Introduction to optimization and operations research Exercises

Michel Bierlaire

September 29, 2024

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1 Modeling

Question 01

The diet problem was one of the first optimization problems studied in the 1930s and 1940s. The problem was motivated by the Army's desire to minimize the cost of feeding GIs in the field while providing then with healthy food.

Given a set of food types, along with the nutrient information and the cost per portion of each food, the objective of the diet problem is to select the number of portions of each food type to purchase so as to minimize the cost of the food while meeting the specified nutritional requirements. Typically, the nutritional requirements are expressed as a minimum and a maximum allowable level for each nutritional component. Other constraints such as minimum and/or maximum number of portions may be included to improve the quality of the menu.

Consider the following simple example. Suppose there are three types of food available: corn, milk and bread. In addition, there are restrictions on the number of calories (between 2000 and 2250) and on the amount of vitamin A (between 5000 and 50,000). The table below shows, for each type of food, the cost per portion, the amount of vitamin A per portion, and the number of calories per portion.

Food	Cost per portion	Vitamin A	Calories
Corn	1.8 CHF	107	72
Milk	2.3 CHF	500	121
Bread	0.5 CHF	0	65

Consider that the maximum number of portions for each type of food is 10. Formulate this problem as an optimization problem by giving

1. the decision variables,
2. the objective function and,
3. the constraints.

Solution 01

1. We aim to select the number of each food type to purchase. Then, the decision variables are :

   - x_1 number of portions of corn to purchase,
   - x_2 number of portions of milk to purchase, and,
   - x_3 number of portions of bread to purchase.

   The vector of decision variables is therefore

   $$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

2. Objective function : minimize the total cost of the food

   $$\min_{x \in \mathbb{R}^3} 1.8x_1 + 2.3x_2 + 0.5x_3.$$

3. Constraints :

   - The obtained food must meet the specified nutritional requirements.
     - Vitamin A :

       $$5000 \leq 107x_1 + 500x_2 \leq 50000.$$

     - Calories :

       $$2000 \leq 72x_1 + 121x_2 + 65x_3 \leq 2250.$$

   - Maximum number of portions:

     $$x_j \leq 10, \quad j = 1, 2, 3.$$

We obtain the following problem.

$$\min_{x \in \mathbb{R}^3} 1.8x_1 + 2.3x_2 + 0.5x_3,$$

subject to

$$\begin{aligned} 107x_1 + 500x_2 &\geq 5000, \\ 107x_1 + 500x_2 &\leq 50000, \\ 72x_1 + 121x_2 + 65x_3 &\geq 2000, \\ 72x_1 + 121x_2 + 65x_3 &\leq 2250, \\ x_1, x_2, x_3 &\leq 10, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The optimal solution is given by

$$x^* = \begin{pmatrix} 1.94 \\ 10 \\ 10 \end{pmatrix},$$

that is 1.94 portions of corn, 10 portions of milk and 10 portions of bread. The total cost is 31.5 CHF.

Question 02

An investment fund is trying to determine how to invest its assets for the following year, in order to maximize its profit. Currently, the fund has 2.5 million CHF that it can invest in state bonds, real estate loans, car loans or scholarship loans. The annual interest rates of the listed investment types are 4% for state bonds, 6% for real estate loans, 8% for car loans and 9% for scholarship loans.

To minimize risks, the investment fund allows only the selection of a strategy satisfying the following restrictions:

- the amount invested in car and scholarship loans must not exceed twice the amount invested in bonds;
- the amount invested in car loans must be larger or equal than the amount invested in scholarship loans;
- the investment in car loans should not exceed the investment in real estate loans by more than 70%.
- at most 50% of the total amount invested should be affected to the scholarship loans.

1. Formulate this problem as an optimization problem by determining
   (a) the decision variables,
   (b) the objective function, and
   (c) the constraints.
2. Write this problem as a minimization problem with only equality constraints and non-negativity constraints over the decision variables.

Solution 02

1. The aim of the investment fund is to determine the amounts that are invested in each investment type. Therefore, the **decision variables** are the amounts in CHF invested in

   - state bonds: x_{sb},
   - real estate loans: x_{re},
   - car loans: $x_{c\ell}$, and,
   - scholarship loans: $x_{s\ell}$.

   The vector of decision variables is therefore

   $$x = \begin{pmatrix} x_{sb} \\ x_{re} \\ x_{c\ell} \\ x_{s\ell} \end{pmatrix} \in \mathbb{R}^4.$$

   As the company wants to maximize its profit, the **objective function** is given by:

   $$\max_{x \in \mathbb{R}^4} 0.04x_{sb} + 0.06x_{re} + 0.08x_{c\ell} + 0.09x_{s\ell}.$$

   The **constraints** of the problem are as follows:

   - the total amount to invest is 2.5 M euro:

     $$x_{sb} + x_{re} + x_{c\ell} + x_{s\ell} = 2'500'000,$$

   - the total amount invested in car and scholarship loans $(x_{c\ell} + x_{s\ell})$ must not exceed twice the amount invested in bonds $(2x_{sb})$:

     $$x_{c\ell} + x_{s\ell} \leq 2x_{sb},$$

   - the amount invested in car loans $(x_{c\ell})$ must be larger or equal than the amount invested in scholarship loans $(x_{s\ell})$:

     $$x_{s\ell} \leq x_{c\ell},$$

- the investment in car loans ($x_{c\ell}$) should not exceed the investment in real estate loans (x_{re}) by more than 70%:

$$x_{c\ell} \leq 1.7 x_{re},$$

- at most 50% of the total amount invested should be affected to the scholarship loans ($x_{s\ell}$):

$$x_{s\ell} \leq 1250'000.$$

Finally, we must impose all the decision variables to be non negative:

$$x_{sb}, x_{re}, x_{c\ell}, x_{s\ell} \geq 0.$$

Putting everything together, we obtain the following optimization problem:

$$\max_{x \in \mathbb{R}^4} 0.04 x_{sb} + 0.06 x_{re} + 0.08 x_{c\ell} + 0.09 x_{s\ell}$$

subject to

$$\begin{aligned} x_{sb} + x_{re} + x_{c\ell} + x_{s\ell} &= 2'500'000, \\ x_{c\ell} + x_{s\ell} &\leq 2 x_{sb}, \\ x_{s\ell} &\leq x_{c\ell}, \\ x_{c\ell} &\leq 1.7 x_{re}, \\ x_{s\ell} &\leq 1'250'000 \\ x_{sb}, x_{re}, x_{c\ell}, x_{s\ell} &\geq 0. \end{aligned}$$

The optimal solution of this problem is given by

$$x^* = \begin{pmatrix} x^*_{sb} \\ x^*_{re} \\ x^*_{c\ell} \\ x^*_{s\ell} \end{pmatrix} = \begin{pmatrix} 696'721.3 \\ 409'836.1 \\ 696'721.3 \\ 696'721.3 \end{pmatrix}.$$

The total profit is 170'902.6 euros.

2. A maximization problem whose objective function is $f(x)$ is equivalent to a minimization problem whose objective function is $-f(x)$. Then:

$$\max_{x \in \mathbb{R}^4} 0.04x_{sb} + 0.06x_{re} + 0.08x_{c\ell} + 0.09x_{s\ell} =$$
$$-\min_{x \in \mathbb{R}^4} -0.04x_{sb} - 0.06x_{re} - 0.08x_{c\ell} - 0.09x_{s\ell}.$$

In order to obtain only equality constraints (with the exception of the non-negativity constraints), we introduce slack variables. For each constraint i, a slack variable y_i is introduced in the specification, and its value is explicitly restricted to be non negative.

$$x_{c\ell} + x_{s\ell} \leq 2x_{sb} \Leftrightarrow \begin{cases} x_{c\ell} + x_{s\ell} - 2x_{sb} + y_1 = 0, \\ y_1 \geq 0. \end{cases}$$

$$x_{s\ell} \leq x_{c\ell} \Leftrightarrow \begin{cases} x_{s\ell} - x_{c\ell} + y_2 = 0, \\ y_2 \geq 0. \end{cases}$$

$$x_{c\ell} \leq 1.7x_{re} \Leftrightarrow \begin{cases} x_{c\ell} - 1.7x_{re} + y_3 = 0, \\ y_3 \geq 0. \end{cases}$$

$$x_{s\ell} \leq 1250'000 \Leftrightarrow \begin{cases} x_{s\ell} - 1'250'000 + y_4 = 0, \\ y_4 \geq 0. \end{cases}$$

Note that we now have 8 decision variables: the four original variables (x_{sb}, x_{re}, $x_{c\ell}$, $x_{s\ell}$) and the four slack variables (y_1, y_2, y_3, y_4). We obtain the following problem:

$$-\min_{x \in \mathbb{R}^4, y \in \mathbb{R}^4} -0.04x_{sb} - 0.06x_{re} - 0.08x_{c\ell} - 0.09x_{s\ell}$$

subject to

$$\begin{aligned} x_{sb} + x_{re} + x_{c\ell} + x_{s\ell} &= 2'500'000, \\ x_{c\ell} + x_{s\ell} - 2x_{sb} + y_1 &= 0, \\ x_{s\ell} - x_{c\ell} + y_2 &= 0, \\ x_{c\ell} - 1.7x_{re} + y_3 &= 0, \\ x_{s\ell} - 1'250'000 + y_4 &= 0, \\ x_{sb}, x_{re}, x_{c\ell}, x_{s\ell}, y_1, y_2, y_3, y_4 &\geq 0. \end{aligned}$$

Question 03

In linear optimization, a problem in standard form is

- a minimization problem,
- where each constraint j is an equality constraint of the form

$$\sum_i a_{ij} x_i = b_j,$$

- all variables are non negative. It is written as:

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Equivalently,

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} c_i x_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} a_{ij} x_i &= b_j, & j &= 1, \ldots, m, \\ x_i &\geq 0, & i &= 1, \ldots, n. \end{aligned}$$

Write the following problem in standard form :

$$\max_{x \in \mathbb{R}^4} -2x_1 + 5x_2 - 7x_3 + 4x_4$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &\geq 7, \\ 8x_1 - 5x_3 + 3x_4 &\leq 8, \\ -2x_2 - 5x_3 + 4x_4 &\leq -6, \\ 1 \leq x_1 &\leq 4, \\ x_2 &\geq -3, \\ x_3 &\in \mathbb{R}, \\ x_4 &\leq 0. \end{aligned}$$

Solution 03

In order to transform the problem into a minimization problem, recall that $\max f(x)$ is equivalent to $-\min -f(x)$. In practice, it is common to consider the problem $\min -f(x)$ instead, and ignore the first "minus". Therefore, we can consider the problem with the objective function:

$$\min_{x \in \mathbb{R}^4} 2x_1 - 5x_2 + 7x_3 - 4x_4.$$

We now reformulate all the constraints of the problem such that we obtain $Ax = b, x \geq 0$.

Consider the first three constraints:

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &\geq 7, \\ 8x_1 - 5x_3 + 3x_4 &\leq 8, \\ -2x_2 - 5x_3 + 4x_4 &\leq -6. \end{aligned}$$

They can be transformed into equality constraints by introducing slack variables $e_1, e_2, e_3 \geq 0$ as follows:

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 - e_1 &= 7, \\ 8x_1 - 5x_3 + 3x_4 + e_2 &= 8, \\ -2x_2 - 5x_3 + 4x_4 + e_3 &= -6. \end{aligned}$$

Note that the slack variable e_1 is associated with a negative sign, because it is introduced in a $\geq$ constraint.

The next constraint

$$1 \leq x_1 \leq 4$$

is equivalent to the two following constraints

$$\begin{aligned} x_1 &\geq 1, \\ x_1 &\leq 4. \end{aligned}$$

Here we use a change of variable:

$$x_1' = x_1 - 1.$$

The constraint $x_1 \geq 1$ becomes

$$x_1' + 1 \geq 1 \iff x_1' \geq 0.$$

This is a non negativity constraint on x_1'. Therefore, we are left only with a lower inequality constraint, that is, $x_1 \leq 4$, which becomes

$$x_1' + 1 \leq 4 \iff x_1' \leq 3.$$

We now introduce the slack variable $e_4 \geq 0$ in order to transform this constraint into an equality constraint:

$$x_1' + e_4 = 3.$$

It remains to replace x_1 with $x_1' + 1$ in the objective function and all the constraints. For the last constraint, $x_2 \geq -3$, we also use a change of variable. We define a new variable $x_2' \geq 0$ as follows

$$x_2' = x_2 + 3.$$

A non-negativity constraint on x_2' ensures that $x_2 \geq -3$. We replace x_2 with $(x_2' - 3)$ in the objective function and all the constraints.

In standard form, all variables must be non negative. However, in the original problem, the variable x_3 can take any real value, positive or negative. Therefore, we introduce two new non negative variables $x_3^+ \geq 0$ and $x_3^- \geq 0$, representing respectively the positive and negative part of x_3. The change of variable is given by

$$x_3 = x_3^+ - x_3^-.$$

We replace x_3 with $(x_3^+ - x_3^-)$ in the objective function and all the constraints.

In the original problem, x_4 must be non positive: $x_4 \leq 0$. To comply with the specification of the standard form, we introduce a new variable $x_4' \geq 0$ defined as

$$x_4' = -x_4.$$

We replace x_4 with $-x_4'$ in the objective function and all the constraints.

The vector of decision variables is now

$$x = \begin{pmatrix} x_1' \\ x_2' \\ x_3^+ \\ x_3^- \\ x_4' \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \in \mathbb{R}^9,$$

all of them being non negative.

By putting everything together, the problem in standard form is:

$$\min_{x \in \mathbb{R}^9} 2(x_1' + 1) - 5(x_2' - 3) + 7(x_3^+ - x_3^-) - 4(-x_4')$$

subject to

$$\begin{aligned} 2(x_1' + 1) + 4(x_2' - 3) - 6(x_3^+ - x_3^-) - e_1 &= 7, \\ 8(x_1' + 1) - 5(x_3^+ - x_3^-) + 3(-x_4') + e_2 &= 8, \\ -2(x_2' - 3) - 5(x_3^+ - x_3^-) + 4(-x_4') + e_3 &= -6, \\ x_1' + e_4 &= 3, \\ x_1', x_2', x_3^+, x_3^-, x_4', e_1, e_2, e_3, e_4 &\geq 0, \end{aligned}$$

that is

$$\min_{x \in \mathbb{R}^9!} 2x_1' - 5x_2' + 7x_3^+ - 7x_3^- + 4x_4' + 17$$

$$\begin{aligned} 2x_1' + 4x_2' - 6x_3^+ + 6x_3^- - e_1 &= 17, \\ 8x_1' - 5x_3^+ + 5x_3^- - 3x_4' + e_2 &= 0, \\ -2x_2' - 5x_3^+ + 5x_3^- - 4x_4' + e_3 &= -12, \\ x_1' + e_4 &= 3, \\ x_1', x_2', x_3^+, x_3^-, x_4', e_1, e_2, e_3, e_4 &\geq 0. \end{aligned}$$

Note:

There is another method to handle the constraint $1 \leq x_1 \leq 4$. Indeed, we can introduce slack variables, $e_5, e_6 \geq 0$ as follows:

$$\begin{aligned} x_1 - e_5 &= 1, \\ x_1 + e_6 &= 4. \end{aligned}$$

Note that the non negativity constraint on x_1 is redundant with the constraint $x_1 - e_5 = 1$. Indeed, since $e_5 \geq 0$, $x_1 = e_5 + 1$ is necessarily non negative. However, in standard form all variables must be non negative. Even if it is redundant.

Question 04

We want to determine the rectangular parallelepiped of unit volume with minimal surface.

1. Formulate this problem as an optimization problem by giving
   (a) the decision variables,
   (b) the objective function and
   (c) the constraints.
2. Write this problem as a maximization problem with only $\leq$ inequality constraints.

Solution 04

1. The rectangular parallelepiped P is characterized by its length, width and height. These are the decision variables:

   - x_1: length of P,
   - x_2: width of P and
   - x_3: height of P.

   Therefore, the vector of decision variables is

   $$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

2. The objective is to minimize the surface of P:

   $$\min_{x \in \mathbb{R}^3} 2(x_1x_2 + x_1x_3 + x_2x_3).$$

3. Constraints :

   - The volume of P must be unitary.

     $$x_1x_2x_3 = 1.$$

   - As x_1, x_2 and x_3 represent physical lengths, they must be non negative:

     $$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_3 &\geq 0. \end{aligned}$$

   We obtain the following optimization problem :

   $$\min 2(x_1x_2 + x_1x_3 + x_2x_3)$$

subject to

$$\begin{aligned} x_1x_2x_3 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_3 &\geq 0. \end{aligned}$$

The optimal solution is given by

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3.$$

4. A minimization problem whose objective function is $f(x)$ is equivalent to a maximization problem whose objective function is $-f(x)$. Then:

$$\min_{x\in\mathbb{R}^3} 2(x_1x_2 + x_1x_3 + x_2x_3) = -\max_{x\in\mathbb{R}^3} -2(x_1x_2 + x_1x_3 + x_2x_3).$$

In order to obtain only $\leq$ inequality constraints, we perform the following operations:

- we multiply each upper inequality by -1 to get a lower inequality:

$$\begin{aligned} x_1 \geq 0 &\iff -x_1 \leq 0, \\ x_2 \geq 0 &\iff -x_2 \leq 0, \\ x_3 \geq 0 &\iff -x_3 \leq 0. \end{aligned}$$

- we replace the equality constraint by two inequality constraints:

$$x_1x_2x_3 = 1 \iff$$

$$\begin{cases} x_1x_2x_3 \leq 1, \\ x_1x_2x_3 \geq 1 \end{cases} \iff \begin{cases} x_1x_2x_3 \leq 1, \\ -x_1x_2x_3 \leq -1. \end{cases}$$

Then, we obtain the following optimization problem.

$$-\max_{x\in\mathbb{R}^3} -2(x_1x_2 + x_1x_3 + x_2x_3)$$

subject to

$$
\begin{aligned}
x_1 x_2 x_3 &\leq 1, \\
-x_1 x_2 x_3 &\leq -1, \\
-x_1 &\leq 0, \\
-x_2 &\leq 0, \\
-x_3 &\leq 0.
\end{aligned}
$$

Question 05

The company Coola-Coola Ltd. wants to design a can of soda of volume 0.33 liters. They need to set the dimensions (in centimeters) of this can to use the minimum amount of aluminium, knowing that the form of the can is a perfect cylinder, and the thickness of the aluminium is the same everywhere. Write the problem as an optimization problem.

Hint

Consider the three steps of modeling:

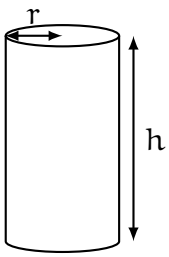
1. Identify the decision variables.
2. Write the objective function.
3. Write the constraints.

Solution 05

We consider the three steps of the modeling process.

Decision variables The design of the cylinder depends on two variables, both expressed in centimeters:

- the radius of the basis: r,
- the height of the cylinder: h.



Objective function Since the thickness of the aluminium is the same at any part of the can, the total surface of the cylinder has to be minimized. The objective function must provide the total surface as a function of the decision variables:

$$f : \mathbb{R}^2 \to \mathbb{R} : f(r, h).$$

- Each basis is a circle of radius r, so its surface is πr^2.
- The side of the can is a rectangle of area $2\pi rh$.

