Teacher: Michel Bierlaire
Cours: Introduction to optimization and operations research

Friday 30 January 2025 (09:15 - 11:15)

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Lennon John

Part 1: multiple choice questions

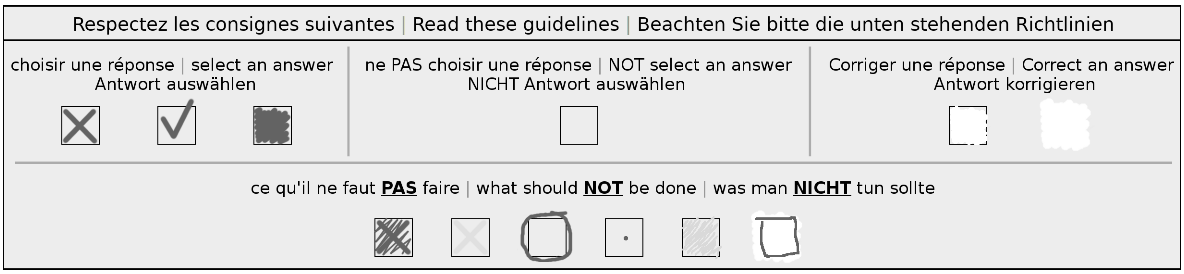
SCIPER : **XXXXX1** Signature :

Wait until the beginning of the exam before turning the page. This document is printed double-sided and has 20 multiple choice questions. Each question has exactly one correct answer. Do not remove the staple. The exam contains 12 pages.

Carefully read the following instructions:

- Place your student card on the table.
- You are allowed to have a **4 pages** (8 sides) **handwritten** summary. Any summary which is not handwritten cannot be used and **will be confiscated**. The summary must clearly show your name and today's date. The summary will be collected at the end of the exam.
- Draft paper is available, so **please do not write on the exam**. Calculators and electronic devices are **not allowed**.
- The exam is corrected electronically. Use a **pen (not a pencil) with black or dark blue ink** and avoid altering and erasing your answers if possible. If you must change an answer, use correction tape (avoid correction fluid). Do not draw an empty box.
- The grading scheme for the multiple choice questions is:

  +1.5 point for a correct answer,
  0 points if no answer is given,
  −0.5 point for a wrong answer.
- If a question contains a mistake, the teacher can remove it from the exam.
- Follow these guidelines for **marking your answers**:



Question 1 : Consider a linear optimization problem $P = \{\min \quad c^T x : Ax = b, x \geq 0\}$ and its dual problem $D = \{\max \quad b^T \lambda : A^T \lambda \leq c\}$. If D' is the dual of D, then which of the following statements is **correct**?

- ☐ If P is unbounded then D' is infeasible.
- ☐ Problems P and D' are equivalent.
- ☐ If P is infeasible D' can be feasible but it is unbounded.
- ☐ Problems D and D' are equivalent.

Question 2 : We need to solve a maximum flow problem on a network where arc capacities are specified, but no costs are associated with the arcs. To model this problem as a transshipment problem aimed at minimizing the total costs on the network, we must introduce an artificial (or dummy) arc into the original network. Which of the following specifications for this artificial arc is **correct**?

- ☐ Lower Bound: 0, Upper Bound: maximum flow of the original network, Cost: 0.
- ☐ Lower Bound: 0, Upper Bound: ∞, Cost: 0.
- ☐ Lower Bound: 1, Upper Bound: ∞, Cost: 1.
- ☐ Lower Bound: 0, Upper Bound: ∞, Cost: -1.

Question 3 : Let us assume that a function $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and has a local minimum at $x = x^*$. Which of the following statements is **correct**?

- ☐ There exists a value $\varepsilon > 0$, such that $f(x) \geq f(x^*)$ for all x such that $|x - x^*| < \varepsilon$.
- ☐ The second derivative matrix of $f(x)$ at x^*, $\nabla^2 f(x^*)$, must be positive definite.
- ☐ Since the function is continuous and differentiable, $f(x^*)$ is the global minimum of $f(x)$ for all x.
- ☐ $f(x)$ is constant around x^*.

