Linear optimization

From geometry to algebra

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Introduction to optimization and operations research

Linear optimization

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} c_i x_i$$

subject to

$$\sum_{i=1}^{n} a_{ji} x_i = b_j, \qquad j = 1, \ldots, m,$$

$$x_i \geq 0, \qquad i = 1, \ldots, n.$$

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$$x_i \geq 0, \qquad i = 1, \dots, n.$$

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = b,$$

$$x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Polyhedron

$$\{x \in \mathbb{R}^n | Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

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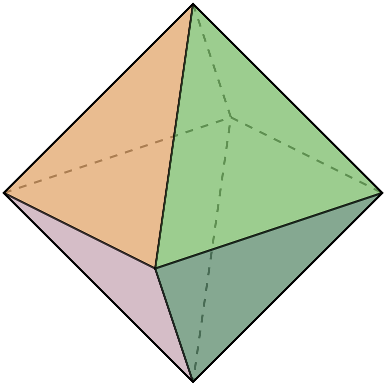


Convex set

A polyhedron $\mathcal{P}$ is a convex set

For all $x, y \in \mathcal{P}$, for all $0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda) y \in \mathcal{P}.$$

Polyhedron representations

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$:

Canonical form

$$\{x \in \mathbb{R}^n | Ax \leq b\}$$

Standard form

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$$

Geometric interpretation

- Canonical form:
$$\{x \in \mathbb{R}^n | Ax \leq b\}.$$
- Include signed and slack variables:
$$\{x^+, x^- \in \mathbb{R}^n, x^s \in \mathbb{R}^m | A(x^+ - x^-) + x^s = b, x^+, x^-, x^s \geq 0\}.$$
$$\{x \in \mathbb{R}^{2n+m} | \tilde{A}x = b, x \geq 0\}$$
- Active constraints:
$$a_j^T x = b_j \iff x_j^s = 0.$$

Vertices

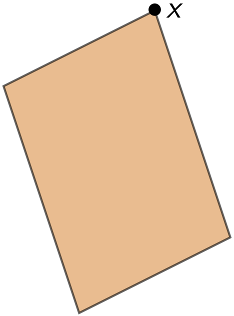
Motivation

- The vertices of a polyhedron play a major role in optimization.
- Often, this is where we will find the optimal solution.

Vertex

x is a vertex of $\mathcal{P}$ if there is no $y, z \in \mathcal{P}$ such that $\exists 0 < \lambda < 1$ and

$$x = \lambda y + (1 - \lambda) z.$$



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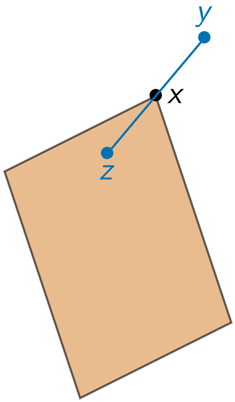

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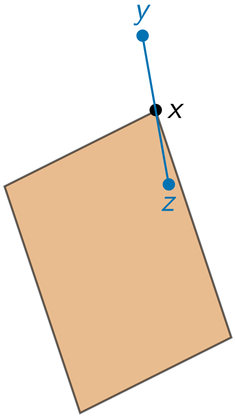

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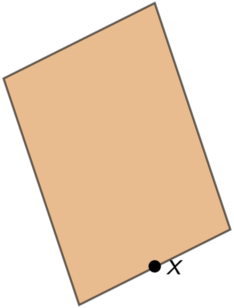

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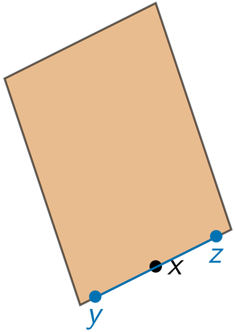


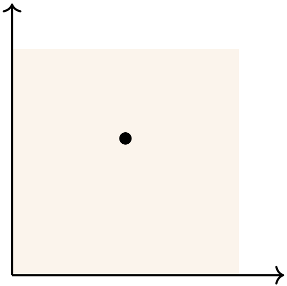
Existence

Theorem 3.37

- Consider a polyhedron $\mathcal{P}$ in standard form.
- If it is not empty, it has at least one vertex.

Idea of the proof

- Start from $x \in \mathcal{P}$.
- Follow a direction pointing to a constraint.
- Activate the first constraint met.
- Repeat in the facet which is of lower dimension.



Feasible directions

Motivation

- Most algorithms are iterative. They move from a feasible point in a given direction.
- We must make sure that it is possible to generate another feasible point along that direction.