Therefore, the objective function to minimize is

$$f(r, h) = 2\pi r^2 + 2\pi rh.$$

Constraints The volume of the can must be 0.33 liters, that is 330 cm^3. The first constraint is therefore:

$$\pi r^2 h = 330.$$

We also need non negativity constraints:

$$r \geq 0,$$
$$h \geq 0.$$

The optimization problem is therefore:

$$\min_{r,h} 2\pi r^2 + 2\pi rh$$

subject to

$$\begin{aligned} \pi r^2 h &= 330, \\ r &\geq 0, \\ h &\geq 0. \end{aligned}$$

The optimal solution of this problem is $r = 3.746$ cm and $h = 7.491$ cm.

Question 06

Given the following optimization problem, transform it in such a way as to obtain a minimization problem in which all decision variables must be non negative and all constraints are defined by lower inequalities.

$$\max -x_1^2 + \sin x_2$$

subject to

$$\begin{aligned} 6x_1 - x_2^2 &\geq 1, \\ x_1^2 + x_2^2 &= 3, \\ x_1 &\geq 2, \\ x_2 &\in \mathbb{R}. \end{aligned}$$

Solution 06

We apply a sequence of simple transformations to the optimization problem.

1. A maximization problem whose objective function is $f(x)$ is equivalent to a minimization problem whose objective function is $-f(x)$:

   $$\operatorname{argmax}_x f(x) = \operatorname{argmin}_x -f(x)$$

   and

   $$\max f(x) = -\min -f(x).$$

   By applying this statement to our problem we obtain:

   $$-\min x_1^2 - \sin x_2$$

   subject to

   $$\begin{aligned} 6x_1 - x_2^2 &\geq 1, \\ x_1^2 + x_2^2 &= 3, \\ x_1 &\geq 2, \\ x_2 &\in \mathbb{R}. \end{aligned}$$

2. Now we transform the constraints. An equality constraint can be written as the combination of two inequalities.

   $$g(x) = 0 \iff \begin{cases} g(x) \leq 0 \\ g(x) \geq 0. \end{cases}$$

   The optimization problem becomes

   $$-\min x_1^2 - \sin x_2$$

   subject to

   $$\begin{aligned} 6x_1 - x_2^2 &\geq 1 \\ x_1^2 + x_2^2 &\leq 3 \\ x_1^2 + x_2^2 &\geq 3 \\ x_1 &\geq 2 \\ x_2 &\geq 0. \end{aligned}$$

3. A constraint defined by a lower inequality can be multiplied by -1 to get an upper inequality:

$$g(x) \leq 0 \Longleftrightarrow -g(x) \geq 0.$$

$$-\min x_1^2 - \sin x_2$$

subject to

$$\begin{aligned} -6x_1 + x_2^2 &\leq -1 \\ x_1^2 + x_2^2 &\leq 3 \\ -x_1^2 - x_2^2 &\leq -3 \\ x_1 &\geq 2 \\ x_2 &\in \mathbb{R}. \end{aligned}$$

4. If a variable x can take any real value, it can be replaced by two non negative artificial variables denoted by x^+ and x^-, such that

$$x = x^+ - x^-.$$

We can impose $x^+ \geq 0$ and $x^- \geq 0$, without loss of generality. We apply this to the variable x_2 in our formulation.

$$-\min x_1^2 - \sin(x_2^+ - x_2^-)$$

subject to

$$\begin{aligned} -6x_1 + (x_2^+ - x_2^-)^2 &\leq -1 \\ x_1^2 + (x_2^+ - x_2^-)^2 &\leq 3 \\ -x_1^2 - (x_2^+ - x_2^-)^2 &\leq -3 \\ x_1 &\geq 2 \\ x_2^+ &\geq 0 \\ x_2^- &\geq 0. \end{aligned}$$

5. In the presence of a constraint $x \geq a$, with $a \in \mathbb{R}$, a simple change of variable

$$x = \hat{x} + a$$

transforms the constraint in

$$\hat{x} \geq 0.$$

We apply this last principle to variable x_1, and we obtain:

$$-\min(\hat{x}_1 + 2)^2 - \sin(x_2^+ - x_2^-)$$

subject to

$$\begin{aligned}
-6(\hat{x}_1 + 2) + (x_2^+ - x_2^-)^2 &\leq -1 \\
(\hat{x}_1 + 2)^2 + (x_2^+ - x_2^-)^2 &\leq 3 \\
-(\hat{x}_1 + 2)^2 - (x_2^+ - x_2^-)^2 &\leq -3 \\
\hat{x}_1 &\geq 0 \\
x_2^+ &\geq 0 \\
x_2^- &\geq 0.
\end{aligned}$$

As requested, it is a minimization problem, all decision variables are non negative and all constraints are defined by lower inequalities. Any solution $(\hat{x}_1, x_2^+, x_2^-)$ of this problem corresponds to the following solution of the original problem:

$$\begin{aligned}
x_1 &= \hat{x}_1 + 2 \\
x_2 &= x_2^+ - x_2^-.
\end{aligned}$$

Question 07

The following optimization problem is not linear, because of the absolute value in the objective function. Transform it into a linear problem in which all decision variables must be non negative.

$$\min |x_1 - x_2|$$

subject to

$$x_1 \geq 0,$$
$$x_2 \geq 0.$$

Solution 07

To solve this exercise, we remember that if a variable x can take any real value, it can be replaced by two non negative artificial variables denoted by x^+ and x^-, such that $x = x^+ - x^-$. We also recall that the absolute value of x is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

In our case, we have that

$$|x_1 - x_2| = \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2, \\ x_2 - x_1 & \text{if } x_1 < x_2. \end{cases}$$

Since x_1 and x_2 are non negative real numbers, the difference $x_1 - x_2$ can take any real value.

Let us study the absolute value by examining the two cases:

- If $x_1 - x_2 \geq 0$, we can define the non negative quantity

$$y^+ = \begin{cases} x_1 - x_2, \text{ if } x_1 - x_2 > 0, \\ 0, \text{ otherwise.} \end{cases}$$

- If $x_1 - x_2 < 0$, we can define the positive quantity

$$y^- = \begin{cases} x_2 - x_1, \text{ if } x_1 - x_2 < 0, \\ 0, \text{ otherwise.} \end{cases}$$

Consequently, the absolute value of the difference can then be written as

$$|x_1 - x_2| = y^+ + y^-.$$

Furthermore, if we impose that $y^+ \geq 0$ and $y^- \geq 0$, we have that our minimization problem can be written as

$$\min_{x_1, x_2, y^+, y^-} (y^+ + y^-)$$

subject to

$$\begin{aligned} y^+ &\geq x_1 - x_2, \\ y^- &\geq x_2 - x_1, \\ x_1 &\geq 0, \\ x_2 &\geq 0, \\ y^+ &\geq 0, \\ y^- &\geq 0. \end{aligned}$$

Note that this formulation is not strictly equivalent to the original one for all feasible solutions. For instance, the feasible solution $x_1 = 0$, $x_2 = 0$, $y^+ = 1$, $y^- = 1$ has objective value 2 in the transformed problem, and 0 in the original one.

But it is equivalent at the optimal solution. Denote the optimal solution of the original problem (x_1^*, x_2^*).

- If $x_1^* - x_2^* > 0$, the lowest possible value for y^+, denoted by $(y^+)^*$, is $x_1^* - x_2^*$ (because of the constraint $y^+ \geq x_1 - x_2$), and the lowest possible value for y^-, denoted by $(y^-)^*$, is 0 (because of the constraint $y^- \geq 0$). Therefore, the objective function of the transformed problem at the optimal solution is

$$(y^+)^* + (y^-)^* = x_1^* - x_2^* + 0 = |x_1^* - x_2^*|.$$

- If $x_1^* - x_2^* < 0$, for similar reasons, we have $(y^+)^* = 0$ and $(y^-)^* = x_2^* - x_1^*$. Therefore, the objective function of the transformed problem at the optimal solution is

$$(y^+)^* + (y^-)^* = 0 + x_2^* - x_1^* = |x_1^* - x_2^*|.$$

- If $x_1^* - x_2^* = 0$, we have $(y^+)^* = 0$ and $(y^-)^* = 0$. Therefore, the objective function of the transformed problem at the optimal solution is

$$(y^+)^* + (y^-)^* = 0 + 0 = |x_1^* - x_2^*|.$$

Question 08

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = ax + b,$$

where a, $b \in \mathbb{R}$ are parameters.

For each of the following optimization problems, discuss the existence of an optimal solution as a function of the parameters a and b.

1.
$$\min_{x \in \mathbb{R}} ax + b.$$

2.
$$\min_{x \in \mathbb{R}} ax + b$$
subject to
$$x + 2 = 1.$$

3.
$$\min_{x \in \mathbb{R}} ax + b$$
subject to
$$x \leq 0.$$

4.
$$\min_{x \in \mathbb{R}} ax + b$$
subject to
$$x^2 \leq 1.$$

Solution 08

1.

$$\min_{x \in \mathbb{R}} ax + b.$$

If $a \neq 0$, the problem is not bounded from below, and the problem has no infimum, and no optimum. The problem is bounded from below if $a = 0$. In that case, the infimum is equal to b:

$$\text{If } a = 0, \inf_{x \in \mathbb{R}} f(x) = b.$$

For any $x \in \mathbb{R}$, we have $f(x) = b$. Therefore, any $x \in \mathbb{R}$ is a global optimum of the problem.

2.

$$\min_{x \in \mathbb{R}} ax + b$$

subject to

$$x + 2 = 1.$$

The only feasible point is $x = -1$, for any value of a and b. It is also the optimum.

3.

$$\min_{x \in \mathbb{R}} ax + b$$

subject to

$$x \leq 0.$$

- If $a > 0$, the problem is unbounded. Indeed, decreasing the value of x also decreases the value of the objective function.
- If $a = 0$, the function is constant, and any feasible x is optimum (see the discussion of the previous case).
- If $a < 0$, b is the best lower bound, that is the infimum. As $f(0) = b$, $x = 0$ is the optimum solution.

4.

$$\min_{x \in \mathbb{R}} ax + b$$

subject to

$$x^2 \leq 1.$$

The constraint $x^2 \leq 1$ is equivalent to impose that $-1 \leq x \leq 1$. In this case the feasible set is compact, and Weierstrass theorem guarantees the existence of a minimum.

- If $a = 0$, the function is constant, and any feasible x is optimum (see the discussion above).
- If $a > 0$, the infimum is $b - a$, and the optimum solution is $x = -1$.
- If $a < 0$, the infimum is $b + a$, and the optimum solution is $x = 1$.

Question 09

The main assistant of the course introduction to optimization and operations research has a busy schedule. She tends to be very stressed. Conscious of the risks of burnout, she wants to organize her time in order to minimize her stress. For each of the activities she undertakes during the week, she has managed to calculate a stress index. Knowing that she wants to establish her schedule on the 5 days of the week only and that she always sleeps 8 hours a day, here is the list of her main activities and the associated stress index:

Work Although her work contract indicates 41 hours of work per week, she often works overtime. When that happens, her stress index goes up by 5 points per hour. On the other hand, each time she does one hour of work less, her stress index decreases by 2 points. However, she can never do less than 38 hours a week.

Sport She needs to play sports and she usually does 6 hours of sport per week. This is the maximum she can do in order to not get injured. On the other hand, if she does less than 6 hours, her stress index increases by 3 points per hour of inactivity. She must imperatively do at least 2 hours of sport per week.

Obligations (eating, washing, shopping, ...) It takes her at least 15 hours a week. For each additional hour, her stress index increases by 1 point.

Leisure (movies, parties, ...) The remaining free time that she has allows her to relax and her stress index drops by 5 points per hour of free time. Also, she decided that she should have at least as many free time hours as half of her hours of work.

We formulate this problem as a linear optimization problem with the following decision variables:

- x^+_{work}: additional hours of work,
- x^-_{work}: missing hours of work,
- x^-_{sport}: missing hours of sport,

- $x^{+}_{obligations}$: additional hours of obligations,
- $x_{leisure}$: hours of leisure.

Identify the objective function, and the constraints, and write the complete linear optimization problem.

Solution 09

Objective function: The objective is to minimize the stress of the assistant. The objective function is then:

$$\min f(x) = 5x^{+}_{work} - 2x^{-}_{work} + 3x^{-}_{sport} + x^{+}_{obligations} - x_{leisure}$$

Constraints: The number of hours allocated to the different activities must be equals to the number of hours at disposal. The time at disposal is given by :

$$\text{Time at disposal} = 5 \times 24 - 5 \times 8 = 80\text{hours}$$

The constraint is therefore given by

$$41 + x^{+}_{work} - x^{-}_{work} + 6 - x^{-}_{sport} + 15 + x^{+}_{obligations} + x_{leisure} = 80$$

The assistant cannot work less than 38 hours per week :

$$41 + x^{+}_{work} - x^{-}_{work} \geq 38$$

She needs to do at least 2 hours of sport per week:

$$6 - x^{-}_{sport} \geq 2$$

The hours for leisure have to be at least as large as half of the hours for work :

$$2x_{leisure} \geq 41 + x^{+}_{work} - x^{-}_{work}$$

The non-negativity constraints:

- $x^{+}_{work} \geq 0$,
- $x^{-}_{work} \geq 0$,
- $x^{-}_{sport} \geq 0$,

- $x^+_{\text{obligations}} \geq 0$,
- $x_{\text{leisure}} \geq 0$.

We obtain the following optimization problem:

$$\min f(x) = 5x^+_{\text{work}} - 2x^-_{\text{work}} + 3x^-_{\text{sport}} + x^+_{\text{obligations}} - x_{\text{leisure}}$$

subject to

$$\begin{aligned}
41 + x^+_{\text{work}} - x^-_{\text{work}} + 6 - x^-_{\text{sport}} + 15 + x^+_{\text{obligations}} + x_{\text{leisure}} &= 80,\\
41 + x^+_{\text{work}} - x^-_{\text{work}} &\geq 38,\\
6 - x^-_{\text{sport}} &\geq 2,\\
2x_{\text{leisure}} - 41 + x^+_{\text{work}} - x^-_{\text{work}} &\geq 0,\\
x^+_{\text{work}} &\geq 0,\\
x^-_{\text{work}} &\geq 0,\\
x^-_{\text{sport}} &\geq 0,\\
x^+_{\text{obligations}} &\geq 0,\\
x_{\text{leisure}} &\geq 0.
\end{aligned}$$

Question 10

Consider the constrained optimization problem:

$$\min_{x \in \mathbb{R}} x^2,$$

subject to

$$1 \leq x \leq 4.$$

Show that $x^* = 1$ verifies the first-order Karush-Kuhn-Tucker optimality conditions.
Hint: write first the problem in the general form used in the lecture:

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to

$$h(x) = 0, \quad h : \mathbb{R}^n \to \mathbb{R}^m, m \geq 0,$$
$$g(x) \leq 0, \quad g : \mathbb{R}^n \to \mathbb{R}^p, p \geq 0.$$

Solution 10

We first write the optimization problem in the general form. As there is only one variable, $n = 1$. As there is no equality constraint, $m = 0$. As there are two inequality constraints, $p = 2$.

$$\min_{x \in \mathbb{R}} x^2$$

subject to

$$1 - x \leq 0,$$
$$x - 4 \leq 0.$$

We associate a Lagrange multiplier with each constraint, and write the Lagrangian:

$$L(x, \mu_1, \mu_2) = x^2 + \mu_1(1 - x) + \mu_2(x - 4).$$

We first verify the conditions $g_i(x^*)\mu_i^* = 0$, where $x^* = 1$. For $i = 1$, we have

$$(1 - x^*)\mu_1^* = (1 - 1)\mu_1^* = 0,$$

irrespectively of the value of μ_1^*. Indeed, the first constraint is *active* at x^*. For $i = 2$, we have

$$(x^* - 4)\mu_2^* = -3\mu_1^* = 0,$$

that establishes that $\mu_2^* = 0$. Indeed, the second constraint in *not active* at x^*. The derivative of the Lagrangian must be equal to zero.

$$\frac{\partial L}{\partial x}(x^*, \mu_1^*, \mu_2^*) = 2x^* - \mu_1^* + \mu_2^* = 2 - \mu_1^* = 0.$$

Therefore, $\mu_1^* = 2$, which is non negative, so that the first-order necessary conditions are verified.

2 Linear optimization: from geometry to algebra

Question 01

Consider the following polyhedron represented in canonical form:

$$\mathcal{P} = \{x \in \mathbb{R}^2 \mid Ax \leq b\},$$

with

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

1. Write explicitly the inequality constraints.
2. Draw the polyhedron. Hint: each constraint defines a half-plane, and the intersection of the half-planes gives the polyhedron. Draw first the lines defined by the equality version of the constraints, and identify on which side each half-place lies.
3. Write the polyhedron in standard form.

Solution 01

1. The inequality constraints are

$$\begin{array}{rcl} -x_1 + x_2 & \leq & 2, \\ x_1 + x_2 & \leq & 4, \\ -x_1 & \leq & 0, \\ -x_2 & \leq & 0. \end{array}$$

2. Each of the four constraints defines a half-plane, and the intersection of the half-planes gives the polyhedron in $\mathbb{R}^2$. We draw first the lines defined by the equality version of the constraints:

$$\begin{array}{rcl} -x_1 + x_2 & = & 2, \\ x_1 + x_2 & = & 4, \\ -x_1 & = & 0, \\ -x_2 & = & 0. \end{array}$$

   In order to identify on which side of the line is the half-plane, we use the gradient with the opposite sign. Indeed, the gradient of a function indicates the direction in which the function increases. Here, as the function must be less or equal to a constant, we need the direction in which the function decreases.

   For instance, for the first constraint, the gradient is the first row of the matrix A. Written as a column vector, it is:

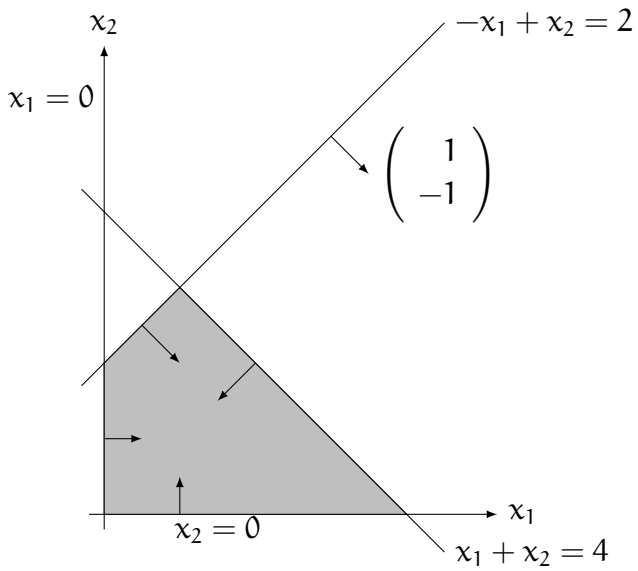
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   Therefore, the direction that shows which side of the line is feasible is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   The other directions are obtained in the same way:

$$d_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; d_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \; d_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; d_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



3. In order to write the standard form of the polyhedron, we first rewrite the inequality as follows:

$$\begin{aligned} -x_1 \quad +x_2 \quad &\leq \quad 2, \\ x_1 \quad +x_2 \quad &\leq \quad 4, \\ x_1 \quad &\geq \quad 0, \\ x_2 \quad &\geq \quad 0. \end{aligned}$$

   As required by the standard form, x_1 and x_2 must be non negative, so that nothing has to be done for that.