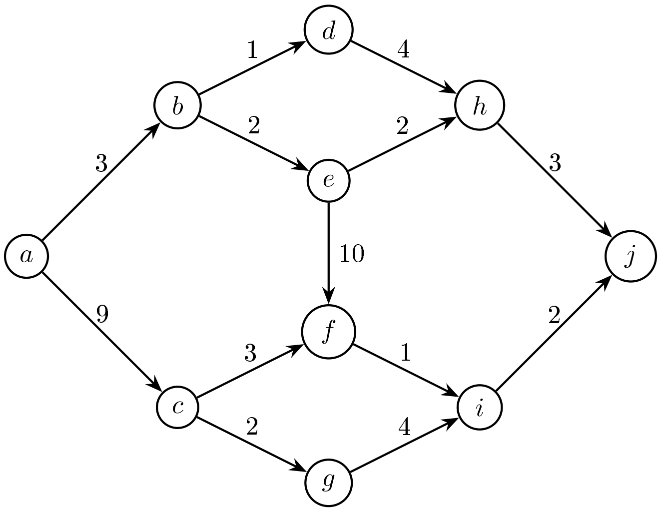
Question 4 : Consider the following simplex tableau for a linear optimization problem:

x_1	x_2	x_3	x_4	x_5	
4	2	-3	1	0	6
1	1	3	0	1	4
-3	-1	0	0	0	0

Which of the following statements is **correct**?

- ☐ The current solution is optimal.
- ☐ x_4 and x_5 are basic variables.
- ☐ The current solution is degenerate.
- ☐ x_1, x_2 and x_3 are basic variables.

Question 5 : Consider the following directed graph. We want to find the shortest path from node a to node j.



At the end of the third iteration of Dijkstra's algorithm, the current state is summarized as follows:

Iteration	S	i	λ_a	λ_b	λ_c	λ_d	λ_e	λ_f	λ_g	λ_h	λ_i	λ_j
3	$\{e, c, h\}$	e	0	3	9	4	5	∞	∞	8	∞	∞

What is the state of Dijkstra's algorithm after the next (fourth) iteration? Select the **correct** answer.

- ☐ $(4,\ \{c, f, h\},\ h,\ \lambda_a = 0,\ \lambda_b = 3,\ \lambda_c = 9,\ \lambda_d = 4,\ \lambda_e = 5,\ \lambda_f = 15,\ \lambda_g = \infty,\ \lambda_h = 7,\ \lambda_i = \infty,\ \lambda_j = \infty)$.
- ☐ $(4,\ \{c, f, h\},\ c,\ \lambda_a = 0,\ \lambda_b = 3,\ \lambda_c = 9,\ \lambda_d = 4,\ \lambda_e = 5,\ \lambda_f = 15,\ \lambda_g = \infty,\ \lambda_h = 7,\ \lambda_i = \infty,\ \lambda_j = \infty)$.
- ☐ $(4,\ \{c, f, h\},\ h,\ \lambda_a = 0,\ \lambda_b = 3,\ \lambda_c = 9,\ \lambda_d = 4,\ \lambda_e = 5,\ \lambda_f = 15,\ \lambda_g = \infty,\ \lambda_h = 8,\ \lambda_i = \infty,\ \lambda_j = \infty)$.
- ☐ $(4,\ \{c, f, h\},\ c,\ \lambda_a = 0,\ \lambda_b = 3,\ \lambda_c = 9,\ \lambda_d = 4,\ \lambda_e = 5,\ \lambda_f = \infty,\ \lambda_g = \infty,\ \lambda_h = 7,\ \lambda_i = \infty,\ \lambda_j = \infty)$.

Question 6 :
Newton's method can experience convergence issues, particularly in regions where the function is flat or poorly scaled. To address this, techniques such as preconditioning and line search are often employed. Consider the function $f(x) = x^4 - 3x^2$, where we aim to minimize f using Newton's method starting from $x_0 = 0.7$.

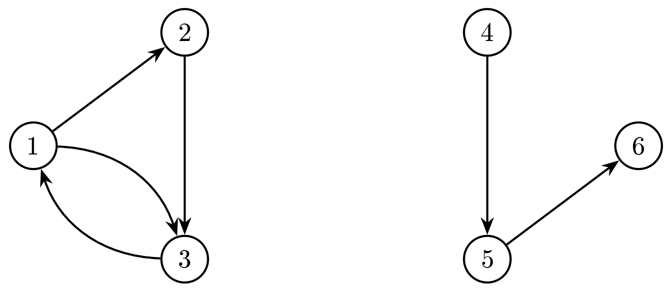
Which of the following statements is **correct**?