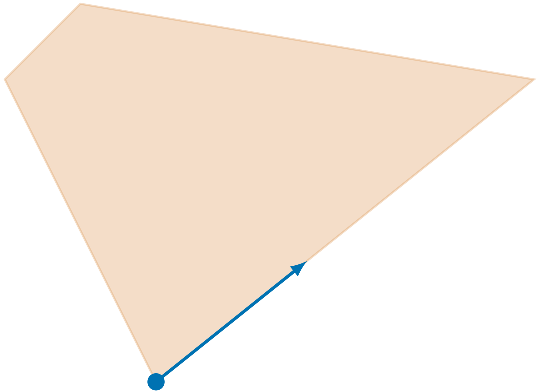
Definition

Consider $Y \subseteq \mathbb{R}^n$ be the feasible set and $x \in Y$.
The direction $d \in \mathbb{R}^n$ is feasible in x if $\exists \eta > 0$ such that

$$x + \alpha d \in Y, \forall 0 < \alpha < \eta.$$

Examples

Examples



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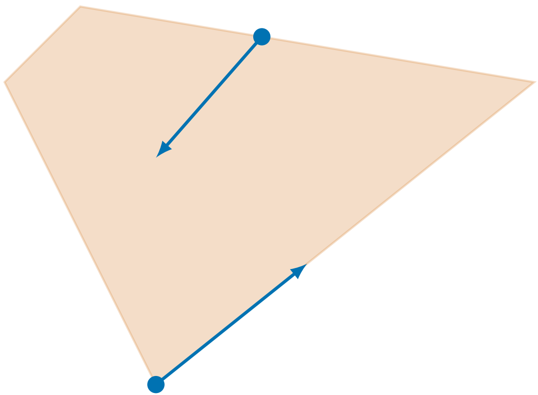
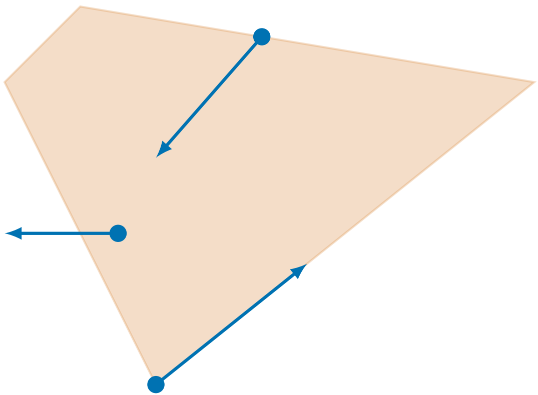
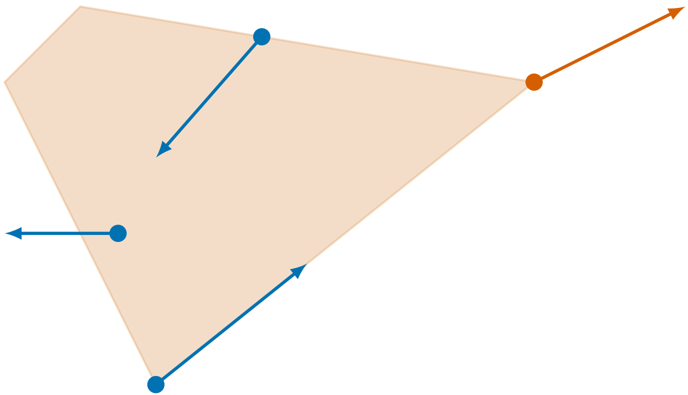


Examples



Examples



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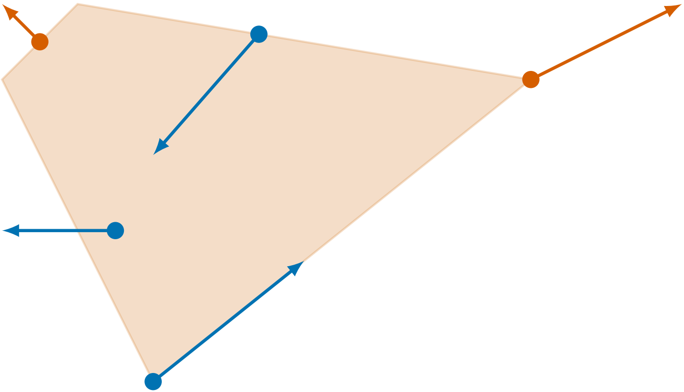
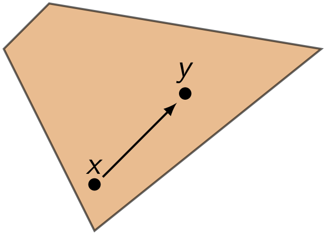


Feasible direction in a convex set

If X is convex, and $x, y \in X$,

$$d = y - x$$

is feasible in x.