   Now, we need to include the slack variables. As there are 2 inequality constraints (different from the non-negativity constraints), we need 2 slack variables:

$$\mathcal{P} = \{x \in \mathbb{R}^2, x^s \in \mathbb{R}^2 \mid Ax + x^s = b, x, x^s \geq 0\}. \tag{1}$$

   We now have 4 variables. The set of constraints characterizing the polyhedron is:

$$\begin{aligned} -x_1 \quad +x_2 \quad +x_3 \quad &= 2, \\ x_1 \quad +x_2 \quad +x_4 \quad &= 4, \\ x_1 \quad &\geq 0, \\ x_2 \quad &\geq 0, \\ x_3 \quad &\geq 0, \\ x_4 \quad &\geq 0. \end{aligned}$$

Question 02

Consider the following polyhedron represented in standard form:

$$\mathcal{P} = \{x \in \mathbb{R}^4 \mid Ax = b, x \geq 0\},$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Write the polyhedron in canonical form.

Solution 02

The canonical form only includes "lesser or equal" inequalities. Therefore, we can transform each equality constraint into two inequality constraints.

The first constraint is

$$x_1 + x_2 + x_3 = 1. \tag{2}$$

It can be written with two inequality constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 1, \\ x_1 + x_2 + x_3 &\geq 1, \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 1, \\ -x_1 - x_2 - x_3 &\leq -1. \end{aligned}$$

The second constraint is

$$x_1 - x_2 + x_4 = 1. \tag{3}$$

It can be written with two inequality constraints:

$$\begin{aligned} x_1 - x_2 + x_4 &\leq 1, \\ x_1 - x_2 + x_4 &\geq 1, \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_1 - x_2 + x_4 &\leq 1, \\ -x_1 + x_2 - x_4 &\leq -1. \end{aligned}$$

Finally, the fact that x_i is non negative can be written as $-x_i \leq 0$, for $i = 1, \ldots, 4$.

Putting everything together, we obtain

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 1, \\ -x_1 - x_2 - x_3 &\leq -1, \\ x_1 - x_2 + x_4 &\leq 1, \\ -x_1 + x_2 - x_4 &\leq -1, \\ -x_1 &\leq 0, \\ -x_2 &\leq 0, \\ -x_3 &\leq 0, \\ -x_4 &\leq 0. \end{aligned}$$

In matrix form, we have

$$\mathcal{P} = \{x \in \mathbb{R}^4 \mid Ax \leq b\},$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that there are other ways to write the polyhedron in canonical form. In particular, if we observe that x_3 and x_4 in the standard form are slack variables, we can eliminate them in the canonical form. Indeed, we can combine (2) with the non negativity constraint of x_3 to obtain:

$$x_3 = 1 - x_1 - x_2 \geq 0,$$

or, equivalently

$$x_1 + x_2 \leq 1.$$

And x_3 disappears from the formulation. Similarly, (3) can be written

$$x_4 = 1 - x_1 + x_2 \geq 0,$$

or, equivalently

$$x_1 - x_2 \leq 1.$$

Putting everything together, we obtain:

$$\begin{aligned} x_1 + x_2 &\leq 1, \\ x_1 - x_2 &\leq 1, \\ -x_1 &\leq 0, \\ -x_2 &\leq 0. \end{aligned}$$

In matrix form, it is:

$$\mathcal{P} = \{x \in \mathbb{R}^2 \mid Ax \leq b\}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

This is now a polyhedron in $\mathbb{R}^2$.

Question 03

Consider the following linear optimization problem:

$$\min_{x \in \mathbb{R}^2} 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 3, \\ x_1 + 2x_2 &\leq 6, \\ x_1 &\geq 1, \\ x_2 &\geq 1. \end{aligned}$$

1. Does the solution $x_1^* = 1$, $x_2^* = 2$ verify all the constraints of the problem? Which constraints are active at (x_1^*, x_2^*)?
2. Answer the same questions with $x_1^* = 2$, $x_2^* = 1.5$.
3. Answer the same questions with $x_1^* = 1$, $x_2^* = 1$.

Solution 03

1. Yes, $x_1^* = 1$, $x_2^* = 2$ is a feasible solution. The verification is done by substitution:

$$\begin{aligned} 1 + 2 = 3 \geq 3, & \quad \checkmark \\ 1 + 2 \cdot 2 = 5 \leq 6, & \quad \checkmark \\ 1 \geq 1, & \quad \checkmark \\ 2 \geq 1. & \quad \checkmark \end{aligned}$$

   The first and the third constraints are active at (x_1^*, x_2^*). They are the two constraints that are verified with equality. The point lies at the intersection of the two lines that have $x_1 + x_2 = 3$ and $x_1 = 1$ as equations and is labeled A in the figure below.

2. Yes, $x_1^* = 2$, $x_2^* = 1.5$ is a feasible solution. The verification is done by substitution:

$$\begin{aligned} 2 + 1.5 = 3.5 \geq 3 & \quad \checkmark \\ 2 + 2 \cdot 1.5 = 5 \leq 6 & \quad \checkmark \\ 2 \geq 1 & \quad \checkmark \\ 1.5 \geq 1. & \quad \checkmark \end{aligned}$$

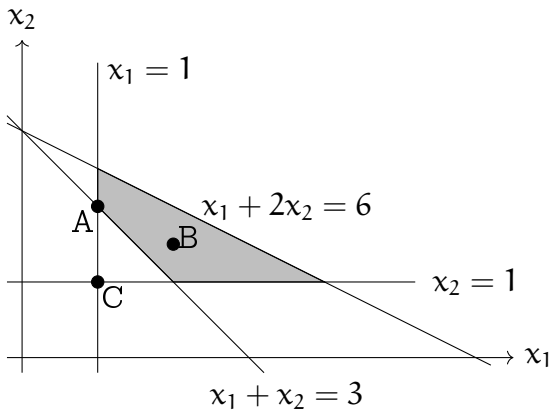
   No constraint is active. Therefore, (x_1^*, x_2^*) is an interior point. It is labeled B in the figure below.

3. The point $x_1^* = 1$, $x_2^* = 1$ is infeasible, as it violates the first constraint.

$$\begin{aligned} 1 + 1 = 2 \not\geq 3 & \quad \text{✗} \\ 1 + 2 \cdot 1 = 3 \leq 6 & \quad \checkmark \\ 1 \geq 1 & \quad \checkmark \\ 1 \geq 1. & \quad \checkmark \end{aligned}$$

   The point is labeled C in the figure below.

The graph below shows the feasible region for the problem.


x_2
$x_1 = 1$
$x_1 + 2x_2 = 6$
A
B
C
$x_2 = 1$
x_1
$x_1 + x_2 = 3$

Question 04

Consider the following linear optimization problem:

$$\min_{x \in \mathbb{R}} 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 3, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

1. Does the solution $x_1^* = 0$, $x_2^* = 3$ verify all the constraints of the problem? Which constraints are active at (x_1^*, x_2^*)?
2. Answer the same questions with $x_1^* = 2$, $x_2^* = 2$.

Solution 04

1. Yes, $x_1^* = 0$, $x_2^* = 3$ is a feasible solution. The verification is done by substitution:

$$\begin{aligned} 0 + 3 = 3 \quad & \checkmark \\ 0 \geq 0 \quad & \checkmark \\ 3 \geq 0. \quad & \checkmark \end{aligned}$$

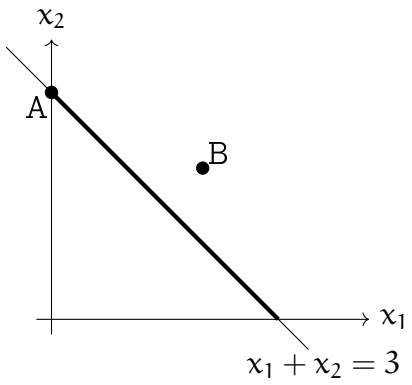
   The first two constraints are active at $\{x_1^*, x_2^*\}$. In particular, notice that equality constraints are always active when verified. The point is labeled A in the figure below.

2. No, $x_1^* = 2$, $x_2^* = 2$ is not a feasible solution, since the equality constraint is not verified:

$$\begin{aligned} 2 + 2 = 3 \quad & \text{✗} \\ 2 \geq 0 \quad & \checkmark \\ 2 \geq 0. \quad & \checkmark \end{aligned}$$

   The point is labeled B in the figure below.

The feasible region for the problem is the segment represented in bold in the following graph.



Question 05

For each of the following systems, find out how many solutions they have. Check also if there are redundant equations that can be removed.

Hint In order to identify the rank of the matrix, you may want to transform it into a row echelon form using elementary row operations. A matrix is in row echelon form when it satisfies the following conditions:

- each leading entry is in a column to the right of the leading entry in the previous row,
- rows with all zero elements, if any, are below rows having a non-zero element.

Some definitions also impose that the first non-zero element in each row is 1. However, this conditions is not needed to obtain the rank of the matrix.

Question 1

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 4x_1 + 5x_2 + 6x_3 &= 0, \\ 7x_1 + 8x_2 &= 0. \end{aligned}$$

Question 2

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 4x_1 + 5x_2 + 6x_3 &= 0, \\ 7x_1 + 8x_2 + 9x_3 &= 0. \end{aligned}$$

Question 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 2x_1 + 4x_2 + 6x_3 &= 0, \\ 5x_1 + 10x_2 + 15x_3 &= 0. \end{aligned}$$

Question 4

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 - x_2 + x_4 &= 1, \\ x_1 - 5x_2 - 2x_3 + 3x_4 &= 1. \end{aligned}$$

Question 5

$$\begin{aligned} x_1 + x_2 &= 0, \\ x_1 + 2x_2 + 2x_3 &= 0, \\ 3x_2 + 3x_3 &= 0. \end{aligned}$$

Solution 05

Question 1

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 4x_1 + 5x_2 + 6x_3 &= 0, \\ 7x_1 + 8x_2 &= 0. \end{aligned}$$

We have 3 equations and 3 variables. In matrix form, we have

$$Ax = b, \text{ with } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We transform the matrix into a row echelon form to identify its rank

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -21 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 9 \end{pmatrix}$$

where the following elementary row operations have been applied:

(1) $r_3 = r_3 - 7r_1$ and $r_2 = r_2 - 4r_1$.

(2) $r_3 = r_3 - 2r_2$.

Note that to obtain a true row echelon form, we would also need to normalize each row to obtain a one as the first leading element. However, it is not necessary in order to obtain the rank of the matrix, which is the number of non zero rows in this representation. As each row contains non zero elements, the rank is 3 here.

The system is compatible, as the vector

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the system. As the matrix is of full rank, there is no redundant equation, and x is the unique solution.

Question 2

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 4x_1 + 5x_2 + 6x_3 &= 0, \\ 7x_1 + 8x_2 + 9x_3 &= 0. \end{aligned}$$

We have 3 equations and 3 variables. In matrix form, we have

$$Ax = b, \text{ with } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We transform the matrix into a row echelon form to identify its rank

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

where the following row operations have been applied:

(1) $r_3 = r_3 - 7r_1$ and $r_2 = r_2 - 4r_1$.

(2) $r_3 = r_3 - 2r_2$.

As only two rows contain non zero elements, the rank of the matrix is 2. The system is compatible, as the vector

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the system. Therefore, one of the constraints is redundant, and there is an infinite number of solutions.

Question 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 2x_1 + 4x_2 + 6x_3 &= 0, \\ 5x_1 + 10x_2 + 15x_3 &= 0. \end{aligned}$$

We have 3 equations and 3 variables. In matrix form, we have

$$Ax = b, \text{ with } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We transform the matrix into a row echelon form to identify its rank

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the following row operations have been applied:

(1) $r_3 = r_3 - 5r_1$ and $r_2 = r_2 - 2r_1$.

As only one row contains non zero elements, the rank of the matrix is 1. The system is compatible, as the vector

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the system. Therefore, two of the constraints are redundant, and there is an infinite number of solutions.

Question 4

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 - x_2 + x_4 &= 1, \\ x_1 - 5x_2 - 2x_3 + 3x_4 &= 1. \end{aligned}$$

We have 3 equations and 4 variables. In matrix form, we have

$$Ax = b, \text{ with } A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -5 & -2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We transform the matrix into a row echelon form to identify its rank

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -5 & -2 & 3 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & -6 & -3 & 3 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the following row operations have been applied:

(1) $r_3 = r_3 - r_1$ and $r_2 = r_2 - r_1$.

(2) $r_3 = r_3 - 3r_2$.

As only two rows contain non zero elements, the rank of the matrix is 2. The system is compatible, as the vector

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the system. Therefore, one of the constraints is redundant, and there is an infinite number of solutions.

Question 5

$$\begin{aligned} x_1 + x_2 &= 0, \\ x_1 + 2x_2 + 2x_3 &= 0, \\ 3x_2 + 3x_3 &= 0. \end{aligned}$$

We have 3 equations and 3 variables. In matrix form, we have

$$Ax = b, \text{ with } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We transform the matrix into a row echelon form to identify its rank

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

where the following row operations have been applied:

(1) $r_2 = r_2 - r_1$.

(2) $r_3 = r_3 - 3r_2$.

As each row contains non zero elements, the rank of the matrix is 3. The system is compatible, as the vector

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the system. Therefore, no constraint is redundant, and x is a unique solution.

Question 06

Consider the polyhedron defined as follows:

$$\mathcal{P} = \{x \in \mathbb{R}^2 \mid -x_1 + x_2 \leq 2, x_1 + x_2 \leq 4, x_1 \geq 0, x_2 \geq 0\}.$$

Determine whether the given directions $\tilde{d} \in \mathbb{R}^2$ are feasible for the given feasible points $\tilde{x}$ of $\mathcal{P}$:

1. $\tilde{x}_a = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\tilde{d}_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
2. $\tilde{x}_b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\tilde{d}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$;
3. $\tilde{x}_c = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\tilde{d}_c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$;
4. $\tilde{x}_d = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\tilde{d}_d = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Hints:

1. If the polyhedron is written in standard form, a direction d is feasible if the following conditions are verified:
   - $Ad = 0$, and
   - $d_i \geq 0$ when $x_i = 0$.
2. To obtain a vector x in standard form, you must calculate the values of the slack variables.
3. To obtain a *direction* d in standard form, you need to consider $y = x + d$. If x^s and y^s are the corresponding vectors in standard form (including the slack variables), the direction in standard form is

$$d^s = y^s - x^s.$$

Solution 06

First, the polyhedron needs to be written in standard form. In order to do so, we need to transform the constraints into equality constraints by introducing the slack variables x_3 and x_4 for the first and second equation, respectively. The constraints are then written as follows:

$$\begin{aligned} -x_1 + x_2 + x_3 &= 2, \\ x_1 + x_2 + x_4 &= 4, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Given a feasible point of the polyhedron, it is possible to analyze if a direction is feasible by checking the following two conditions:

- $Ad = 0$, and
- $d_i \geq 0$ when $x_i = 0$,

where

$$A \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

Let's verify if these conditions are satisfied for the four cases.

1. $\tilde{x}_a = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\tilde{d}_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   We first need to compute the values of the slack variables x_3 and x_4 and we obtain $x_3 = 0$ and $x_4 = 2$. Therefore, we have

   $$x_a = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix},$$

   which is indeed a feasible point.

We do the same for the vector $\tilde{x}_a + \tilde{d}_a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. In standard form, it is $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Therefore, the direction in standard form is

$$d_a = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

In order to satisfy the first condition, we must have

$$Ad_a = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is verified. For the second condition, we look at the entries 1 and 3 of d_a, because $(x_a)_1 = (x_a)_3 = 0$ and the only zero entries. As $(d_a)_1 = 1$ and $(d_a)_3 = 1$, both entries of the direction are non negative and we conclude that the direction is feasible at the given point.

2. $\tilde{x}_b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\tilde{d}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.
In this case, it is immediate to see that the direction is not feasible, as the second condition is not verified for the first entry: $(\tilde{d}_b)_1 = -1 < 0$ and $(\tilde{x}_b)_1 = 0$.

3. $\tilde{x}_c = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\tilde{d}_c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
We compute the values of the slack variables: $x_3 = 2$, $x_4 = 0$, so that

$$x_c = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

We do the same for the vector $\tilde{x}_c + \tilde{d}_c = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. In standard form, it is $\begin{pmatrix} 2 \\ 4 \\ 0 \\ -2 \end{pmatrix}$. Therefore, the direction in standard form is

$$d_c = \begin{pmatrix} 2 \\ 4 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix}.$$

For the first condition to be satisfied, we must have

$$Ad = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is verified. For the second condition, we need to look at the entry 4 of d_c, as $(x_c)_4 = 0$ is the only zero entry of x_c. As $(d_c)_4 = -2 < 0$, the second condition is not satisfied, and the direction is not feasible at the given point.

4. $\tilde{x}_d = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\tilde{d}_d = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

We compute the values of the slack variables: $x_3 = 5$, $x_4 = 1$, so that

$$x_d = \begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

We do the same for the vector $\tilde{x}_d + \tilde{d}_d = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. In standard form,

it is $\begin{pmatrix} 3 \\ 2 \\ 3 \\ -1 \end{pmatrix}$. Therefore, the direction in standard form is

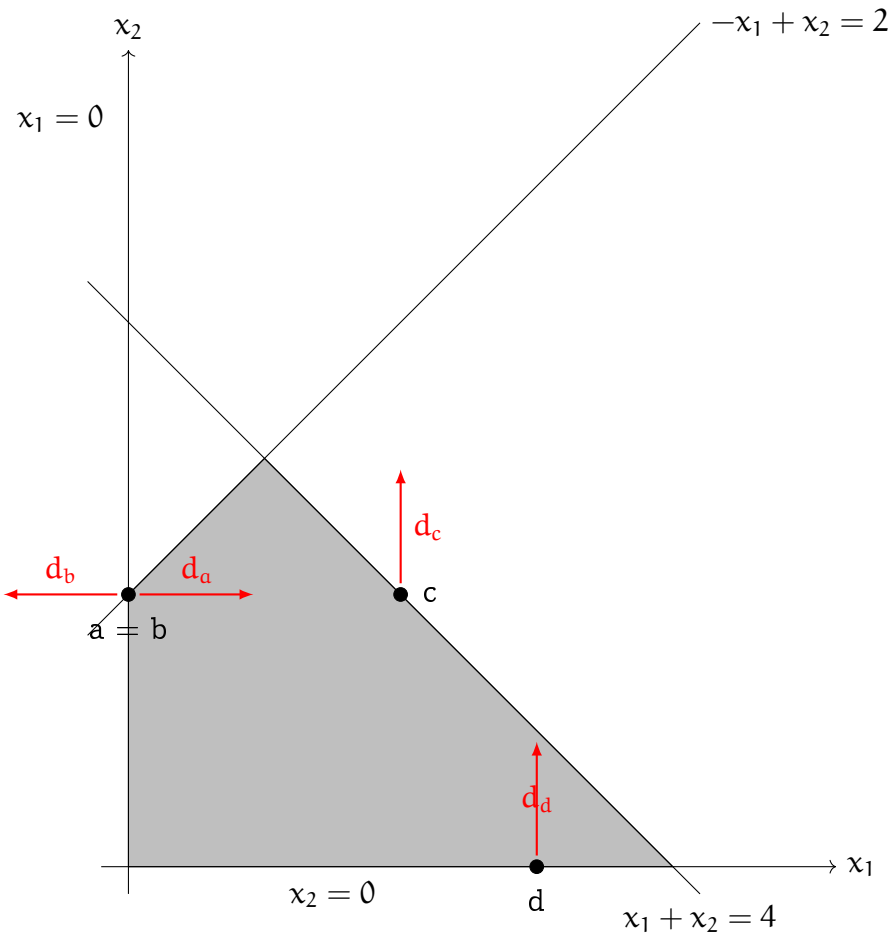
$$d_d = \begin{pmatrix} 3 \\ 2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix}.$$

For the first condition to be satisfied, we must have

$$Ad_d = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is verified. For the second condition, we need to look at the entry 2 of d_d, as $x_2 = 0$ is the only zero entry of x. As $d_2 = 2 \geq 0$, the second condition is satisfied as well, and the direction is feasible at the given point.