- ☐ Performing an inexact line search to control the step size at each iteration can help ensure convergence, as it prevents taking overly large steps and overly short steps.
- ☐ Using a preconditioner that rescales $f(x)$ will guarantee convergence to a local minimum.
- ☐ Changing the initial guess x_0 to -0.7 will ensure better convergence of Newton's method to a global minimum.
- ☐ No modification is necessary since Newton's method will always converge if $f(x)$ is twice differentiable.

Question 7 :
Let us assume that we are minimizing the objective function of an integer linear optimization problem using a branch and bound algorithm and the lower bound of a subproblem exceeds the current best-known solution's value. What is the **correct** action to take?

- ☐ The algorithm terminates and returns the best-known solution.
- ☐ The subproblem is solved exactly to find the optimal solution.
- ☐ The subproblem is further divided into smaller subproblems by branching on fractional variables.
- ☐ The subproblem is not explored further.

Question 8 : Below is a directed graph with 6 nodes, representing a network for a transshipment problem.



Identify the **correct** incidence matrix that describe the divergence constraints for this network problem, assuming that the lower bound on the flow for each arc is 0 and there is no upper bound on the flow.

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Question 9 : Consider the following constraints:

$$\begin{aligned} -2x_1 + x_2 &\leq 2, \\ 3x_1 + x_2 &\geq 2, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

How many constraints are active at $(x_1, x_2) = (0, 2)$?

- ☐ 4.
- ☐ 1.
- ☐ 3.
- ☐ 2.

Question 10 : Consider the following optimal solution to a linear optimization problem $x = (0, 0, \frac{1}{3}, \frac{1}{7})^T$. Based on this information, what can we conclude about the dual problem?

- ☐ The dual variables corresponding to primal variables x_3 and x_4 will be equal to 0.
- ☐ The last two constraints of the dual problem are active.
- ☐ The dual variables at the given point are $\lambda = (3, 7, 0, 0)^T$.
- ☐ The first two constraints of the dual problem are active.

Question 11 : Consider the following simplex tableau for a linear optimization problem:

x_1	x_2	x_3	x_4	x_5	
2	−1	0	1	0	8
1	−1	3	0	1	6
2	−3	1	0	0	0

Which of the following statements is **correct** for the next step in the simplex procedure?

- ☐ The algorithm stops.
- ☐ x_2 enters the basis and x_4 leaves the basis.
- ☐ x_2 enters the basis and x_5 leaves the basis.
- ☐ x_1 enters the basis and x_4 leaves the basis.

Question 12 : In the simplex method, Bland's rule specifies that, when choosing a non-basic variables to enter the basis, the variable with the smallest index among those with a negative reduced cost must be selected first. For what reason is Bland's rule used ?

- ☐ To ensure that the simplex algorithm will not cycle on degenerate vertices, and will terminate in a finite number of iterations.
- ☐ To minimize the total number of iterations that are needed to find the optimal solution.
- ☐ To improve the value of the objective function by the largest possible amount at each iteration.
- ☐ To avoid degenerate vertices during the execution of the simplex algorithm.

Question 13 : Consider the following table, where each row represents a step of the Dijkstra's algorithm.

Iteration	S	i	λ_a	λ_b	λ_c	λ_d	λ_e	λ_f	λ_g	λ_h
0	{a}	a	0	∞	∞	∞	∞	∞	∞	∞
1	{c,b}	b	0	2	4	∞	∞	∞	∞	∞
2	{c,d,e}	c	0	2	4	5	7	∞	∞	∞
3	{d,e}	d	0	2	4	5	6.5	∞	∞	∞
4	{e,g,f}	e	0	2	4	5	6.5	7	9	∞
5	{h,g,f}	f	0	2	4	5	6.5	7	9	9.5
6	{h,g}	h	0	2	4	5	6.5	7	9	9
7	{g}	g	0	2	4	5	6.5	7	9	9
8	{}	-	0	2	4	5	6.5	7	9	9

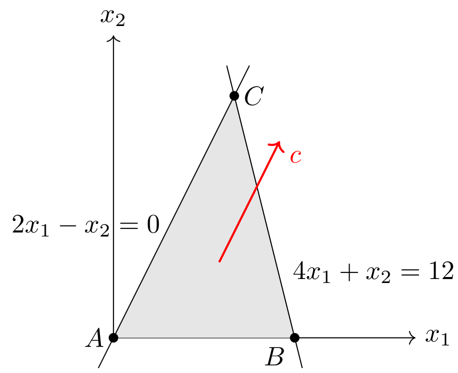
Using the table, determine the shortest path from node a to node h?