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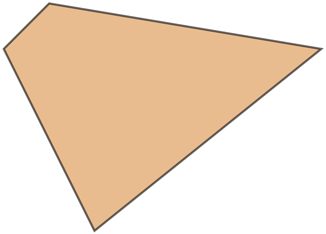

Polyhedron in standard form

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b,\ x \geq 0\}$$

$x^+ \in \mathcal{P}$, $d \in \mathbb{R}^n$, $\alpha > 0$.
Two conditions for d to be feasible

$$b = A(x^+ + \alpha d) = Ax^+ + \alpha Ad = b + \alpha Ad$$

Polyhedron in standard form

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, \ x \geq 0\}$$

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Two conditions for d to be feasible

$$b = A(x^+ + \alpha d) = Ax^+ + \alpha Ad = b + \alpha Ad$$

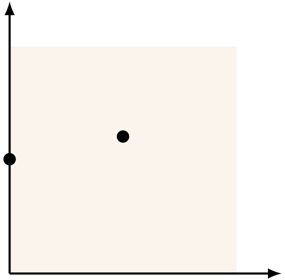
Theorem 3.13: first condition

$$Ad = 0.$$

Polyhedron in standard form

Theorem 3.13: second condition

- If $x_i^+ > 0$, $\forall i$: every direction is feasible.
- If $\exists i$ such that $x_i^+ = 0$, then $d_i \geq 0$.



Standard form

Motivation

- It is convenient to write linear constraints in standard form.
- All inequality constraints are non negativity constraints.
- The rest are equality constraints.

Standard form

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Equality constraints

$$Ax = b$$

Number of x such that $Ax = b$

- 0: incompatible
- 1: non singular
- ∞: underdetermined: the only interesting one

Rank

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \quad \text{rank}(A) = 2.$$

Compatible: $x_1 = 0,\ x_2 = 0,\ x_3 = 5,\ x_4 = 2.$

$$\begin{array}{ccccccc} x_1 & -x_2 & & +x_4 & = & 2 \\ & & x_3 & -x_4 & = & 3 \\ x_1 & -x_2 & +x_3 & & = & 5 \end{array}$$

$$x_4 = 2 - x_1 + x_2$$
$$x_3 - 2 + x_1 - x_2 = 3$$

Redundant constraints

- Consider a compatible system $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- $\text{Rank}(A) = r < m$.
- Then there exists $m - r$ redundant constraints that can be removed.

`Theorem 3.6`

It can be assumed that A is of full rank.

Elimination of constraints

Motivation

- If the polyhedron is in standard form,
- we use the equality constraints to eliminate some of them.

Example

$$\min x_1 + x_2 + x_3 + x_4$$

subject to

$$x_1 + x_2 + x_3 = 1$$
$$x_1 - x_2 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

$$x_3 = 1 - x_1 - x_2$$
$$x_4 = 1 - x_1 + x_2$$

$$\min x_1 + x_2 + 1 - x_1 - x_2 + 1 - x_1 + x_2 = -x_1 + x_2 + 2$$

Warning: $x_1 = 3$,$x_2 = 1$, $x_3 = -3$, $x_4 = -1$

Terminology: x_3, x_4: basic variables

Method

$$Ax = b,\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n, \mathrm{rank}(A) = m.$$

- Select m columns of A linearly independent to form $B \in \mathbb{R}^{m\times m}$:

$$AP = (B\ N) \text{ with } PP^T = I$$

- Rewrite the equality constraints:

$$Ax = (AP)(P^T x) = Bx_B + Nx_n = b$$

- Eliminate the basic variables:

$$x_B = B^{-1}(b - Nx_N).$$

Example

$$\begin{array}{ccccc} x_1 & +x_2 & +x_3 & & = & 1 \\ x_1 & -x_2 & & +x_4 & = & 1. \end{array}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eliminate x_3 and x_4.

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$AP = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Example in Python

```
matrix_a: np.ndarray = np.array(
    [[1, 1, 1, 0], [1, -1, 0, 1]]
)
renumbering: list[int] = [2, 3, 0, 1]
identity_matrix: np.ndarray = np.eye(4)
permutation_matrix: np.ndarray = identity_matrix[:,
   renumbering]
```

```
>> [[0. 0. 1. 0.]
    [0. 0. 0. 1.]
    [1. 0. 0. 0.]
    [0. 1. 0. 0.]]
```

Example in Python

```
>> [[ 1  1  1  0]
    [ 1 -1  0  1]]
permuted_matrix_a: np.ndarray = matrix_a @
   permutation_matrix
>> [[ 1  0  1  1]
    [ 0  1  1 -1]]
```