The following figure presents the four studied cases.


x_2
$-x_1 + x_2 = 2$
$x_1 = 0$
d_c
d_b
d_a
c
a = b
d_d
x_1
$x_2 = 0$
d
$x_1 + x_2 = 4$

Question 07

Let $X \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ an interior point of X. It means that there exists $\varepsilon > 0$ such that

$$\|z - x\| \leq \varepsilon \Longrightarrow z \in X.$$

Show that any direction $d \in \mathbb{R}^n$ is feasible in x.

Solution 07

x is an interior point of X, that is, there exists $\varepsilon > 0$ such that

$$\|z - x\| \leq \varepsilon \Longrightarrow z \in X.$$

It means that the (convex) ball of radius ε and centered at x is completely in X.

Consider an arbitrary direction $d \neq 0$, and the point

$$z = x + \varepsilon \frac{d}{\| d \|}.$$

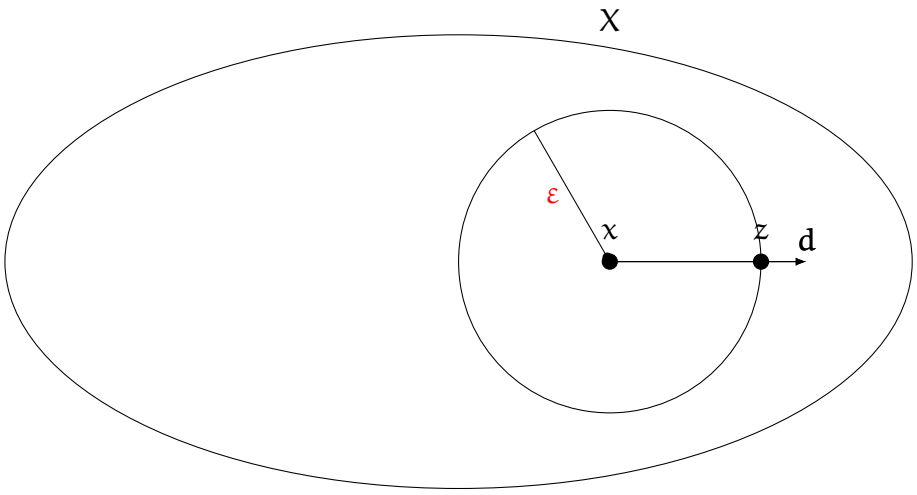
The point z is at the intersection of the direction d and the border of the ball. As

$$\|z - x\| = \left\| \varepsilon \frac{d}{\| d \|} \right\| = \varepsilon,$$

the definition of the interior point guarantees that $z \in X$. By convexity of the ball, all points $x + \alpha d$ with

$$0 \leq \alpha \leq \frac{\varepsilon}{\|d\|}$$

also belong to the ball, and therefore, to X, proving that the direction d is feasible.



Question 08

Consider the following optimization problem:

$$\min f(x_1, x_2, x_3, x_4) = x_1^2 + \sin(x_3 - x_2) + x_4 + 1$$

subject to

$$x_1 + x_2 + x_3 = 1,$$
$$x_1 - x_2 + x_4 = 1.$$

Use two different ways to eliminate the constraints. For each case, identify the basic and non basic variables.

Solution 08

The optimization problem is

$$\min f(x_1, x_2, x_3, x_4) = x_1^2 + \sin(x_3 - x_2) + x_4 + 1$$

subject to

$$x_1 + x_2 + x_3 = 1,$$
$$x_1 - x_2 + x_4 = 1.$$

As there are two constraints, two variables have to be selected to be in the basis. The number of possibilities to select 2 variables out of 4 is 6. We consider each of them below.

1. One possible way to rewrite the constraints is to express x_3 and x_4 as a function of x_1 and x_2:

   $$x_3 = 1 - x_1 - x_2,$$
   $$x_4 = 1 - x_1 + x_2.$$

   The optimization problem can be rewritten so as to depend only on two variables x_1 and x_2:

   $$\min_{x \in \mathbb{R}^2} f(x_1, x_2) = x_1^2 + \sin(-x_1 - 2x_2 + 1) - x_1 + x_2 + 2,$$

   without constraint. Note the presence of the linear terms in x_1 and x_2. This function is unbounded.

   The basic variables are x_3 and x_4 and the non basic variables are x_1 and x_2.

2. A second option is to express x_1 and x_2 as a function of x_3 and x_4:

   $$x_1 = 1 - \frac{x_3}{2} - \frac{x_4}{2},$$
   $$x_2 = -\frac{x_3}{2} + \frac{x_4}{2}.$$

   The optimization problem can be rewritten so as to depend only on two variables x_1 and x_2:

   $$\min f(x_3, x_4) = (1 - \frac{x_3}{2} - \frac{x_4}{2})^2 + \sin(\frac{3x_3}{2} - \frac{x_4}{2}) + x_4 + 1,$$

   without constraint. This function, again, is unbounded.

   The basic variables are x_1 and x_2 and the non basic variables are x_3 and x_4.

3. We can express x_1 and x_3 as a function of x_2 and x_4:

$$x_1 = 1 + x_2 - x_4,$$
$$x_3 = -2x_2 + x_4.$$

$$\min f(x_2, x_4) = (1 + x_2 - x_4)^2 + \sin(-3x_2 + x_4) + x_4 + 1.$$

We obtain again an unbounded function.

The basic variables are x_1 and x_3, and the non basic variables are x_2 and x_4.

4. We can express x_1 and x_4 as a function of x_2 and x_3:

$$x_1 = 1 - x_2 - x_3,$$
$$x_4 = 2x_2 + x_3.$$

We obtain

$$\min f(x_2, x_3) = (1 - x_2 - x_3)^2 + \sin(x_3 - x_2) + 2x_2 + x_3 + 1,$$

and the objective function is again unbounded.

The basic variables are x_1 and x_4, and the non basic variables are x_2 and x_3.

5. We can express x_2 and x_3 as a function of x_1 and x_4:

$$x_2 = -1 + x_1 + x_4,$$
$$x_3 = 2 - 2x_1 - x_4.$$

We obtain

$$\min f(x_1, x_4) = x_1^2 + \sin(3 - 3x_1 - 2x_4) + x_4 + 1$$

and the objective function is again unbounded.

The basic variables are x_2 and x_3, and the non basic variables are x_1 and x_4.

6. We can express x_2 and x_4 as a function of x_1 and x_3:

$$x_2 = 1 - x_1 - x_3,$$
$$x_4 = 2 - 2x_1 - x_3.$$

We obtain

$$\min f(x_1, x_3) = x_1^2 + \sin(2x_3 - 1 + x_1) - 2x_1 - x_3 + 3,$$

and the objective function is again unbounded.

The basic variables are x_2 and x_4, and the non basic variables are x_1 and x_3.

Question 09

Consider the following system of linear constraints:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1, \\ x_1 - x_3 &= -1, \\ 2x_1 + x_2 + x_4 &= 0. \end{aligned}$$

Propose a procedure to eliminate as many variables as possible, except x_1. Identify the basic and non basic variables, and write explicitly the equation

$$Ax = AP(P^T x) = Bx_B + Nx_N = b,$$

including the matrix P.

Solution 09

The constraints can be written as $Ax = b$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The system is compatible as

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is feasible.

In principle, in the presence of three constraints, we should select 3 variables out of the 4 to be basic variables. But for each of the 4 possibilities, the corresponding basic matrix is not invertible, precluding from eliminating the constraints. The reason is that the constraint matrix is not of full rank. The third constraint happens to be the sum of the two other constraints. It is therefore redundant and must be removed. We obtain the system with the following data:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

As we have two constraints, we can eliminate two variables. There are 6 ways to do it. But because x_1 cannot be in the basis, we need to select two variables out of three. And there are three ways to do it.

1. x_2 and x_3 in the basis. The basic matrix is obtained by extracting columns 2 and 3 from A:

   $$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

   which is invertible and happens to be equal to its own inverse:

   $$B^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

In order to apply the generic formula

$$AP = \begin{pmatrix} B & N \end{pmatrix},$$

we need to permute the columns of the matrix so that columns 2 and 3 become the two first columns of the permuted matrix. We use the following permutation matrix,

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $PP^T = I$ and

$$AP = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

The constraints can now be written as

$$Ax = AP(P^T x) = Bx_B + Nx_N = b,$$

and we obtain

$$\begin{aligned} x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} &= B^{-1}(b - Nx_n) \\ &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \right) \\ &= \begin{pmatrix} -2x_1 - x_4 \\ x_1 + 1 \end{pmatrix}. \end{aligned}$$

The basic variables are x_2 and x_3, and the non basic variables are x_1 and x_4.

2. x_2 and x_4 in the basis. The basic matrix is obtained by extracting columns 2 and 4 from A:

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

This matrix is not invertible, and the elimination of constraints cannot be performed.

3. x_3 and x_4 in the basis. The basic matrix is obtained by extracting columns 2 and 4 from A:

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is invertible, and

$$B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

In order to apply the generic formula

$$AP = \begin{pmatrix} B & N \end{pmatrix},$$

we need to permute the columns of the matrix so that columns 3 and 4 become the two first columns of the permuted matrix. We use the following permutation matrix,

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

so that $PP^T = I$ and

$$AP = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

The constraints can now be written as

$$Ax = AP(P^T x) = Bx_B + Nx_N = b,$$

and we obtain

$$\begin{aligned} x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} &= B^{-1}(b - Nx_n) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 + x_1 \\ -2x_1 - x_2 \end{pmatrix}. \end{aligned}$$

The basic variables are x_3 and x_4, and the non basic variables are x_1 and x_2.

Question 10

Consider the following set of constraints:

$$\begin{aligned} x_1 + x_2 &\geq 1, \\ x_1 + x_2 &\leq 2, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

1. Draw the polyhedron in $\mathbb{R}^2$.
2. Write the polyhedron in standard form.
3. Enumerate all potential bases. For each of them, check if it is a valid basis. If so, if it is feasible. If so, identify the corresponding vertex on the polyhedron.

Solution 10

1. The polyhedron is represented in Figure 1

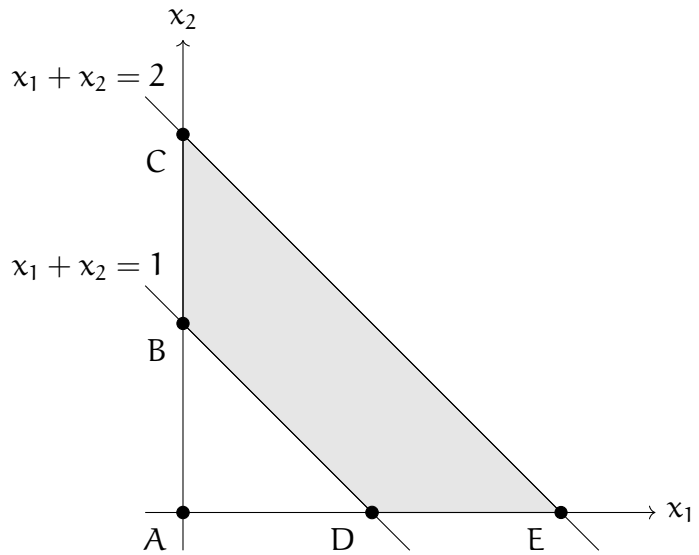


Figure 1: The constraint polyhedron

Note that it has four vertices, labeled B, C, D and E.

2. The polyhedron can be written in standard form by introducing non negative slack variables, and rewriting the constraints in the form of equality constraints:

$$\begin{aligned} -x_1 - x_2 + x_3 &= -1, \\ x_1 + x_2 + x_4 &= 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

The equality constraints in matrix form are

$$Ax = b, \text{ where } A \in \mathbb{R}^{2\times 4},\ b \in \mathbb{R}^2,\ x \in \mathbb{R}^4,$$

$$A = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

3. This is a problem with $n = 4$ variables and $m = 2$ equality constraints. Therefore, the total number of potential bases is 6. Let's investigate them.

   (a) x_3 and x_4 in the basis. We consider the third and fourth columns of matrix A which form the matrix

   $$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   It is invertible, therefore it is a valid basis. We set the non basic variables x_1 and x_2 to zero. Then:

   $$x_B = B^{-1}b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

   x_B has a negative entry. The basic solution is not a *feasible* basic solution. Therefore, it does not correspond to a vertex of the polyhedron. Setting x_1 and x_2 to zero corresponds to point $A = (0, 0)$ in Figure 1. At that point, two constraints are active: $x_1 \geq 0$ and $x_2 \geq 0$, as it is a basic solution. But it is clearly outside of the polyhedron, and it is not a vertex.

   (b) x_1 and x_3 in the basis. We consider the first and third columns of matrix A which form the matrix

   $$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   This matrix is invertible. Therefore, we have a valid basis. We set the non basic variables x_2 and x_4 to zero. Then:

   $$B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

   x_B is non negative, and we have a feasible basic solution. It corresponds to a vertex of the polyhedron. This is the point $E =$

$(2,0)$ in Figure 1. At that point, two constraints are active: $x_1 + x_2 \leq 2$ and $x_2 \geq 0$. In standard form, it means that constraints $x_4 \geq 0$ and $x_2 \geq 0$ are active. It shows that setting the non basic variables to zero is equivalent to impose the corresponding constraints to be active.

(c) x_1 and x_2 in the basis. We consider the first and second columns of matrix A which form the matrix

$$B = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix B is singular. Therefore, this is not a valid basis. From a geometrical point of view, a basis is at the intersection of two constraints. But, here, the non basic variables x_3 and x_4 correspond to two parallel constraints. They have no intersection.

(d) x_1 and x_4 in the basis. We consider the first and fourth columns of matrix A which form the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The basic matrix is invertible. It is therefore a valid basis. We set the non basic variables x_2 and x_3 to zero. Then:

$$B^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

x_B is non negative, and we have a feasible basic solution. It corresponds to the point $D = (1,0)$ in Figure 1.

(e) x_2 and x_3 in the basis. We consider the second and third columns of matrix A which form the matrix

$$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is invertible, and we have a valid basis. We set the non basic variables x_1 and x_4 to zero. Then:

$$B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

It is non negative, and corresponds to the point $C = (0, 2)$ in Figure 1.

(f) x_2 and x_4 in the basis. We consider the second and fourth columns of matrix A which form the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is invertible, and we have a valid basis. We set the non basic variables x_1 and x_3 to zero. Then:

$$B^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

x_B is non negative. We have a feasible basic solution. It is a vertex of the polyhedron and corresponds to the point $B = (0, 1)$ in Figure 1.

Question 11

Consider a linear optimization problem such that there is more than one optimal solution. We claim that, in this case, there are at least two vertices that are optimal. Is this claim true or not? Prove it, or find a counter-example.

Solution 11

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^2} x_1$$

subject to

$$x_1, x_2 \geq 0.$$

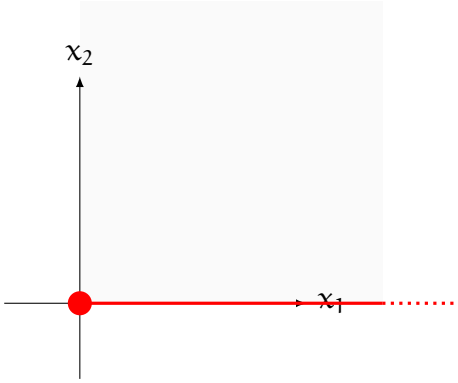
For any $\alpha \geq 0$, the solution

$$\begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

is optimal. Indeed, it is feasible, and x_1 achieves its smallest possible value. Therefore, the problem has an inifinite number of optimal solutions. The feasible set is the non negative quadrant. It has only one vertex:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

as illustrated in the following figure:



Consequently, the claim is not true.

Question 12

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^2} x_1 + x_2$$

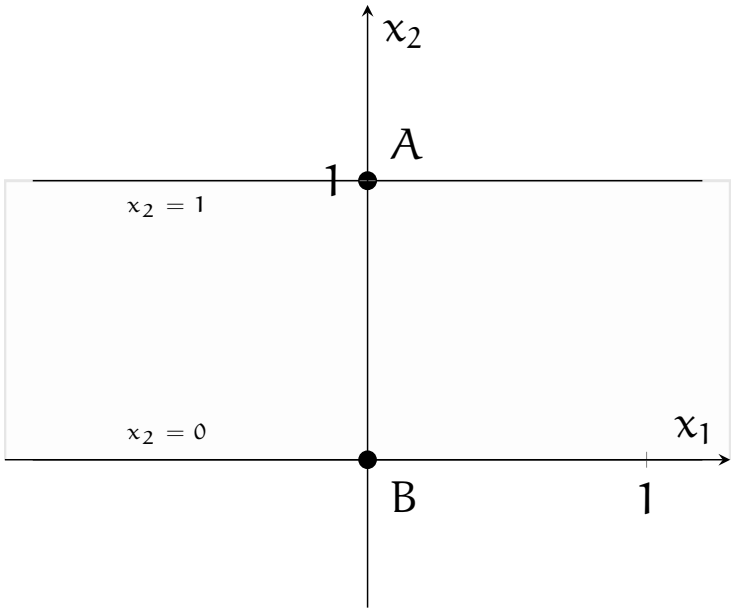
subject to

$$0 \leq x_2 \leq 1.$$

Write the problem in standard form, and identify all the basic solutions. Mention those that are feasible.

Solution 12

The constraint polyhedron is represented in the figure below:



Note that this polyhedron in $\mathbb{R}^2$ has no vertex. In order to write the problem in standard form, we need to replace x_1 by the difference of two non negative variables

$$x_1 = x_1^+ - x_1^-,$$

and we need to introduce a slack variable for the constraint $x_2 \leq 1$. We obtain

$$\min_{x \in \mathbb{R}^4} x_1^+ - x_1^- + x_2$$

subject to

$$\begin{aligned} x_2 + x_3 &= 1, \\ x_1^+, x_1^-, x_2, x_3 &\geq 0. \end{aligned}$$

In matrix form, we have

$$\min_{x \in \mathbb{R}^4} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix},$$
$$b = 1,$$
$$c = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$
$$x = \begin{pmatrix} x_1^+ \\ x_1^- \\ x_2 \\ x_3 \end{pmatrix}.$$

This is a problem with $n = 4$ variables and $m = 1$ constraints. Therefore, there are four potential bases.

