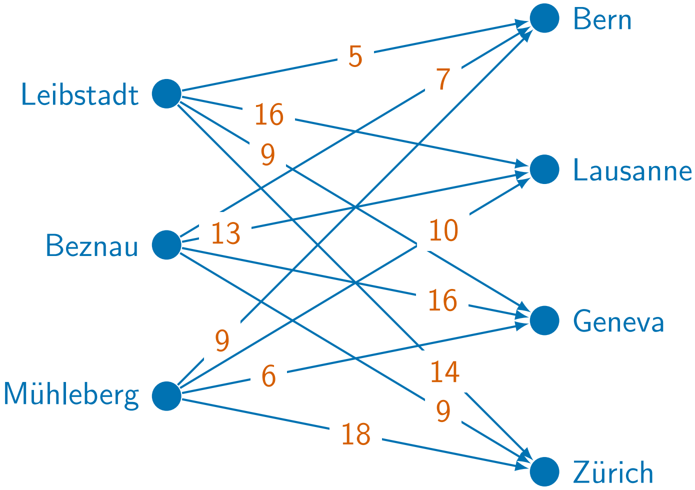
Network

Comments

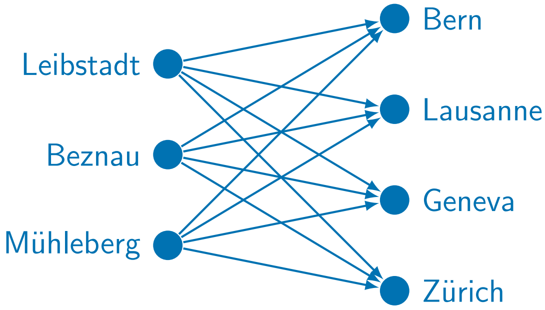
- No “real” network.
- We create a network representation.
- One set of nodes for the suppliers.
- One set of nodes for the customers.

Network

Network


Bern
Lausanne
Geneva
Zürich
Leibstadt
Beznau
Mühleberg
5
7
16
9
13
10
16
9
6
14
9
18

Transhipment



- cost on each arc: c_{ij}
- supply for supplier nodes: s_i
- supply for customer nodes: $-t_j$
- lower bound on each arc: 0
- upper bound on each arc: $+\infty$

Transhipment

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{m_o} \sum_{j=1}^{m_d} c_{ij} x_{ij}$$

subject to

$$\begin{aligned}
\sum_{j=1}^{m_d} x_{ij} &= s_i, & i &= 1,\dots,m_o,\\
\sum_{i=1}^{m_o} x_{ij} &= t_j, & j &= 1,\dots,m_d,\\
x_{ij} &\geq 0, & i &= 1,\dots,m_o,\ j = 1,\dots,m_d,\\
x_{ij} &= 0, & &\text{if } i \text{ does not serve } j.
\end{aligned}$$

The assignment problem

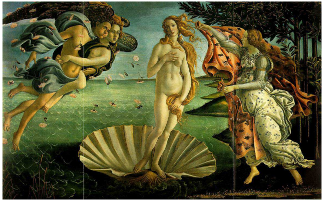
Motivation

The assignment problem consists in assignment n tasks to n resources at minimal cost.

Heirs

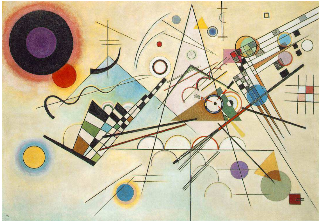
- After the death of her husband, my grandmother has discovered four masterpieces in her attic.
- She does not want to keep them.
- She would like to sell each of them to one of her four children.
- Each child made an offer for the masterpieces of interest.
- She wants to sell exactly one masterpiece to each child.
- Which masterpiece should she sell to which child, to maximize her revenues?