Example

$$AP = (B|N) = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$\begin{aligned} x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} &= B^{-1}(b - Nx_N) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \end{pmatrix} \end{aligned}$$

Algebraic representation of a vertex

Intuition

- At a vertex, constraints are active.
- Standard form: $x_i = 0$.
- To find a vertex:
  1. Select and eliminate basic variables.
  2. Set all non basic variables to 0.
  3. Check feasibility.

$$AP = (B|N)$$

$$x = P \begin{pmatrix} B^{-1}(b - Nx_N) \\ x_N \end{pmatrix} = P \begin{pmatrix} B^{-1}b \\ 0_{\mathbb{R}^{n-m}} \end{pmatrix}.$$

$$B^{-1}b \geq 0.$$

`Theorem 3.35`

Definition

Consider

- $\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$,
- $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $n \geq m$,
- $x \in \mathbb{R}^n$ such that $Ax = b$,
- a set of indices $j_1, \dots, j_m$.

Definition

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- a set of indices $j_1, \ldots, j_m$.

x and the indices form a basic solution if

1. $B = (A_{j_1} \cdots A_{j_m})$ is non singular,
2. $x_i = 0$ if $i \neq j_1, \ldots, j_m$.

Definition

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x and the indices form a basic solution if

1. $B = (A_{j_1} \cdots A_{j_m})$ is non singular,
2. $x_i = 0$ if $i \neq j_1, \ldots, j_m$.

If

$$x_B = B^{-1} b \geq 0,$$

it is a feasible basic solution.

Equivalence

Theorem 3.40

$x^* \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only if it is a feasible basic solution.

Example

Polyhedron

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | x_1 + x_2 \leq 1, 3x_1 + 10x_2 \leq 15, x_1 \geq 0, x_2 \geq 0 \right\}.$$

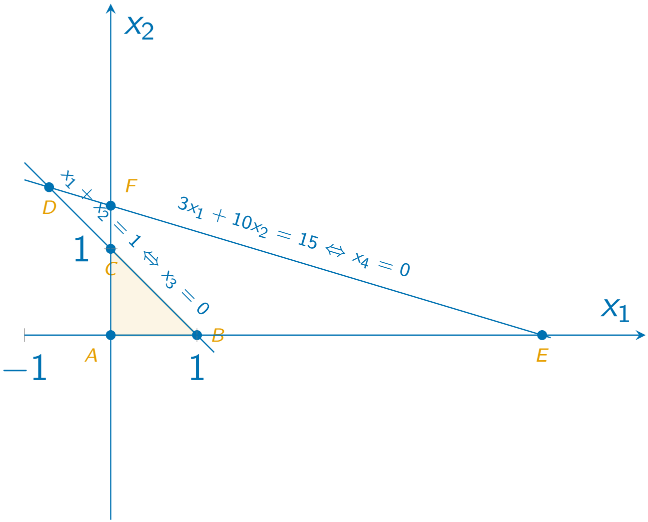
Polyhedron in standard form

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 | Ax = b, x \geq 0 \right\},$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 15 \end{pmatrix}.$$

Example



x_2
x_1
$x_1 + x_2 = 1 \Leftrightarrow x_3 = 0$
$3x_1 + 10x_2 = 15 \Leftrightarrow x_4 = 0$
F
D
C
A
B
E
1
-1
1

Example

- Basic solution with x_3 and x_4 in the basis ($j_1 = 3, j_2 = 4$).

$$B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 15 \end{pmatrix} ; x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 15 \end{pmatrix} .$$

  This basic solution is feasible and corresponds to point $A = (0, 0)$ in the figure.

- Basic solution with x_1 and x_4 in the basis ($j_1 = 1, j_2 = 4$).

$$B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 12 \end{pmatrix} ; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 12 \end{pmatrix} .$$

  This basic solution is feasible and corresponds to point $B = (1, 0)$.

- Basic solution with x_2 and x_4 in the basis ($j_1 = 2, j_2 = 4$).

$$B = \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix} ; B^{-1} = \begin{pmatrix} 1 & 0 \\ -10 & 1 \end{pmatrix} ; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 5 \end{pmatrix} ; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix} .$$

  This basic solution is feasible and corresponds to point $C = (0, 1)$ in the figure.

Example

- Basic solution with x_1 and x_2 in the basis ($j_1 = 1, j_2 = 2$).

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 10 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{10}{7} & -\frac{1}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -\frac{5}{7} \\ \frac{12}{7} \end{pmatrix}; x = \begin{pmatrix} -\frac{5}{7} \\ \frac{12}{7} \\ 0 \\ 0 \end{pmatrix}.$$

  This basic solution is not feasible because $B^{-1}b \not\geq 0$. It corresponds to point $D = (-\frac{5}{7}, \frac{12}{7})$ in the figure.