1. x_1^+ in the basis. In this case $B = 0$. It is not invertible. Therefore, it is not a valid basis.

2. x_1^- in the basis. In this case $B = 0$. It is not invertible. Therefore, it is not a valid basis.

3. x_2 is in the basis. In this case, $B = 1$ and

   $$x_2 = B^{-1}b = 1.$$

   It is feasible. The vertex

   $$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

   corresponds to point A on the figure. Note that it is a vertex of the polyhedron in $\mathbb{R}^4$, even it does not correspond to a vertex of the original polyhedron in $\mathbb{R}^2$.

4. x_3 is in the basis. In this case, $B = 1$ and

   $$x_3 = B^{-1}b = 1.$$

It is feasible. The vertex

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

corresponds to point B on the figure. Again, it is a vertex of the polyhedron in $\mathbb{R}^4$, even it does not correspond to a vertex of the original polyhedron in $\mathbb{R}^2$.

Question 13

Consider the following polyhedron

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | x_1 + x_2 \leq 1, 3x_1 + 10x_2 \leq 15, x_1 \geq 0, x_2 \geq 0 \right\}.$$

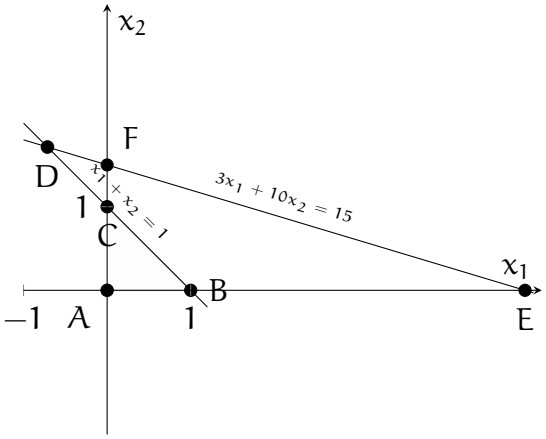
Find all possible basic solutions, and identify which ones are feasible.

Solution 13

We represent the polyhedron

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 | x_1 + x_2 \leq 1, 3x_1 + 10x_2 \leq 15, x_1 \geq 0, x_2 \geq 0 \right\}$$

in the following figure:



The corresponding polyhedron in standard form is

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 | Ax = b, x \geq 0 \right\},$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 15 \end{pmatrix},$$

with the variables x_3 and x_4 being slack variables. Each basic solution is obtained by selecting 2 variables out of 4 to be in the basis. There is a total of 6 possible selections of basic variables.
The matrix $B = (A_{j1}, ..., A_{jm})$ is composed of columns $j_1, ..., j_m$ of the matrix A.

1. Basic solution with x_1 and x_2 in the basis ($j_1 = 1, j_2 = 2$).

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 10 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{10}{7} & -\frac{1}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -\frac{5}{7} \\ \frac{12}{7} \end{pmatrix}; x = \begin{pmatrix} -\frac{5}{7} \\ \frac{12}{7} \\ 0 \\ 0 \end{pmatrix}.$$

This basic solution is not feasible because $B^{-1}b \not\geqslant 0$. It corresponds to point $D = (-\frac{5}{7}, \frac{12}{7})$ in the figure.

2. Basic solution with x_1 and x_3 in the basis $(j_1 = 1, j_2 = 3)$.

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 5 \\ -4 \end{pmatrix}; x = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 0 \end{pmatrix}.$$

This basic solution is not feasible because $B^{-1}b \not\geqslant 0$. It corresponds to point $E = (5, 0)$.

3. Basic solution with x_1 and x_4 in the basis $(j_1 = 1, j_2 = 4)$.

$$B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 12 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 12 \end{pmatrix}.$$

This basic solution is feasible and corresponds to point $B = (1, 0)$.

4. Basic solution with x_2 and x_3 in the basis $(j_1 = 2, j_2 = 3)$.

$$B = \begin{pmatrix} 1 & 1 \\ 10 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & \frac{1}{10} \\ 1 & -\frac{1}{10} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}; x = \begin{pmatrix} 0 \\ \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

This basic solution is not feasible because $B^{-1}b \not\geqslant 0$. It corresponds to point $F = (0, \frac{3}{2})$ in the figure.

5. Basic solution with x_2 and x_4 in the basis $(j_1 = 2, j_2 = 4)$.

$$B = \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ -10 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 5 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix}.$$

This basic solution is feasible and corresponds to point $C = (0, 1)$ in the figure.

6. Basic solution with x_3 and x_4 in the basis $(j_1 = 3, j_2 = 4)$.

$$B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 15 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 15 \end{pmatrix}.$$

   This basic solution is feasible and corresponds to point $A = (0, 0)$ in the figure.

In summary, out of the six basic solutions, only three are feasible and correspond to vertices of the constraint polyhedron.

Question 14

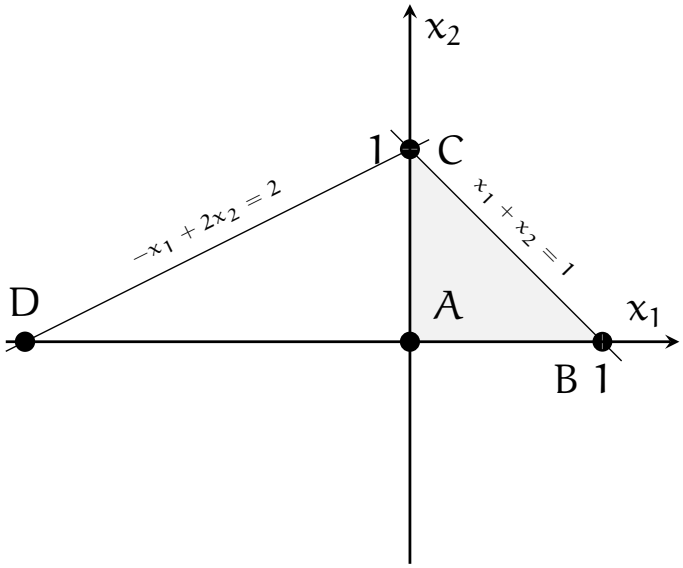
Consider the polyhedron

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 | x_1 + x_2 \leq 1, -x_1 + 2x_2 \leq 2, x_1 \geq 0, x_2 \geq 0 \right\}.$$

1. Represent it graphically.
2. Identify the corresponding polyhedron in standard form.
3. Enumerate the feasible basic solutions, and identify the corresponding vertex on the polyhedron.
4. For each feasible basic solution, mention if it is degenerate or not. Remember that a feasible basic solution is degenerate if at least one of the basic variables is zero.

Solution 14

The polyhedron P is represented below:



The corresponding polyhedron in standard form is obtained by introducing non negative slack variables, and rewriting the constraints in the form of equality constraints. We obtain a polyhedron in $\mathbb{R}^4$:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 | x_1 + x_2 + x_3 = 1, -x_1 + 2x_2 + x_4 = 2, x_1, x_2, x_3, x_4 \geq 0 \right\}.$$

The equality constraints in matrix form are

$$Ax = b, \text{ where } A \in \mathbb{R}^{2\times 4},\ b \in \mathbb{R}^2,\ x \in \mathbb{R}^4,$$
$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We have a problem with $n = 4$ variables and $m = 2$ equality constraints. There are 6 ways to select 2 basic variables out of 4.

1. Basic solution where x_1 and x_2 are in the basis:

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the point $C = (0, 1)$. It is a degenerate solution as one of the basic variables (x_1) is zero. There are 5 active constraints, two equality constraints, and three inequality constraints: $x_1 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

2. Basic solution where x_1 and x_3 are in the basis

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -2 \\ 3 \end{pmatrix}; x = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

The basic solution is not feasible, as $x_1 < 0$. It corresponds to the point $D = (-2, 0)$.

3. Basic solution where x_1 and x_4 are in the basis

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the point $B = (1, 0)$. It is not degenerate as no basic variables is zero.

4. Basic solution where x_2 and x_3 are in the basis

$$B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the point $C = (0, 1)$. It is a degenerate solution as one of the basic variables (x_3) is zero. There are 5 active constraints, two equality constraints, and three inequality constraints: $x_1 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

5. Basic solution where x_2 and x_4 are in the basis

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the point $C = (0, 1)$. It is a degenerate solution as one of the basic variables (x_4) is zero. There are 5 active constraints, two equality constraints, and three inequality constraints: $x_1 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

6. Basic solution where x_3 and x_4 are in the basis

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the point $A = (0, 0)$. It is not degenerate as no basic variable is zero.

In presence of degeneracy, a vertex may correspond to multiple feasible basic solutions. Here the vertex

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

corresponds to the bases (x_1, x_2), (x_2, x_3) and (x_2, x_4). Five constraints are active and, in dimension 4, four constraints are sufficient to identify a vertex.

In the original polyhedron, it corresponds to the point $C = (1, 0)$. At that point, three constraints are active:

- $x_1 + x_2 \leq 1$,
- $-x_1 + 2x_2 \leq 2$,
- $x_2 \geq 0$.

In dimension 2, two constraints are sufficient to identify a vertex.

Question 15

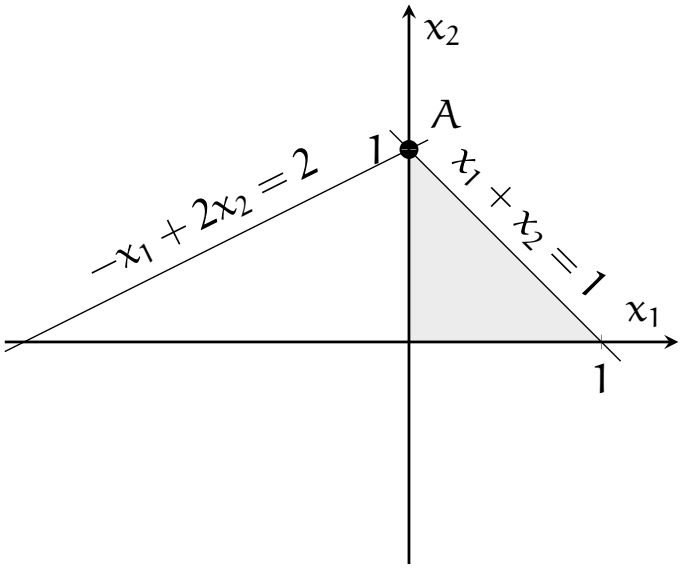
Consider the polyhedron defined by the following constraints:

$$\begin{aligned} x_1 + x_2 &\leq 1, \\ -x_1 + 2x_2 &\leq 2, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Draw the polyhedron.
2. Write the polyhedron in standard form.
3. If there are degenerate vertices, identify them on the graphical representation and provide the corresponding values of the variables.
4. Consider the case where the variables x_1 and x_2 are in the basis. Calculate all basic directions, determine if they are feasible of not, and represent them on the picture.

Solution 15

1. The polyhedron is represented as follows:



2. In standard form, after introducing the two slack variables x_3 and x_4, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ -x_1 + 2x_2 + x_4 &= 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

3. The vertex A corresponds to $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, and is degenerate. Indeed, there are more than $n = 2$ constraints that are active at that vertex. There are two active constraints at each of the two other vertices. Consequently, they are not degenerate.

4. The equality constraints in matrix form are

$$Ax = b, \text{ where } A \in \mathbb{R}^{2\times 4},\ b \in \mathbb{R}^2,\ x \in \mathbb{R}^4,$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We have a problem with $n = 4$ variables and $m = 2$ equality constraints. We consider the basic solution where x_1 and x_2 are in the basis:

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The basic entries of the basic direction corresponding to the non basic variable x_3 is calculated as

$$-B^{-1}A_3 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

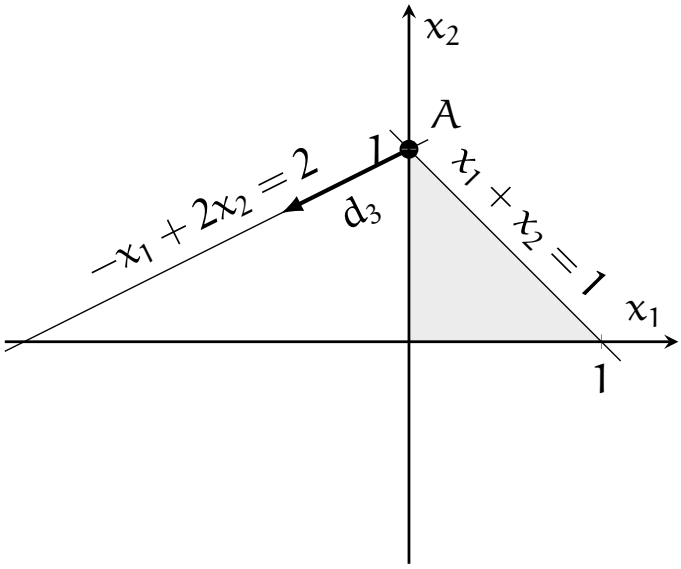
Therefore, the basic direction is

$$d_3 = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}.$$

Moving along d_3, we obtain

$$x^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}\alpha \\ 1 - \frac{1}{3}\alpha \\ \alpha \\ 0 \end{pmatrix},$$

which is infeasible for any $\alpha > 0$. x_1 becomes negative as long as we start following the direction. Therefore, the basic direction d_3 is infeasible. It is represented in the following figure.



The basic entries of the basic direction corresponding to the non basic variable x_4 is calculated as

$$-B^{-1}A_4 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

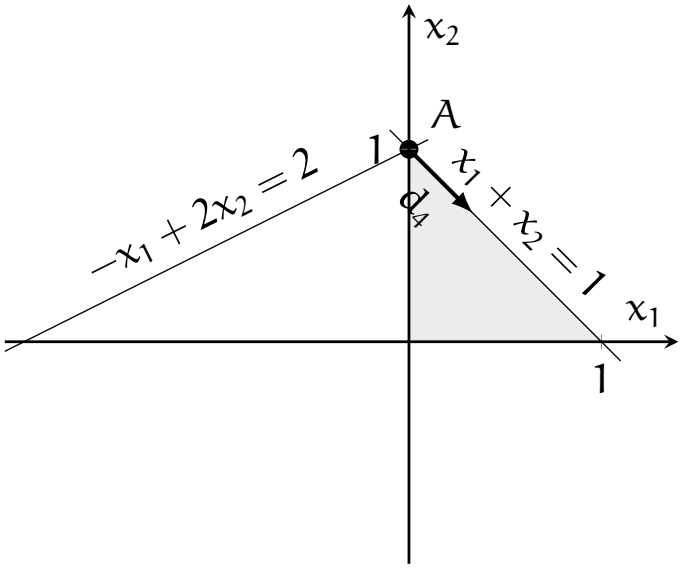
Therefore, the basic direction is

$$d_4 = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}.$$

Moving along d_4, we obtain

$$x^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\alpha \\ 1 - \frac{1}{3}\alpha \\ 0 \\ \alpha \end{pmatrix},$$

which is feasible for $0 \leq \alpha \leq 3$. Therefore, the basic direction d_4 is feasible. It is represented in the following figure.


x2
A
1
$-x_1 + 2x_2 = 2$
d_4
$t_1 + t_2 = 1$
x1
1

Question 16

Consider the following polyhedron represented in standard form:

$$P = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} | Ax = b, x \geq 0 \right\},$$

with $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find the basic directions for the following feasible basic solutions, and mention if they are feasible:

1. Basic solution with x_2 and x_4 in the basis $(j_1 = 2, j_2 = 4)$:

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

2. Basic solution with x_1 and x_4 in the basis $(j_1 = 1, j_2 = 4)$:

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Basic solution with x_1 and x_2 in the basis $(j_1 = 1, j_2 = 2)$:

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution 16

1. We consider the given polyhedron and the feasible basic solution with x_2 and x_4 in the basis. The corresponding feasible basic solution is

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

and the permutation of the columns to have all basic variables first is defined by the matrix

$$P = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

The basic direction corresponding to the non basic variable x_1 is

$$d_1 = P \begin{pmatrix} -B^{-1}A_1 \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix},$$

and the basic direction corresponding to the non basic variable x_3 is

$$d_3 = P \begin{pmatrix} -B^{-1}A_3 \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Moving along d_1 with a step $\alpha \geq 0$, we obtain the point

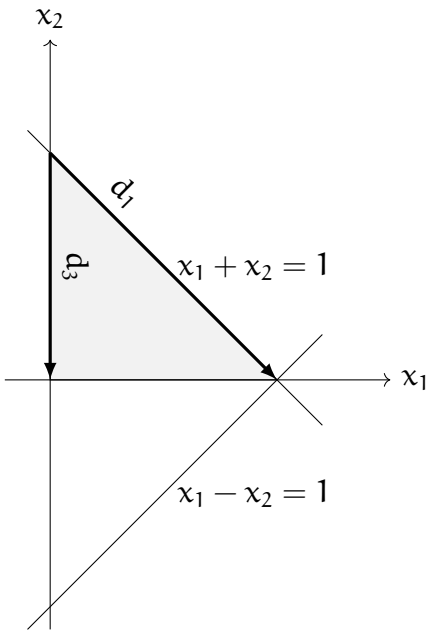
$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 - \alpha \\ 0 \\ 2 - 2\alpha \end{pmatrix},$$

that is feasible for each $0 \leq \alpha \leq 1$. Therefore, the direction is feasible. Moving along d_3 with a step $\alpha \geq 0$, we obtain the point

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1-\alpha \\ \alpha \\ 2-\alpha \end{pmatrix}$$

that is feasible for each $0 \leq \alpha \leq 1$. Therefore, the direction is also feasible.

These two directions are shown in the following figure at point $(0, 1)$, where their length corresponds to $\alpha = 1$.



2. We now consider the feasible basic solution where x_1 and x_4 are in the basis. The feasible basic solution is

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the permutation matrix is

$$P = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

The basic direction corresponding to the non basic variable x_2 is

$$\tilde{d}_2 = P \begin{pmatrix} -B^{-1}A_2 \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

and the basic direction corresponding to the non basic variable x_3 is

$$\tilde{d}_3 = P \begin{pmatrix} -B^{-1}A_3 \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

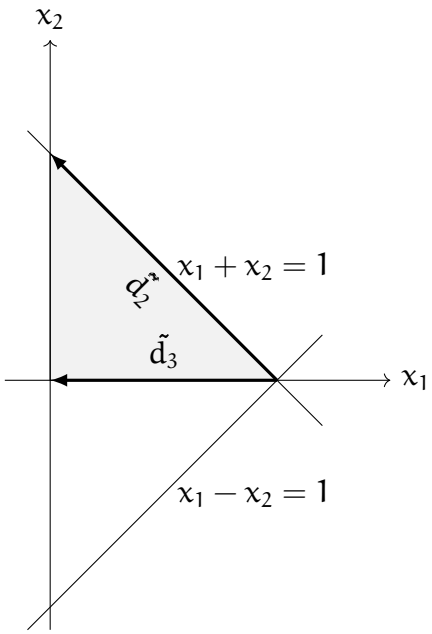
Moving along $\tilde{d}_2$ with a step $\alpha \geq 0$, we obtain the point

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ \alpha \\ 0 \\ 2\alpha \end{pmatrix},$$

that is feasible for any $0 \leq \alpha \leq 1$. Therefore, the direction is feasible. Moving along $\tilde{d}_3$ with a step $\alpha \geq 0$, we obtain the point

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ 0 \\ \alpha \\ \alpha \end{pmatrix},$$

that is feasible for any $0 \leq \alpha \leq 1$. Therefore, the direction is feasible. These two directions are shown in the following figure at point $(1,0)$, where their length corresponds to $\alpha = 1$.