- ☐ $a \to f \to h$.
- ☐ $a \to b \to d \to f \to h$.
- ☐ $a \to f \to e \to h$.
- ☐ $a \to e \to f \to h$.

Question 14 : Consider the following linear optimization problem:

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \leq 0, \\ & 4x_1 + x_2 \leq 12, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The feasible region is represented in the diagram below, with vertices labeled as A, B, and C. The current basic feasible solution is at vertex A. The cost vector c direction is indicated.



Let x_3, x_4 be the slack variables corresponding to the constraints $4x_1+x_2 \leq 12$ and $2x_1-x_2 \leq 0$, respectively. What can we say about the next step of the simplex algorithm from vertex A ?

- ☐ If x_4 leaves the basis, the next vertex will be C.
- ☐ The next iteration of the simplex will lead to vertex C which is optimal.
- ☐ If x_3 leaves the basis, the next vertex will be B.
- ☐ A is a degenerate vertex and we might stay at A.

Question 15 : Consider the following linear optimization problem:

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + 4x_2 \geq 8, \\ & x_1 \geq 0, \\ & x_2 \leq 0. \end{aligned}$$

What is the equivalent problem in standard form?

- [ ] $$\begin{aligned} \text{minimize} \quad & 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 - 4x_2 + s_1 = 8, \\ & x_1 \geq 0, \\ & -x_2 \geq 0, \\ & s_1 \geq 0. \end{aligned}$$

- [ ] $$\begin{aligned} \text{minimize} \quad & 3x_1 - 5x_2 \\ \text{subject to} \quad & 2x_1 - 4x_2 - s_1 = 8, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & s_1 \geq 0. \end{aligned}$$

- [ ] $$\begin{aligned} \text{minimize} \quad & 3x_1 + 5x_2 + s_1 \\ \text{subject to} \quad & 2x_1 + 4x_2 + s_1 = 8, \\ & s_1 \geq 0. \end{aligned}$$

- [ ] $$\begin{aligned} \text{minimize} \quad & 3x_1 - 5x_2 \\ \text{subject to} \quad & 2x_1 + 4x_2 + s_1 = 8, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & s_1 \geq 0. \end{aligned}$$

Question 16 :

Consider the following polyhedron in standard form $P = \{Ax = b : x \geq 0\}$ where A has m rows and n columns. We separate the set of variables into m basic variables and $n - m$ non-basic variables, in such a way that the corresponding basic matrix is non singular. Which of the following statements is **correct** for a basic feasible solution?

- [ ] The non-basic variables can be expressed as a function of the basic variables.
- [ ] The basic variables are all equal to zero.
- [ ] If a basic variable is non-zero, the corresponding non-basic variable must be zero.
- [ ] The non-basic variables are all equal to zero.

Question 17 : Consider a linear optimization problem that has a degenerate feasible basic solution. What statement is **correct**?

- ☐ All reduced costs are nonnegative at the degenerate feasible basic solution.
- ☐ Some basic variables in the solution are equal to zero at the degenerate feasible basic solution.
- ☐ The objective function value is the same at all vertices.
- ☐ The problem has no optimal solution.

Question 18 : In linear optimization, if the primal problem is unbounded, what can be said about the dual problem?

- ☐ Nothing can be said about the dual yet.
- ☐ The dual problem has a finite optimal solution.
- ☐ The dual problem is infeasible.
- ☐ The dual problem is unbounded.

Question 19 : Which of the following statements about a tree is **false**?

- ☐ A tree is a connected graph with no cycle.
- ☐ A tree can have multiple connected components.
- ☐ Every tree with n nodes has exactly $n-1$ edges.
- ☐ A tree is a special case of a graph.

Question 20 :
Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is a differentiable nonlinear function, $x_k \in \mathbb{R}^n$ is a point, and $d_k \in \mathbb{R}^n$ is a descent direction for g at x_k (i.e., $\nabla g(x_k)^T d_k < 0$). Consider a step length α' that satisfies the inequality:

$$g(x_k + \alpha' d_k) \leq g(x_k) + \alpha' \varepsilon \, \nabla g(x_k)^T d_k,$$

for $0 < \varepsilon < 1$.
Which statement is **correct** about α'?

- ☐ α' always satisfies both Wolfe conditions.
- ☐ α' satisfies the first Wolfe condition but not necessarily the second Wolfe condition.
- ☐ α' satisfies the second Wolfe condition but not necessarily the first Wolfe condition.
- ☐ Nothing can be concluded about the Wolfe conditions at α' in the general case.