- Basic solution with x_1 and x_3 in the basis ($j_1 = 1, j_2 = 3$).

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 5 \\ -4 \end{pmatrix}; x = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 0 \end{pmatrix}.$$

  This basic solution is not feasible because $B^{-1}b \not\geq 0$. It corresponds to point $E = (5, 0)$.

- Basic solution with x_2 and x_3 in the basis ($j_1 = 2, j_2 = 3$).

$$B = \begin{pmatrix} 1 & 1 \\ 10 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & \frac{1}{10} \\ 1 & -\frac{1}{10} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}; x = \begin{pmatrix} 0 \\ \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

  This basic solution is not feasible because $B^{-1}b \not\geq 0$. It corresponds to point $F = (0, \frac{3}{2})$ in the figure.

Simple Python implementation

```python
def calculate_basic_solution(
    A: np.ndarray, b: np.ndarray, basic_indices:
        list[int]
) -> np.ndarray:
    m, n = A.shape
    basis = A[:, basic_indices]
    x_basic: np.ndarray = np.linalg.solve(basis, b)
    basic_solution: np.ndarray = np.zeros(n)
    basic_solution[basic_indices] = x_basic
    return basic_solution

A = np.array([[1, 1, 1, 0], [3, 10, 0, 1]])
b = np.array([1, 15])
solution_1 = calculate_basic_solution(A, b, [2, 3])
>> Basic solution 1:  [ 0.   0.   1. 15.]
```

Degeneracy

Context

- The concepts of vertex and basic feasible solutions are equivalent.
- But there is not necessarily a bijection between the two sets.
- A vertex may correspond to several basic feasible solutions.

Example

Consider the polyhedron in $\mathbb{R}^2$:

$$
\begin{aligned}
x_1 + x_2 &\leq 1 \\
0 \leq x_1 &\leq 1 \\
0 \leq x_2 &\leq 1
\end{aligned}
$$

We consider the equivalent polyhedron in standard form in $\mathbb{R}^5$:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
x_1 + x_4 &= 1 \\
x_2 + x_5 &= 1 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
$$

Example: first basis

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \; b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

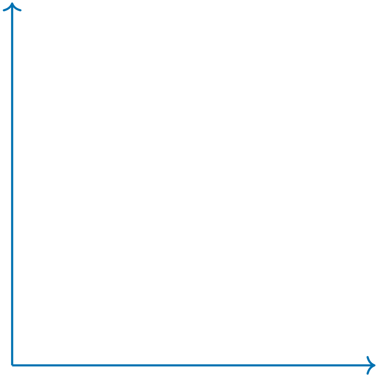
$$x_B = B^{-1} b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \; x = \begin{pmatrix} 1 \\ \\ 0 \\ \\ 1 \end{pmatrix} \geq 0$$

Example: second basis

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_B = B^{-1} b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ \\ \\ 0 \\ 1 \end{pmatrix} \geq 0$$

Example



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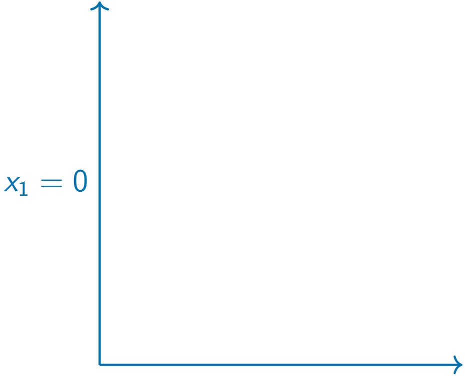

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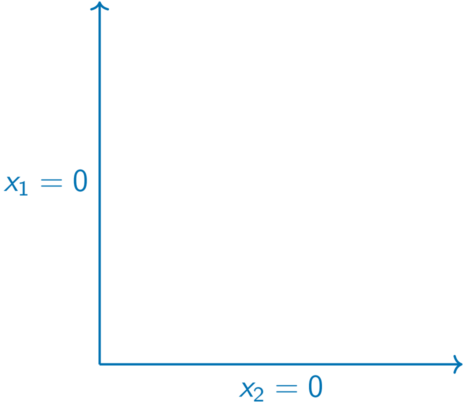

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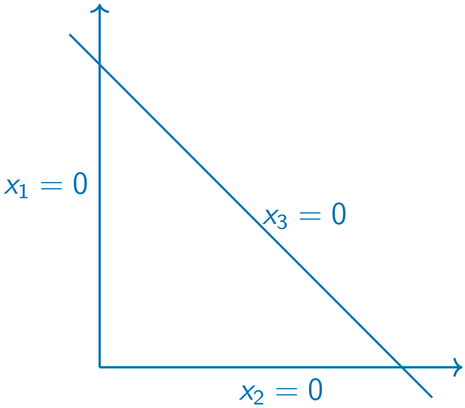