3. Finally, consider the feasible basic solution where x_1 and x_2 are in the basis. Then,

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } P = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = I.$$

The basic direction corresponding to the non basic variable x_3 is

$$\hat{d_3} = \begin{pmatrix} -B^{-1}A_3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix},$$

and the basic direction corresponding to the non basic variable x_4 is

$$\hat{d_4} = \begin{pmatrix} -B^{-1}A_4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}.$$

Moving along $\hat{d}_3$ with a step $\alpha \geq 0$, we obtain the point

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\alpha \\ -\frac{1}{2}\alpha \\ \alpha \\ 0 \end{pmatrix},$$

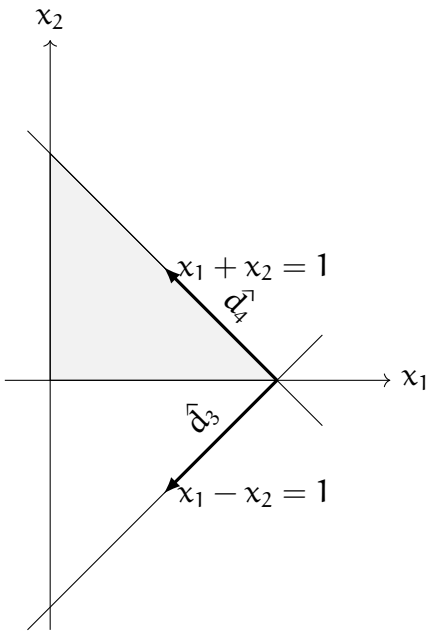
that is not feasible for any $\alpha > 0$ as the second entry is negative. Therefore, the direction is **not** feasible.

Moving along $\hat{d}_4$ with a step $\alpha \geq 0$, we obtain the point

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\alpha \\ \frac{1}{2}\alpha \\ 0 \\ \alpha \end{pmatrix},$$

that is feasible for any $0 \leq \alpha \leq 2$. Therefore, the direction is feasible.

These two directions are shown in the following figure at point $(1, 0)$, where their length corresponds to $\alpha = 1$. Note that $\hat{d}_4$ is a feasible direction, whereas $\hat{d}_3$ is not.



Question 17

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^2} x_1 + x_2$$

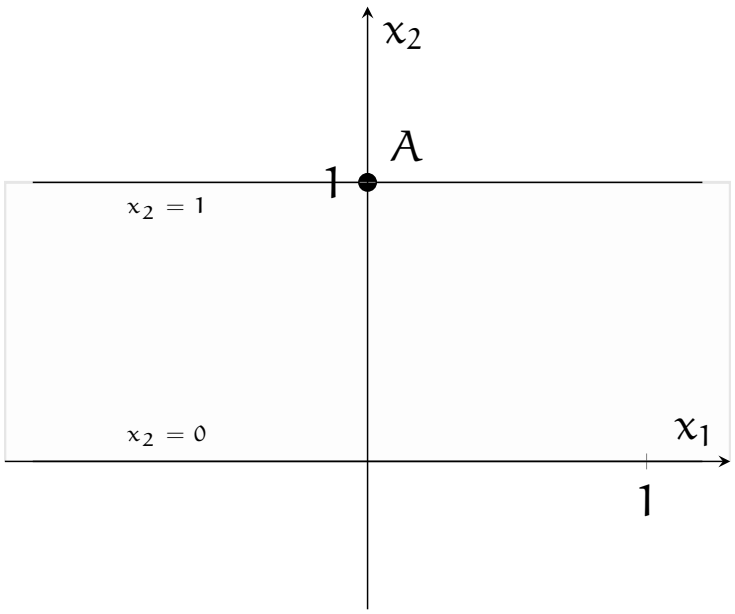
subject to

$$0 \leq x_2 \leq 1.$$

Write the problem in standard form, and consider the feasible basic solution where x_2 is in the basis. Calculate all the basic directions. Verify if they are feasible, and represent them on a figure.

Solution 17

The constraint polyhedron is represented in the figure below:



In order to write the problem in standard form, we need to replace x_1 by the difference of two non negative variables

$$x_1 = x_1^+ - x_1^-,$$

and we need to introduce a slack variable for the constraint $x_2 \leq 1$. We obtain

$$\min_{x \in \mathbb{R}^4} x_1^+ - x_1^- + x_2$$

subject to

$$\begin{aligned} x_2 + x_3 &= 1, \\ x_1^+, x_1^-, x_2, x_3 &\geq 0. \end{aligned}$$

In matrix form, we have

$$\min_{x \in \mathbb{R}^4} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix},$$
$$b = 1,$$
$$c = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$
$$x = \begin{pmatrix} x_1^+ \\ x_1^- \\ x_2 \\ x_3 \end{pmatrix}.$$

This is a problem with $n = 4$ variables and $m = 1$ constraints. So the basis matrix is of dimension one, and is a scalar. If x_2 is in the basis, $B = 1$ and we have

$$x_2 = B^{-1}b = 1,$$

and the feasible basic solution is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and corresponds to point A on the figure. As there are three non basic variables, there are three basic directions.

1. The basic direction associated with x_1^+ is

$$d_1^+ = \begin{pmatrix} 1 \\ 0 \\ d_B \\ 0 \end{pmatrix},$$

where

$$d_B = -B^{-1}A_1 = 0,$$

as the variable x_1^+ is associated with column 1 of A, so that

$$d_1^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

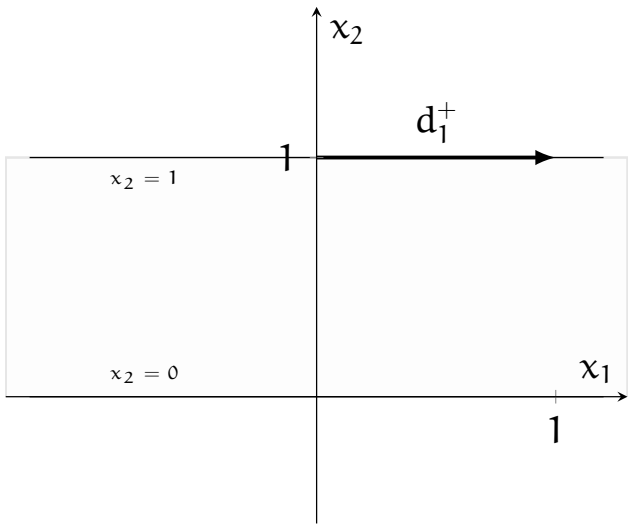
Performing a step α in this direction leads to the point

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The point is feasible for any value of $\alpha \geq 0$, so that the direction is feasible. In the original variables, with $x_1 = x_1^+ - x_1^-$, it corresponds to a direction

$$\begin{pmatrix} 1-0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

as illustrated on the following figure:



2. The basic direction associated with x_1^- is

$$d_1^- = \begin{pmatrix} 0 \\ 1 \\ d_B \\ 0 \end{pmatrix},$$

where

$$d_B = -B^{-1}A_2 = 0,$$

as the variable x_1^- is associated with column 2 of A, so that

$$d_1^- = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

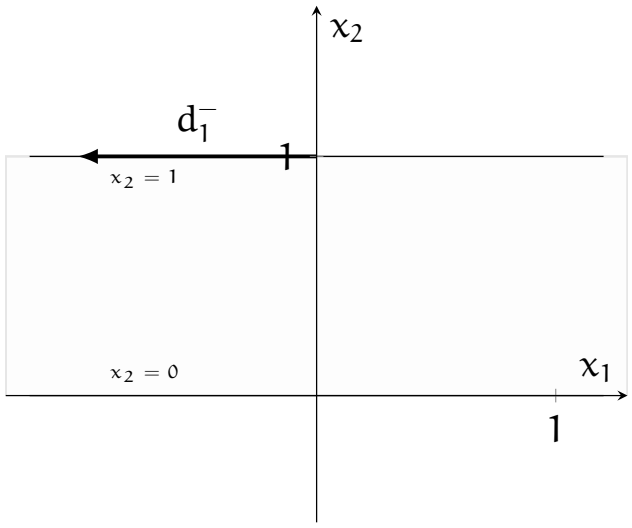
Performing a step α in this direction leads to the point

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ 1 \\ 0 \end{pmatrix}.$$

The point is feasible for any value of $\alpha \geq 0$, so that the direction is feasible. In the original variables, with $x_1 = x_1^+ - x_1^-$, it corresponds to a direction

$$\begin{pmatrix} 0 - 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

as illustrated on the following figure:



3. The basic direction associated with x_3 is

$$d_3 = \begin{pmatrix} 0 \\ 0 \\ d_B \\ 1 \end{pmatrix},$$

where

$$d_B = -B^{-1}A_4 = -1,$$

as the variable x_3 is associated with column 4 of A, so that

$$d_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

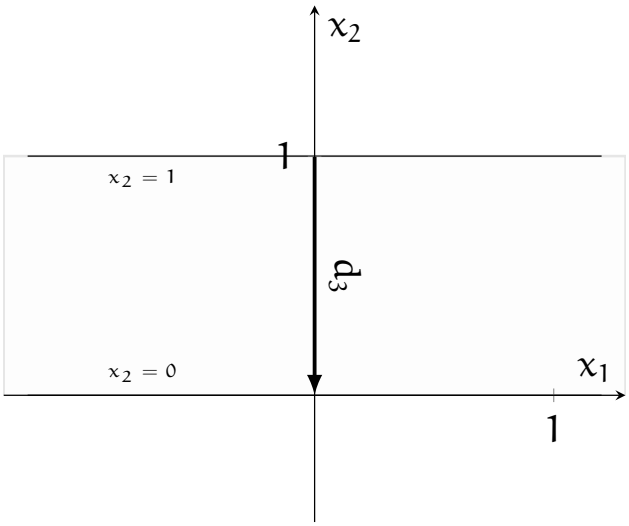
Performing a step α in this direction leads to the point

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1-\alpha \\ \alpha \end{pmatrix}.$$

The point is feasible for any value of α such that $0 \leq \alpha \leq 1$, so that the direction is feasible. In the original variables, with $x_1 = x_1^+ - x_1^-$, it corresponds to a direction

$$\begin{pmatrix} 0-0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

as illustrated on the following figure:


x2
1
x2 = 1
g3
x2 = 0
x1
1

Question 18

Consider the optimization problem $\max x_1 + x_2$, where (x_1, x_2) is constrained to belong to the polyhedron represented in Figure 2.

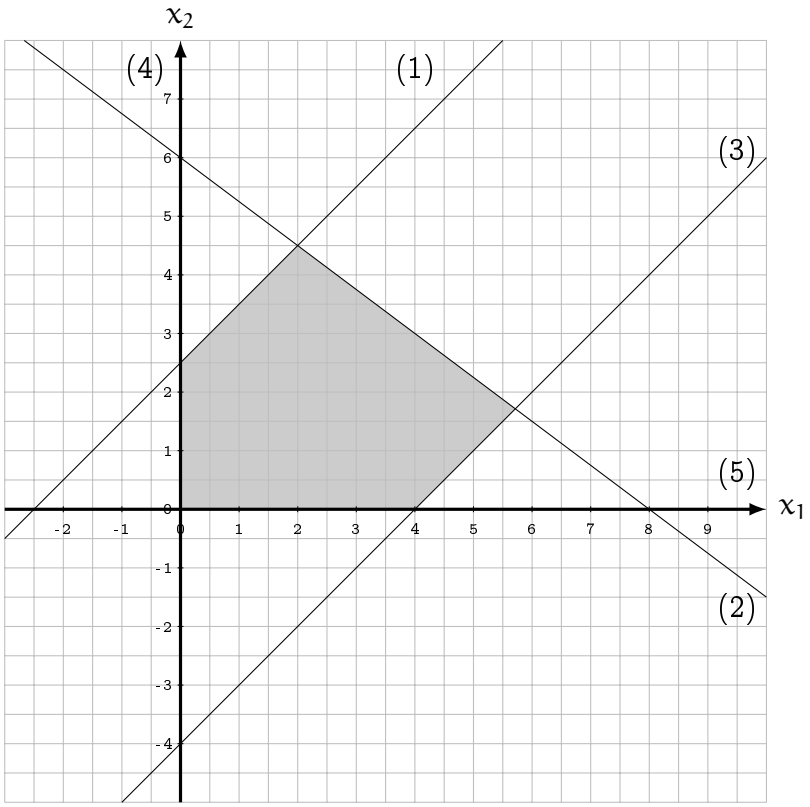


Figure 2: Feasible Domain

1. Formulate the 5 constraints that determine the feasible domain represented in gray in Figure 2.
2. Write the corresponding linear optimization problem in standard form. Give explicitly the matrix A and the vectors b and c.
3. List all basic solutions of the problem. For each basic solution:
   - identify the solution on Figure 2,

- determine if the solution is feasible.

4. Why does point $(3, 2)$ represent a feasible solution, but not a basic solution?

5. Is the direction $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ feasible for the given feasible point $x = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$? Justify.

Solution 18

1. The constraints are as follows:

$$\begin{aligned} x_1 - x_2 &\geq -2.5, && (1)\\ 3x_1 + 4x_2 &\leq 24, && (2)\\ x_1 - x_2 &\leq 4, && (3)\\ x_1 &\geq 0, && (4)\\ x_2 &\geq 0. && (5) \end{aligned}$$

2. We now write the problem in standard form. We include the following slack variables to the inequality constraints to obtain equality constraints:

   - x_3 for constraint (1),
   - x_4 for constraint (2), and,
   - x_5 for constraint (3).

   Moreover, a maximization problem whose objective function is $f(x)$ is equivalent to a minimization problem whose objective function is $-f(x)$. Then we obtain the following problem.

$$\min_{x \in \mathbb{R}^5} -x_1 - x_2$$

   subject to

$$\begin{array}{rrrrrcll} x_1 & -x_2 & -x_3 & & & = & -2.5, & \\ 3x_1 & +4x_2 & & +x_4 & & = & 24, & \\ x_1 & -x_2 & & & +x_5 & = & 4, & \\ & & & & x_i & \geq & 0, & i = 1, \ldots, 5. \end{array}$$

   The number of equality constraints is $m = 3$ and the number of decision variables is $n = 5$. The matrix $A \in \mathbb{R}^{m \times n}$ and the vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given by

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} -2.5 \\ 24 \\ 4 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Each basic solution is obtained by selecting $m = 3$ variables out of $n = 5$ to be in the basis. The $n - m = 2$ non basic variables equal to 0. There is a total of 10 possible selections of basic variables. We detail the computation of some of them.

   - Basic solution with x_1, x_2 and x_3 in the basis $(j_1 = 1, j_2 = 2, j_3 = 3)$.

$$B = \begin{pmatrix} 1 & -1 & -1 \\ 3 & 4 & 0 \\ 1 & -1 & 0 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} 0 & 1/7 & 4/7 \\ 0 & 1/7 & -3/7 \\ -1 & 0 & 1 \end{pmatrix};$$

$$x_B = B^{-1}b = \begin{pmatrix} 40/7 \\ 12/7 \\ 13/7 \end{pmatrix}; \quad x = \begin{pmatrix} 40/7 \\ 12/7 \\ 13/7 \\ 0 \\ 0 \end{pmatrix}.$$

     This basic solution is feasible since $B^{-1}b \geq 0$. It corresponds to point C in Figure 3 which is the intersection of constraints (2) and (3). Note that activating constraints (2) and (3) in the original problem is equivalent to setting to zero the slack variables x_4 and x_5 in the problem in standard form. These are the non basic variables that are set to zero.

   - Basic solution with x_1, x_2 and x_5 in the basis $(j_1 = 1, j_2 = 2, j_3 = 5)$.

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 4 & 0 \\ 1 & -1 & 1 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} 4/7 & 1/7 & 0 \\ -3/7 & 1/7 & 0 \\ -1 & 0 & 1 \end{pmatrix};$$

$$x_B = B^{-1}b = \begin{pmatrix} 2 \\ 4.5 \\ 6.5 \end{pmatrix}; \quad x = \begin{pmatrix} 2 \\ 4.5 \\ 0 \\ 0 \\ 6.5 \end{pmatrix}.$$

     This basic solution is feasible as $B^{-1}b \geq 0$. It corresponds to vertex H in Figure 3 which is the intersection of constraints (1)

and (2). Note that activating constraints (1) and (2) in the original problem is equivalent to setting to zero the slack variables x_3 and x_4 in the problem in standard form. These are the non basic variables that are set to zero.

- Basic solution with x_1, x_3 and x_5 in the basis $(j_1 = 1, j_2 = 3, j_3 = 5)$.

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} 0 & 1/3 & 0 \\ -1 & 1/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix};$$

$$x_B = B^{-1}b = \begin{pmatrix} 8 \\ 10.5 \\ -4 \end{pmatrix}; \quad x = \begin{pmatrix} 8 \\ 0 \\ 10.5 \\ 0 \\ -4 \end{pmatrix}.$$

This basic solution is not feasible since $B^{-1}b \ngeq 0$. It corresponds to vertex E in Figure 3 which is the intersection of constraints (2) and (5). Note that activating constraints (2) and (5) in the original problem is equivalent to setting to zero the slack variable x_4 and the variable x_2 in the problem in standard form. These are the non basic variables that are set to zero.

The following table gives all the basic solutions. Each basic solution is obtained by fixing $n - m = 2$ non basic variables, which equal 0, and calculating the other variables using $x_B = B^{-1}b$.

Point	x_1	x_2	x_3	x_4	x_5	Feasible?
A	0	6	-3.5	0	10	no
B	0	0	2.5	24	4	yes
C	40/7	12/7	13/7	0	0	yes
D	4	0	6.5	12	0	yes
E	8	0	10.5	0	-4	no
F	0	-4	6.5	40	0	no
G	0	2.5	0	14	6.5	yes
H	2	4.5	0	0	6.5	yes
I	-2.5	0	0	31.5	6.5	no
–	–	–	0	–	0	

Note: It is impossible to have the case $x_3 = x_5 = 0$. In other words, x_3 and x_5 can't be non basic variables in the same time. Indeed, in this case the basic matrix is

$$B = (A_1, A_2, A_4) = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 4 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

which is a singular matrix. Therefore, there is no corresponding basic solution. In fact, having $x_3 = x_5 = 0$ means that constraints (1) and (3) are active at the same time, which is impossible since these two constraints are parallel. This case is represented with dashes in the table.

- In Figure 3, the basic solutions are given with the same letters as on the table above: from A to I. They correspond to the intersections of the constraints.
- A solution is feasible if it satisfies all the constraints of the problem. In order to calculate the basic solutions, we used the formula $x_B = B^{-1}b$. The non negativity constraints still need to be satisfied. The feasible solutions satisfies

  $$x_i \geq 0, \quad \forall i \in 1, \ldots, 5.$$

  Therefore, the feasible basic solutions corresponds to the following points:

  B(0, 0), C(40/7, 12/7), D(4, 0), G(0, 2.5) and H(2, 4.5),

  that happen to be the vertices of the polyhedron.