Heirs

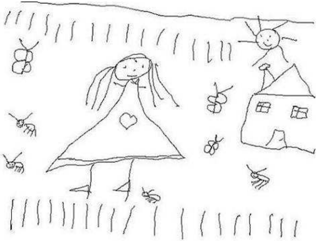


Botticelli, 1485

Bruegel, 1558



Kandinsky, 1923

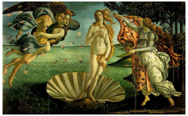


Bierlaire, 1971

Offers (kEuros)

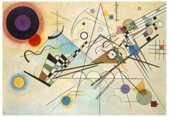
	Botticelli	Bruegel	Kandinsky	Bierlaire
Harry	8000	11000	—	—
Ron	9000	13000	12000	—
Hermione	9000	—	11000	0.01
Ginny	—	14000	12000	—

Network

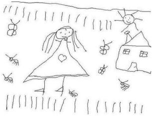


Harry

Ron

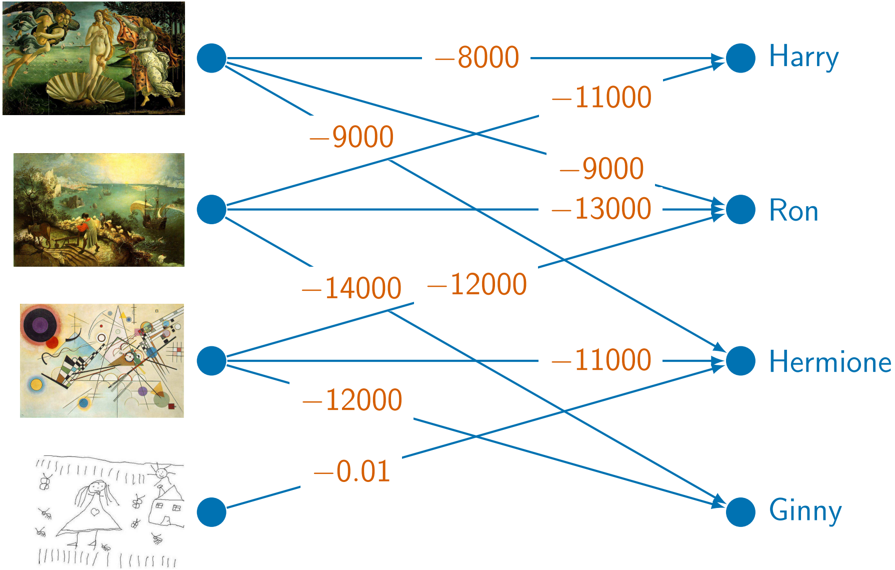


Hermione



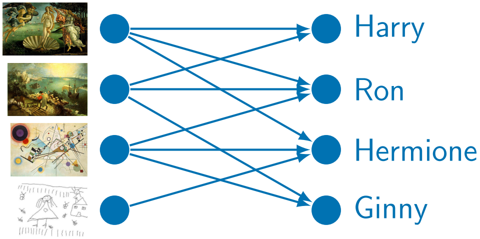
Ginny

Network



−8000
−11000
−9000
−9000
−13000
−14000
−12000
−11000
−12000
−0.01
Harry
Ron
Hermione
Ginny

Transhipment



- cost on each arc: $c_{ij} = -p_{ij}$
- supply for each resource node: 1
- supply for each task node: -1
- lower bound on each arc: 0
- upper bound on each arc: 1

Transhipment

$$\min_{x \in \mathbb{R}^{n^2}} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, \ldots, n, \tag{5}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \ldots, n, \tag{6}$$

$$x_{ij} \geq 0, \qquad i = 1, \ldots, n, \ j = 1, \ldots, n, \tag{7}$$

$$x_{ij} \leq 1, \qquad i = 1, \ldots, n, \ j = 1, \ldots, n. \tag{8}$$

The variable x_{ij} is to take on the value 1 if resource i is assigned to task j, and 0 otherwise.

Summary

- Transhipment problem: linear optimization problem.
- Special structure.
- Property: total unimodularity of the incidence matrix.
- Consequence: if the data is integer, the optimal solution is also integer.
- Shortest path problem.
- Maximum flow problem.
- Transportation problem.
- Assignment problem.