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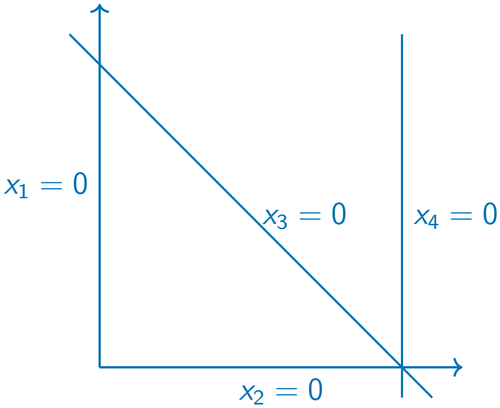
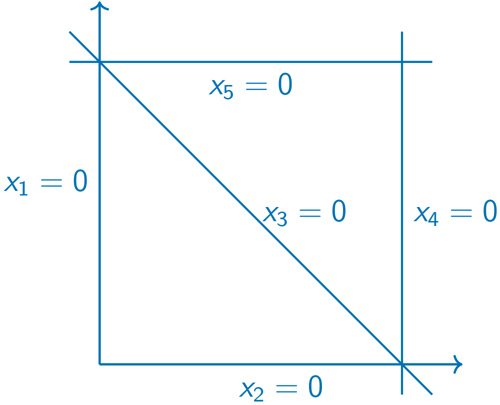
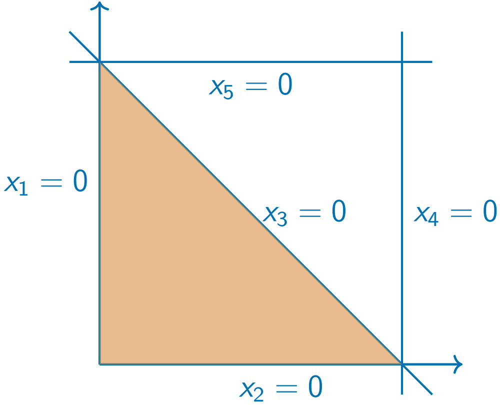


Example



Example



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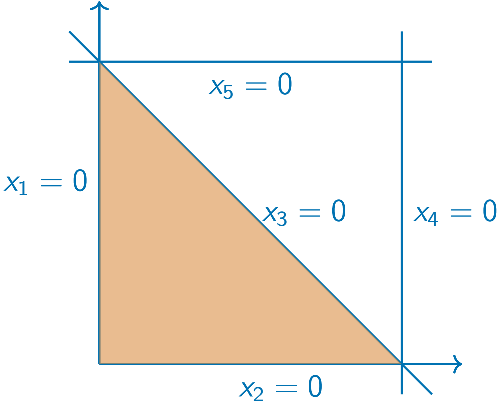

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis 1: [1,3,5]
Basis 2: [1,4,5]

Definition

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\},\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m.$$

A basic solution x is degenerate if

- more than n constraints are active at x, or
- more than $n - m$ components of x are 0.

Note: the equality constraints are always active

Basic directions

Motivation

- The concept of basic and non basic variables allowed us to identify the vertices of the polyhedron.
- Let's now use them to identify feasible directions.

Feasible direction

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} d_B \\ d_N \end{pmatrix}$$

Reminder

d is feasible at x if

$$Ad = 0,$$
$$d_i \geq 0 \text{ if } x_i = 0.$$

$$d_N = (0, 0, 0, 1, 0, 0, 0)$$

$$Ad = Bd_B + Nd_N$$

$$= Bd_B + \sum_{j=m+1}^{n} A_j d_j = Bd_B + A_p = 0,$$

$$d_B = -B^{-1}A_p.$$

Feasible direction

Theorem 3.44

If x is non degenerate, any basic direction is feasible.

Idea

d is feasible at x if

$$Ad = 0,$$
$$d_i \geq 0 \text{ if } x_i = 0.$$

If x is non degenerate, only non basic variables are 0.

Reduced costs

Motivation

- Let's now look at the objective function $c^T x$.
- Its gradient c provides information about its slope.
- In linear optimization, the gradient/slope is constant.
- We are interested in the slope of the objective function along feasible directions.