4. When considering the problem in standard form, the point (3, 2) corresponds to the point $(x_1, x_2, x_3, x_4, x_5) = (3, 2, 3.5, 7, 3)$. This is a feasible solution because all the constraints are satisfied.

   However, in a basic solution at least n constraints must be active. In standard form, $n = 5$. There are $m = 3$ equality constraints that are active. Therefore, at least two variables (the non basic variables) should be zero at a basic solution. It is not the case for point (3, 2), so it is not a basic solution.

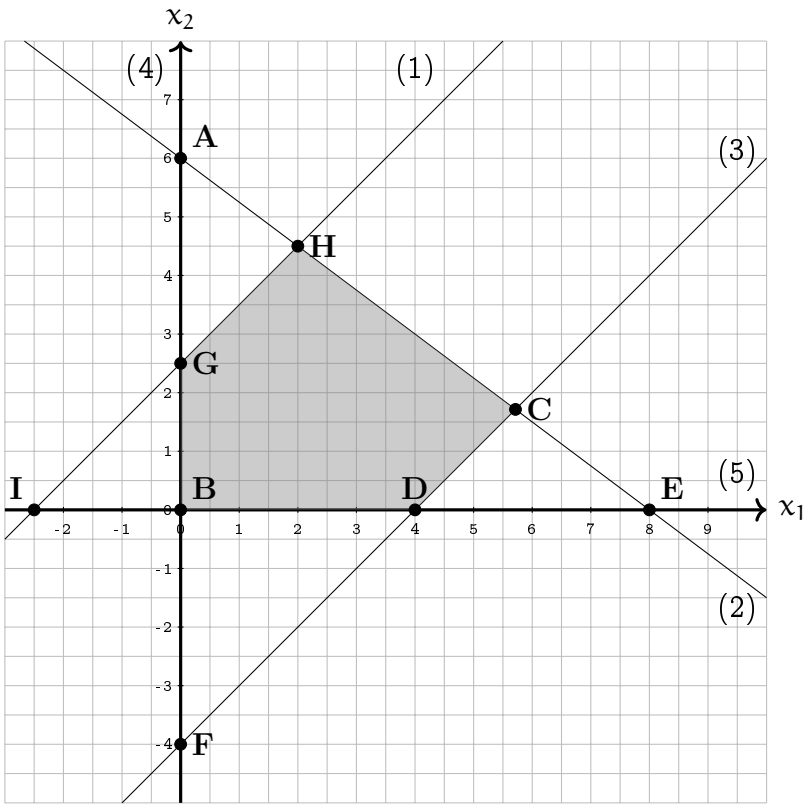


Figure 3: Basic solutions

5. Given a feasible point of the polyhedron, it is possible to analyze if a direction is feasible by checking the following two conditions:

   - $Ad = 0$ and
   - $d_i \geq 0$ when $x_i = 0$,

   where

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix}.$$

We verify if these conditions are satisfied for the case $x = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We first need to compute the values of the slack variables x_3, x_4 and x_5. We obtain $x_3 = 6.5$, $x_4 = 12$ and $x_5 = 0$. Therefore, we have

$$x = \begin{pmatrix} 4 \\ 0 \\ 6.5 \\ 12 \\ 0 \end{pmatrix}.$$

In order to satisfy the first condition, we must have

$$Ad = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix} = \begin{pmatrix} -d_3 \\ 7 + d_4 \\ d_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

which is verified by setting $d_3 = 0$, $d_4 = -7$ and $d_5 = 0$. Therefore,

$$d = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -7 \\ 0 \end{pmatrix}.$$

satisfies the first condition. For the second, we need to look at the entries 2 and 5 of d since $x_2 = x_5 = 0$. As both entries are non negative, we conclude that the direction is feasible at the given point.

Question 19

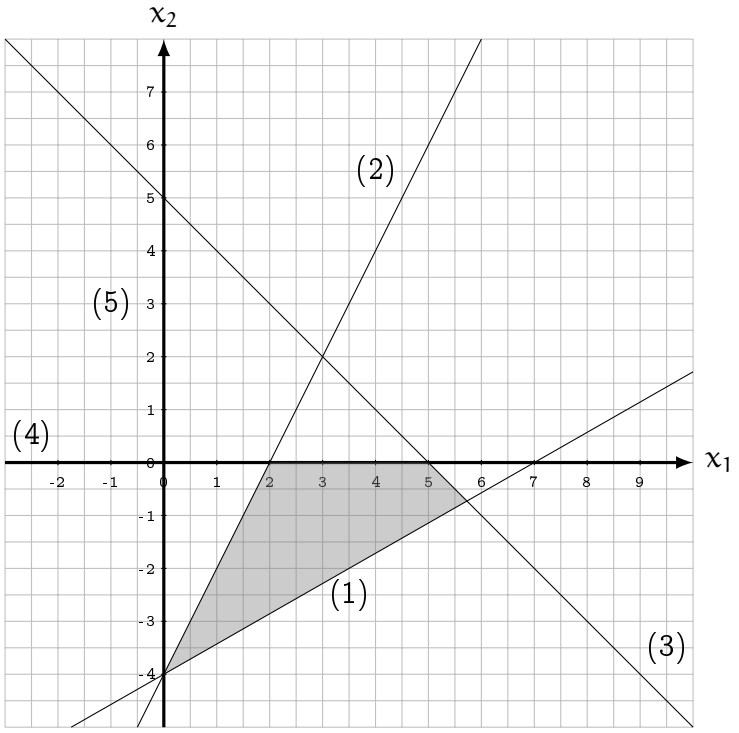


Figure 4: Feasible Domain

Consider the optimization problem $\min 2x_1 - x_2$, where (x_1, x_2) is constrained to belong to the polyhedron represented in Figure 4.

1. Formulate the 5 constraints that determine the feasible domain represented in gray in Figure 4.

2. Write the corresponding linear optimization problem in standard form. Give explicitly the matrix A and the vectors b and c.

3. List the basic solutions of the problem. For each basic solution:

   - identify the solution on Figure 4,
   - determine if the solution is feasible.

4. Is there any degenerate feasible solution? Justify.

5. Why does point $(2, -2)$ represent a feasible solution, but not a basic solution?

6. Consider the solution on Figure 4 where constraints (2) and (5) are active, and the corresponding basic solution in the problem in standard form where the variables associated with these constraints are out of the basis.

   (a) Give the basis matrix.

   (b) Calculate all basic directions. Are the basic directions feasible? Justify your answer.

Solution 19

1. The five constraints are:

$$\begin{aligned} 4/7x_1 - x_2 &\leq 4, && (1)\\ 2x_1 - x_2 &\geq 4, && (2)\\ x_1 + x_2 &\leq 5, && (3)\\ x_2 &\leq 0, && (4)\\ x_1 &\geq 0. && (5) \end{aligned}$$

2. Constraints (1) to (3), can be transformed into equality constraints by introducing slack variables $e_1, e_2, e_3 \geq 0$ as follows:

$$\begin{aligned} 4/7x_1 - x_2 + e_1 &= 4.\\ 2x_1 - x_2 - e_2 &= 4.\\ x_1 + x_2 + e_3 &= 5. \end{aligned}$$

   For constraint (4) we use a change of variable. We define a new variable $x_2' \geq 0$ as

$$x_2' = -x_2.$$

   The non-negativity constraint on x_2' ensures that $x_2 \leq 0$.

   We replace x_2 with $(-x_2')$ in the objective function and all the constraints.

   By putting everything together, the problem in standard form is:

$$\min 2x_1 + x_2'$$

   subject to

$$\begin{aligned} 4/7x_1 + x_2' + e_1 &= 4,\\ 2x_1 + x_2' - e_2 &= 4,\\ x_1 - x_2' + e_3 &= 5,\\ x_1, x_2', e_1, e_2, e_3 &\geq 0. \end{aligned}$$

The number of equality constraints is $m = 3$ and the number of decision variables is $n = 5$. The matrix $A \in \mathbb{R}^{m \times n}$ and the vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given by

$$A = \begin{pmatrix} 4/7 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}, \qquad c = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Each basic solution is obtained by selecting $m = 3$ variables out of $n = 5$ to be in the basis. The $n - m = 2$ non basic variables equal 0. There is a total of 10 possible selections of basic variables. We detail the calculation of some of them:

   - Basic solution with x_1, x_2' and x_3 in the basis $(j_1 = 1, j_2 = 2, j_3 = 3)$.

   $$B = \begin{pmatrix} 4/7 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} 0 & 1/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 1 & -11/21 & 10/21 \end{pmatrix};$$

   $$x_B = B^{-1}b = \begin{pmatrix} 3 \\ -2 \\ 30/7 \end{pmatrix}; \quad x = \begin{pmatrix} 3 \\ -2 \\ 30/7 \\ 0 \\ 0 \end{pmatrix}.$$

   This basic solution is infeasible since $B^{-1}b \not\geqslant 0$. It corresponds to point B in Figure 5 which is the intersection of constraints (2) and (3).

   - Basic solution with x_1, x_2' and x_5 in the basis $(j_1 = 1, j_2 = 2, j_3 = 5)$.

   $$B = \begin{pmatrix} 4/7 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} -7/10 & 7/10 & 0 \\ 7/5 & -2/5 & 0 \\ 21/10 & -11/10 & 1 \end{pmatrix};$$

$$x_B = B^{-1}b = \begin{pmatrix} 0 \\ 4 \\ 9 \end{pmatrix}; \quad x = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 9 \end{pmatrix}.$$

This basic solution is feasible as $B^{-1}b \geq 0$. It corresponds to vertex F in Figure 5 which is the intersection of constraints (1) and (2).

- Basic solution with x_1, x_3 and x_5 in the basis $(j_1 = 1, j_2 = 3, j_3 = 5)$.

$$B = \begin{pmatrix} 4/7 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \quad B^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & -2/7 & 0 \\ 0 & -1/2 & 1 \end{pmatrix};$$

$$x_B = B^{-1}b = \begin{pmatrix} 2 \\ 20/7 \\ 3 \end{pmatrix}; \quad x = \begin{pmatrix} 2 \\ 0 \\ 20/7 \\ 0 \\ 3 \end{pmatrix}.$$

This basic solution is feasible since $B^{-1}b \geq 0$. It corresponds to vertex D in Figure 5 which is the intersection of constraints (2) and (4).

The following table gives all the basic solutions. Each basic solution is obtained by fixing $n - m = 2$ non basic variables, which equal 0, and calculating the other variables using $x_B = B^{-1}b$.

Point	x_1	x_2'	e_1	e_2	e_3	Feasible?
A	0	- 5	9	-9	0	no
B	3	-2	30/7	0	0	no
C	5	0	8/7	6	0	yes
D	2	0	20/7	0	3	yes
E	7	0	0	10	-2	no
F	0	4	0	0	9	yes
G	63/11	8/11	0	90/11	0	yes
H	0	0	4	-4	5	no

- In Figure 5, the basic solutions are given with the same letters as on the table above: from A to H. They correspond to the intersections of the constraints.

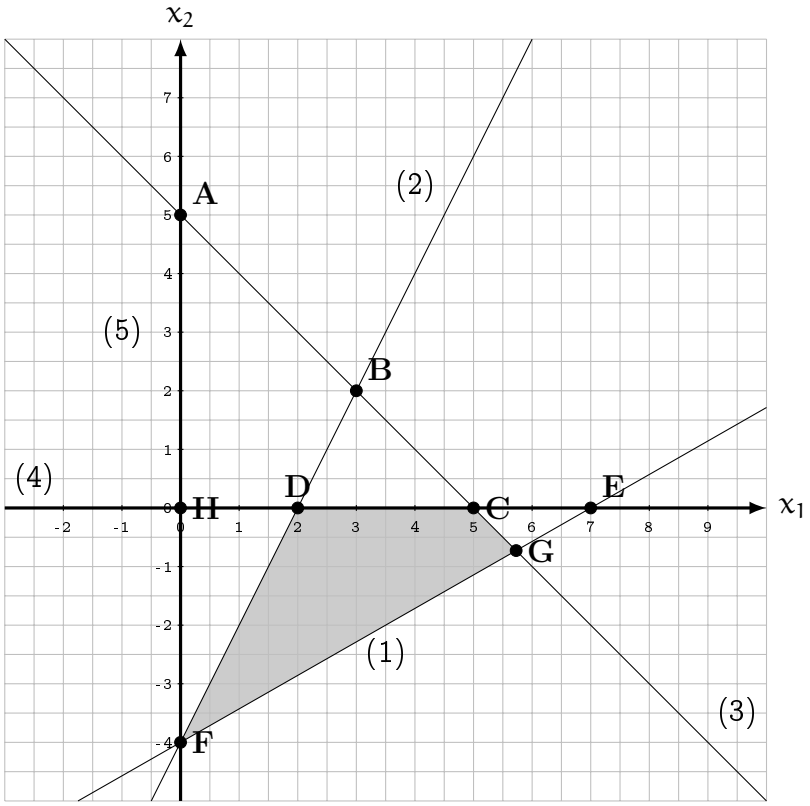


Figure 5: Basic solutions

- A solution is feasible if it satisfies all the constraints of the problem. In order to calculate the basic solutions, we used the formula $x_B = B^{-1}b$. The non negativity constraints still need to be satisfied. The feasible solutions satisfy

$$x_1, x_2', e_1, e_2, e_3 \geq 0.$$

Therefore, the feasible basic solutions correspond to the following points:

D(2, 0), C(5, 0), G(63/11, -8/11) and F(0, -4).

They correspond to the vertices of the polyhedron.

4. $F(0,-4)$ corresponds to the solution $(x_1, x_2', e_1, e_2, e_3) = (0,4,0,0,9)$. It is a degenerate basic feasible solution. Indeed, three variables x_1, e_1 and e_2 are zero. This is strictly greater then $n - m = 5 - 3 = 2$. At the vertex $F(0,-4)$, three constraints are active: (1), (2) and (5). Activating two of them is sufficient to identify the vertex. The third one corresponds to a basic variables, that happens to be zero, as the corresponding constraint is active. This point corresponds to the three following bases:

$$B = \begin{pmatrix} 4/7 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

5. When considering the problem in standard form, the point $(2,-2)$ corresponds to the point $(x_1,\ x_2',\ e_1,\ e_2,\ e_3) = (2,2,6/7,2,5)$. This is a feasible solution because all the constraints are satisfied.

   However, in a basic solution at least n constraints must be active. In standard form, $n = 5$. There are $m = 3$ equality constraints that are active. Therefore, at least two variables (the non basic variables) should be zero at a basic solution. It is not the case for the point $(2,-2)$, so it is not a basic solution.

6. Constraints (2) and (5) are active at point $F(0,-4)$ which corresponds to the solution $(x_1, x_2', e_1, e_2, e_3) = (0,4,0,0,9)$ of the problem in standard form.

   (a) The corresponding basis matrix is

   $$B = (x_2', e_1, e_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

   (b) The basic direction corresponding to the non basic variable x_1 is

   $$d_1 = P \begin{pmatrix} -B^{-1}A_1 \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{7} \\ 2 \\ 1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} -2 \\ \frac{10}{7} \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ \frac{10}{7} \\ 0 \\ -3 \end{pmatrix}.$$

This direction is feasible since for x_1, e_1 and e_2 the corresponding $d_{1i} \geq 0$.

The basic direction corresponding to the non basic variable e_2 is

$$d_2 = P \begin{pmatrix} -B^{-1}A_4 \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

This direction is not feasible since $e_1 = 0$ and $d_{1e1} = -1 < 0$.

Question 20

Consider a linear optimization problem. Which of the following statements is false?

(a) Several basic solutions may correspond to the same vertex of the feasible region.

(b) For each feasible basic solution, there exists at least one feasible basic direction.

(c) If a feasible basic solution is non degenerate, all the corresponding basic directions are feasible.

Solution 20

Answer: (b).

Answer (a) is true. In case of a degenerate solution, several basic solutions may correspond to the same vertex of the feasible region. Answer (c) is also true. Basic directions associated with non degenerate feasible basic solutions are always feasible directions. Refer to Theorem 3.44 (p.90) for the proof of this statement. Answer (b) is false. When we have a degenerate basic solution, it is not always true that there exists at least one feasible basic direction. An example is given by the linear optimization problem in standard form characterized by the following data:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The degenerate feasible basic solution $x^T = (0, 0, 0, 2)$ does not have any feasible basic direction. This is actually the only feasible solution of the problem, and therefore also the optimal solution.

Question 21

Which of the following statements about a feasible basic solution is true for a minimization problem?

(a) If $\bar{c} < 0$, that is, all reduced costs are negative, the solution is optimal.

(b) If the values of the basic variables are negative, the solution is optimal.

(c) If $\bar{c}_N > 0$, that is, the reduced costs of the non basic variables are positive, the solution is optimal.

Solution 21

Answer: (c).

Answer (a) is false. If all reduced costs are negative, it means that all basic directions are decreasing. It is not possible at a minimum of the problem in the general case. The only instance where it may happen is when all basic directions are infeasible. Refer to the video on *optimality conditions* for more details. Answer (b) is false because if the values of the basic variables are negative, then the solution is infeasible, since the non negativity constraints are not respected. It cannot therefore be optimal. Answer (c) is correct. Indeed, the reduced costs of the basic variables are zero, so that all reduced costs are non negative (sufficient optimality conditions).

Question 22

Consider a linear optimization problem. Which of the following statements is false?

(a) Several basic solutions may correspond to the same vertex of the feasible region.

(b) For each feasible basic solution, there exists at least one feasible basic direction.

(c) If a feasible basic solution is non degenerate, all the associated basic directions are feasible.

Solution 22

Answer: (b).

Answer (a) is true. In case of a degenerate solution, several basic solutions may correspond to the same vertex of the feasible region. Answer (c) is also true. Basic directions associated with non degenerate feasible basic solutions are always feasible directions. Refer to Theorem 3.44 (p.90) for the proof of this statement. Answer (b) is false. When we have a degenerate basic solution, it is not always true that there exists at least one feasible basic direction. An example is given by the following optimization problem in standard form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The degenerate feasible basic solution $x^T = (0, 0, 0, 2)$ does not have any feasible basic direction. This is the only feasible solution of the problem, and therefore also the optimal solution.