Basis representation

$$min_{x \in \mathbb{R}^n} f(x) = c^T x$$

subject to

$$Ax = b$$

$$x \geq 0.$$

$$A \in \mathbb{R}^{m \times n}$$

$$b \in \mathbb{R}^m$$

$$c \in \mathbb{R}^n.$$

Basis representation

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Basis

$$A = (B|N)$$

Basis representation

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$$Ax = b$$
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Basis

$$A = (B|N)$$

Basic solution

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

Basis representation

$$min_{x\in\mathbb{R}^n} f(x) = c^T x$$

subject to

$$Ax = b$$
$$x \geq 0.$$

$$A \in \mathbb{R}^{m\times n}$$
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Basis

$$A = (B|N)$$

Basic solution

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

Basic direction

$$d_j = \begin{pmatrix} (d_j)_B \\ (d_j)_N \end{pmatrix} = \begin{pmatrix} -B^{-1}A_j \\ e_j \end{pmatrix}$$

Basic directions

Basic direction

$$d_j = \begin{pmatrix} (d_j)_B \\ (d_j)_N \end{pmatrix} = \begin{pmatrix} -B^{-1}A_j \\ e_j \end{pmatrix}$$

Slope along the basic direction

$$\nabla f(x)^T d_j = c^T d_j = c_B^T (d_j)_B + c_N^T (d_j)_N = -c_B^T B^{-1} A_j + c_j$$

Reduced costs

Definition

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j$$

Non basic variables

Slope of the objective function along the corresponding basic direction.

Basic variables

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j = c_j - c_B^T e_j = c_j - c_j = 0$$

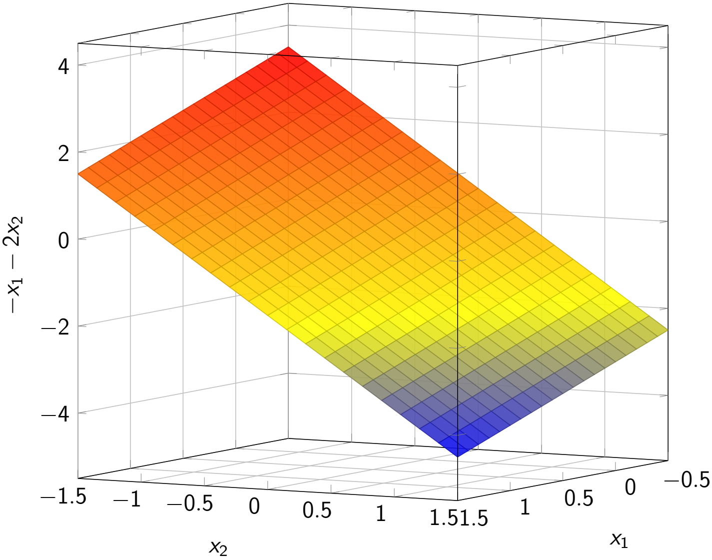
Matrix form

$$\bar{c} = c - A^T B^{-T} c_B.$$

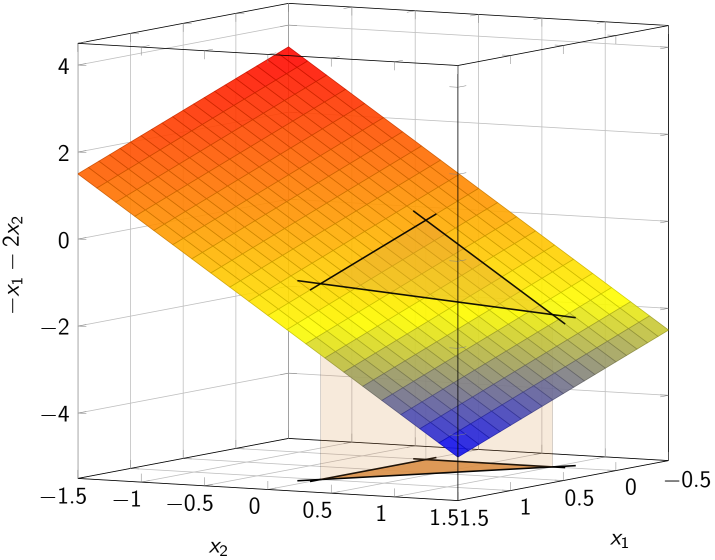
Sufficient optimality conditions

If $\bar{c} \geq 0$, then x^* is optimal.

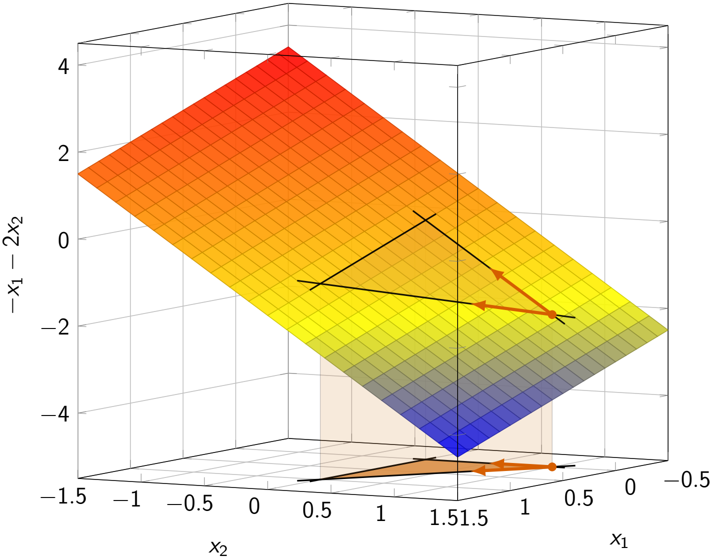
Sufficient optimality conditions



Sufficient optimality conditions



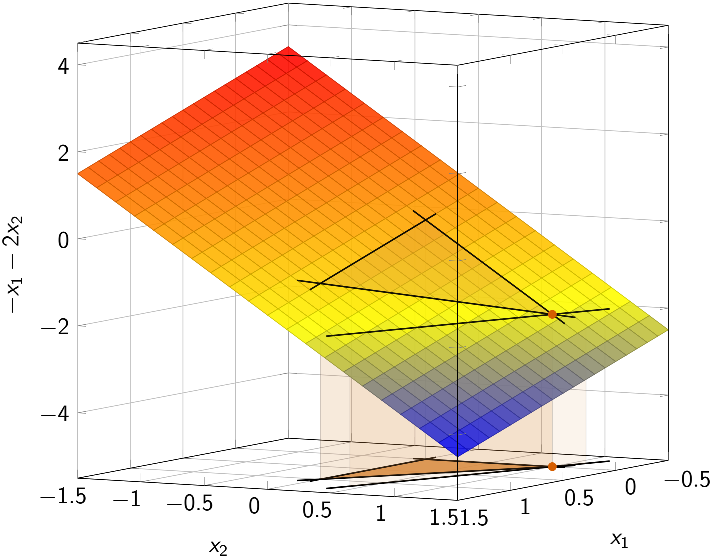
Sufficient optimality conditions



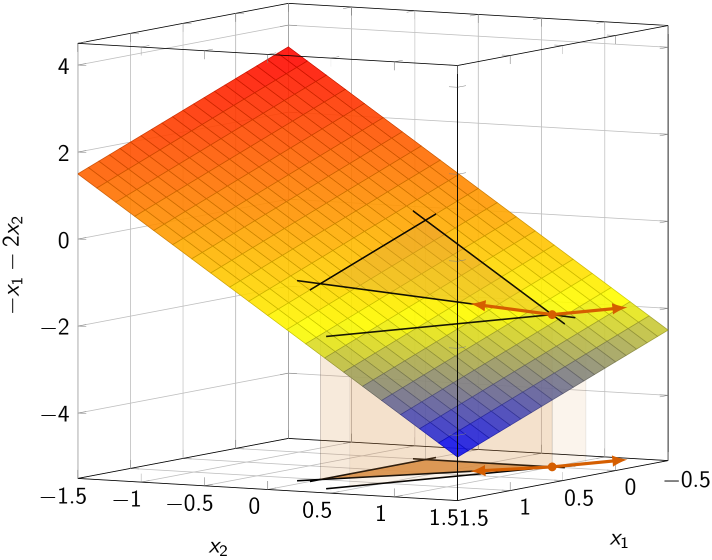
Necessary optimality conditions

- $\bar{c} \geq 0$ is not necessary at an optimal solution.
- Consider a slightly different example.

Necessary optimality conditions



Necessary optimality conditions



Necessary optimality conditions

- x^* is a non degenerate feasible basic solution.

Necessary optimality conditions

- x^* is a non degenerate feasible basic solution.
- If x^* is optimal, then

$$\bar{c} = c - A^T B^{-T} c_b \geq 0.$$

Necessary optimality conditions

Warning

- If the basic solution is degenerate, it is possible that $\bar{c}_j < 0$ for some j.

Necessary optimality conditions

Warning

- If the basic solution is degenerate, it is possible that $\bar{c}_j < 0$ for some j.
- It means that this is a descent direction.

Necessary optimality conditions

Warning

- If the basic solution is degenerate, it is possible that $\bar{c}_j < 0$ for some j.
- It means that this is a descent direction.
- As x^* is optimal, d_j is infeasible.

Summary

- Constraints = polyhedron.
- Active constraints.
- Feasible directions.
- Vertices and feasible basic solution.
- Degeneracy.
- Basic directions.
- Reduced costs.