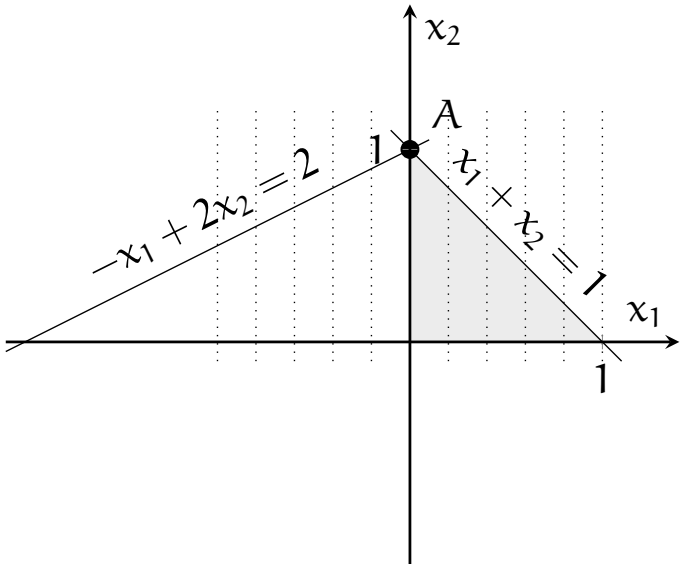
Question 23

Consider the linear optimization problem

$$\min x_1$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 1, \\ -x_1 + 2x_2 &\leq 2, \\ x_1, x_2 &\geq 0. \end{aligned}$$



In standard form, after introducing the two slack variables x_3 and x_4, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ -x_1 + 2x_2 + x_4 &= 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

The vertex A corresponds to $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, and is degenerate. Consider the basis where the variables x_1 and x_2 are in the basis.

1. Calculate all basic directions and represent them on the picture. Calculate the slope of the objective function along each of them.

2. Calculate all the reduced costs.

Solution 23

The problem in matrix form is

$$\min_x c^T x$$

subject to

$$Ax = b,$$

where

$$A \in \mathbb{R}^{2\times 4},\ b \in \mathbb{R}^2,\ c \in \mathbb{R}^4,\ x \in \mathbb{R}^4,$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have a problem with $n = 4$ variables and $m = 2$ equality constraints. We consider the basic solution where x_1 and x_2 are in the basis:

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

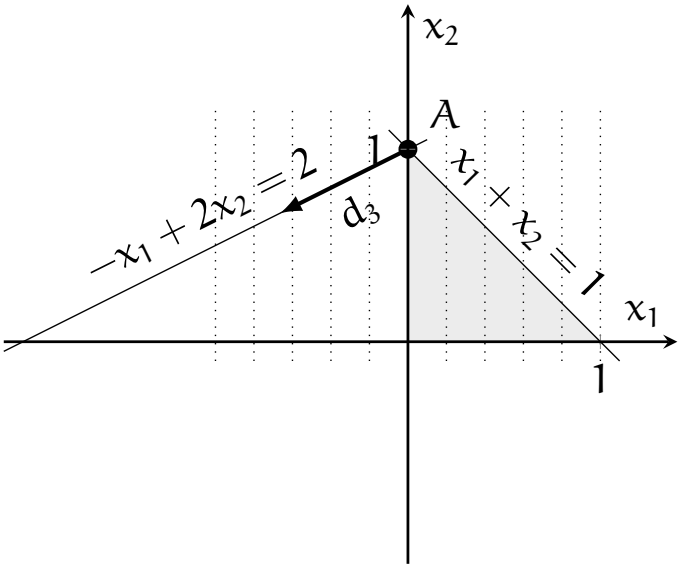
The basic entries of the basic direction corresponding to the non basic variable x_3 is calculated as

$$-B^{-1}A_3 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Therefore, the basic direction is

$$d_3 = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}.$$

It is represented in the following figure.



The slope of the objective function is given by the directional derivative. As

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we have

$$d_3^T c = -\frac{2}{3}.$$

It is negative. It means that the objective function is decreasing along that direction. The reduced cost is

$$\bar{c}_3 = c_3 - c_B^T B^{-1} A_3 = 0 - (1\ 0) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{2}{3},$$

and we confirm that it is indeed the slope of the function in the basic direction.

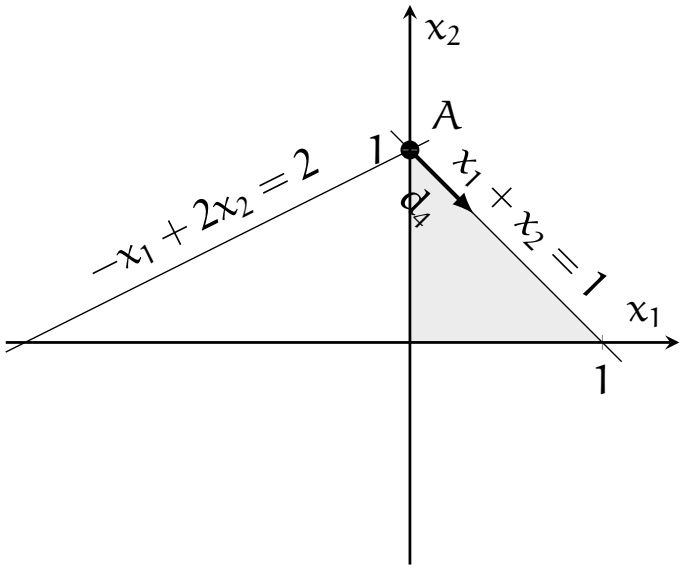
The basic entries of the basic direction corresponding to the non basic variable x_4 is calculated as

$$-B^{-1} A_4 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Therefore, the basic direction is

$$d_4 = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}.$$

It is represented in the following figure.



The slope of the objective function is given by the directional derivative. As

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we have

$$d_4^T c = \frac{1}{3}.$$

It is positive. It means that the objective function is increasing along that direction. The reduced cost is

$$\bar{c}_4 = c_4 - c_B^T B^{-1} A_4 = 0 - (1\ 0) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3},$$

and we confirm that it is indeed the slope of the function in the basic direction.

We have calculated the reduced costs for the two non basic variables. We now need to calculate them for the basic variables.

$$\bar{c}_1 = c_1 - c_B^T B^{-1} A_1 = 1 - (1\ 0) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0,$$

and

$$\bar{c}_2 = c_2 - c_B^T B^{-1} A_2 = 0 - (1\ 0) \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0.$$

This confirms the fact that the reduced costs of basic variables is always zero.

Question 24

Consider the optimization problem

$$\min x_1$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ -x_1 + 2x_2 + x_4 &= 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Consider the basis where the variables x_1 and x_2 are in the basis, and calculate all reduced costs for that basis. Remember that the reduced cost associated with a non basic variable x_p is defined as

$$\bar{c}_p = c_p - c_B^T B^{-1} A_p,$$

where c_B are the entries of the vector c correpsonding to the basic variables, and A_p is the pth column of A.

Solution 24

First, we note that the reduced costs corresponding to the basic variables are zero. Therefore, $\bar{c}_1 = \bar{c}_2 = 0$.

The reduced cost for the non basic variable x_3 is defined as

$$\bar{c}_3 = c_3 - c_B^T B^{-1} A_3 = c_3 + c_B^T (d_3)_B,$$

where c_3 is the coefficient of x_3 in the objective function ($c_3 = 0$), c_B are the entries of the cost vector corresponding to the basic variables:

$$c_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $(d_3)_B$ are the basic entries of the basic direction d_3:

$$(d_3)_B = -B^{-1} A_3 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Therefore,

$$\bar{c}_3 = c_B^T (d_3)_B = -\frac{2}{3}.$$

Note that it is negative. It means that the objective function is decreasing along that direction. Each step of size 1 into this direction will decrease the value of the objective function by $2/3$.

The reduced cost for the non basic variable x_4 is defined as

$$\bar{c}_4 = c_4 - c_B^T B^{-1} A_4 = c_4 + c_B^T (d_4)_B,$$

where c_4 is the coefficient of x_4 in the objective function ($c_4 = 0$), c_B are the entries of the cost vector corresponding to the basic variables:

$$c_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $(d_4)_B$ are the basic entries of the basic direction d_4:

$$(d_4)_B = -B^{-1} A_4 = -\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Therefore,

$$\bar{c}_4 = c_B^T (d_4)_B = \frac{1}{3}.$$

Note that it is positive. It means that the objective function is increasing along that direction. Each step of size 1 into this direction will increase the value of the objective function by $1/3$.

As a final note, it happens that the point $(0, 1, 0, 0)$ is an optimal solution of the problem. Indeed, it is feasible, and x_1 is at its minimum value. Still, not all reduced costs are non negative. But the direction d_3 corresponding to the negative reduced cost is infeasible (as seen in the previous exercise). It is an example that shows that the condition "all reduced costs non negative" is a sufficient condition for optimality, but not a necessary condition. In particular, Theorem 6.30 does not apply as the feasible basic solution is degenerate.

Question 25

Consider the polyhedron

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1 - x_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \right\}.$$

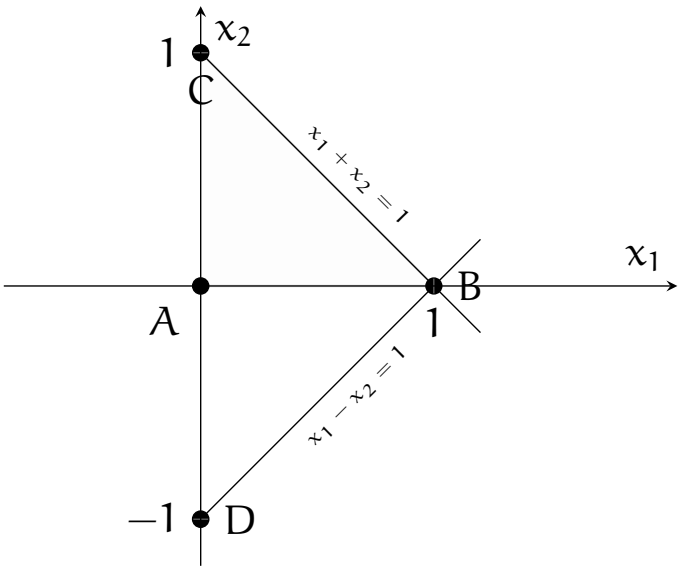
1. Represent it graphically.
2. Identify the corresponding polyhedron in standard form.
3. Enumerate the feasible basic solutions, and identify the corresponding vertex on the polyhedron.
4. For each feasible basic solution, mention if it is degenerate or not.

Solution 25

We represent the polyhedron

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1 - x_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \right\}.$$

in the following figure:



The corresponding polyhedron in standard form is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 | Ax = b, x \geq 0 \right\},$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with the variables x_3 and x_4 being slack variables. Each basic solution is obtained by selecting 2 variables out of 4 to be in the basis. There is total of 6 possible selections of basic variables.
The matrix $B = (A_{j1}, ..., A_{jm})$ is composed of columns $j_1, ..., j_m$ of the matrix A.

1. Basic solution with x_1 and x_2 in the basis $(j_1 = 1, j_2 = 2)$.

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   This basic solution is feasible and corresponds to point $B = (1, 0)$ in the figure.

2. Basic solution with x_1 and x_3 in the basis $(j_1 = 1, j_2 = 3)$.

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   This basic solution is feasible and also corresponds to point $B = (1, 0)$.

3. Basic solution with x_1 and x_4 in the basis $(j_1 = 1, j_2 = 4)$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   This basic solution is feasible and corresponds to point $B = (1, 0)$.

4. Basic solution with x_2 and x_3 in the basis $(j_1 = 2, j_2 = 3)$.

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}.$$

   This basic solution is not feasible because $B^{-1}b \ngeq 0$. It corresponds to point $D = (0, -1)$ in the figure.

5. Basic solution with x_2 and x_4 in the basis $(j_1 = 2, j_2 = 4)$

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

   This basic solution is feasible and corresponds to point $C = (0, 1)$ in the figure.

6. Basic solution with x_3 and x_4 in the basis $(j_1 = 3, j_2 = 4)$

$$B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

   This basic solution is feasible and corresponds to point $A = (0, 0)$ in the figure.

In summary, out of the six basic solutions, three are degenerate, and they correspond to the same vertex, identified by five active constraints: two equality constraints, and three inequality constraints: $x_2 \geq 0$, $x_3 \geq 0$, and $x_4 \geq 0$. In the original polyhedron Q, among the six basic solutions,

- one corresponds to the vertex A, where the two constraints $x_1 \geq 0$ and $x_2 \geq 0$, associated with the non basic variables, are active,
- three correspond to the vertex B, where the **three** constraints $x_2 \geq 0$, $x_1 + x_2 \geq 1$, and $x_1 - x_2 \geq 1$ are active,
- one corresponds to the vertex C, where the two constraints $x_1 \geq 0$ and $x_1 + x_2 \leq 0$, associated with the non basic variables, are active,
- one is infeasible, and corresponds to point D, where the two constraints $x_1 \geq 0$ and $x_1 - x_2 \leq 1$ are active.

The three basic solutions associated with vertex B are degenerate. From an algebraic point of view, one of the basic variables is equal to zero. From a geometrical point of view, more than two constraints are active.

3 The simplex algorithm

Question 01

Consider the following linear optimization problem:

$$\min -2x_1 + 2x_2$$

subject to:

$$x_1 + x_2 \leq 5, \tag{4}$$
$$-x_1 + 2x_2 \leq 3, \tag{5}$$
$$x_1 \geq 0, \tag{6}$$
$$x_2 \geq 0. \tag{7}$$

1. Represent the feasible region graphically.
2. Identify the optimal solution using the graphical method.
3. Solve the problem using the simplex algorithm.

Solution 01

1. The feasible region is shown in gray in Figure 6.

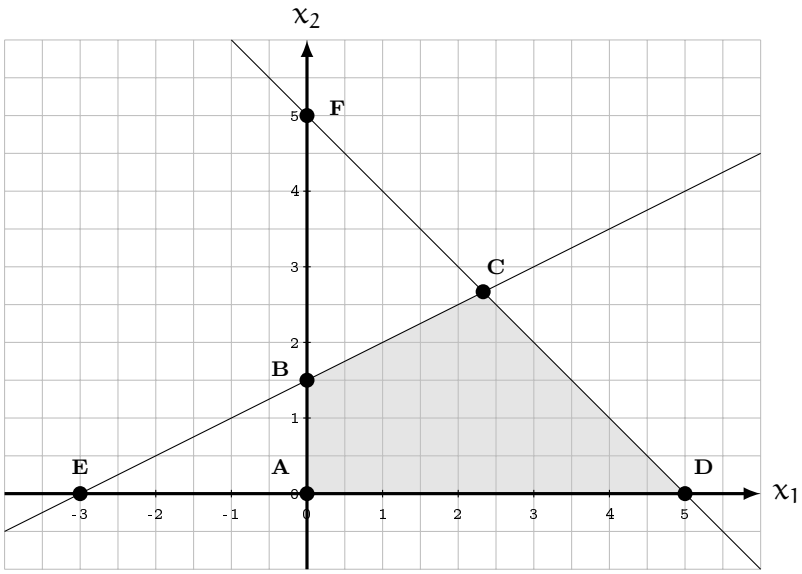


Figure 6: Feasible domain

2. We have to add the level lines corresponding to different values of the objective function. We start by drawing an arbitrary level line intersecting the feasible domain, say $f = -2x_1 + 2x_2 = 0$. The gradient corresponds to the steepest ascent direction. As we minimize, we need to identify the steepest descent direction, that is

$$-\nabla f(x) = -c = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

We move this level line parallel to itself as far as possible in the direction of $-c$ as long as it intersects the feasible domain. The minimum value that the objective function can take is $f = -10$. The optimal solution corresponds to vertex D $(5, 0)$, as shown in Figure 7.

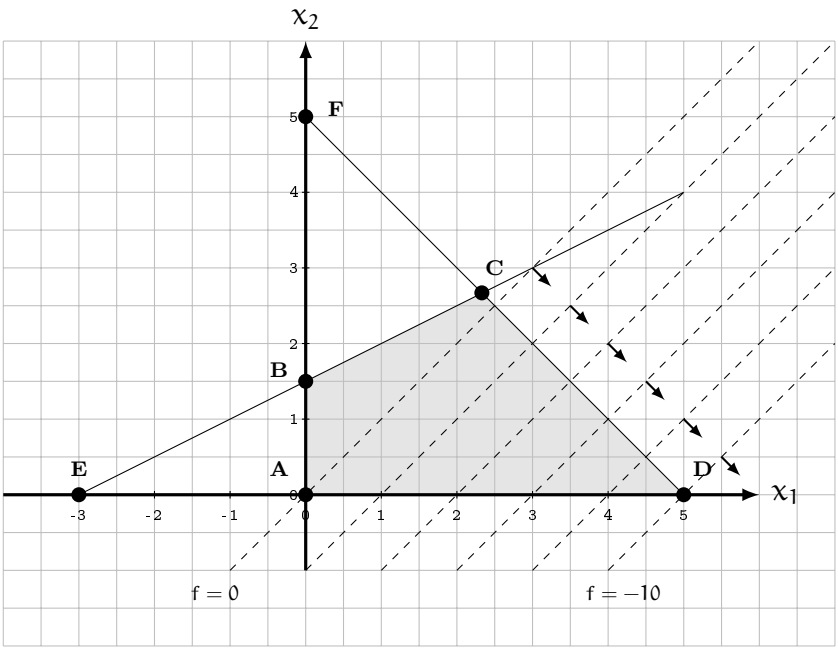


Figure 7: Graphical method

3. To solve the problem using the simplex method, we first put the problem in standard form. In order to do this, we introduce the slack variables x_3 and x_4 to get the following formulation:

$$\min -2x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ -x_1 + 2x_2 + x_4 &= 3, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

We can then identify the matrix A, the vector b and the vector c:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

Before we begin the optimization procedure, we need to find a starting solution. As $b \geq 0$, selecting the slack variables to be in the basis leads to a feasible basic solution. Therefore, $x_1 = x_2 = 0$ can be used as a starting point.

The initial basic indices are

$$J^0 = (3, 4).$$

Iteration 1

Let's consider the vertex where all slack variables are in the basis. The corresponding basic matrix is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is the identity matrix. Therefore, $B^{-1} = I$ as well. As the basic variables are slack variables, they do not appear in the objective function, and $c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Consequently, the objective value is $c_B^T x_B = 0$. The reduced costs for the non basic variables are

$$\begin{array}{rcccl} \bar{c}_1 & = & c_1 - c_B^T B^{-1} A_1 & = & -2, \\ \bar{c}_2 & = & c_2 - c_B^T B^{-1} A_2 & = & 2, \end{array}$$

where A_1, A_2 are the first and second columns of the matrix A and c_B is the part of vector c corresponding to the basic indices.

x_1 is the only non basic variable associated with a negative reduced cost. Therefore, it enters the basis. We have:

$$x_B = B^{-1} b = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad x = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 3 \end{pmatrix}.$$

The basic entries of the basic direction are

$$d_B = -B^{-1} A_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

so that the basic direction is

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

We calculate the distances to the constraint:

$$\begin{aligned} \alpha_3 &= -5/(-1) = 5 \\ \alpha_4 &= +\infty \end{aligned}$$

The smallest value is $\alpha_3 = 5$. Therefore, x_3 exits the basis. The new set of basic indices is therefore

$$J^1 = (1, 4).$$

Iteration 2

The basic matrix is

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We calculate the reduced costs, using the fact that

$$c_B = \begin{pmatrix} c_1 \\ c_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T B^{-1} A_2 = 2 - (-2, 0) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4, \\ \bar{c}_3 &= c_3 - c_B^T B^{-1} A_3 = 0 - (-2, 0) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2. \end{aligned}$$

As both reduced costs are non negative, we have identified an optimal vertex. We have

$$x_b = B^{-1} b = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \quad x^* = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 8 \end{pmatrix},$$

with optimal objective value

$$c^T x^* = -10.$$

Note that it is the same solution as identified by the graphical method.

Question 02

Consider a linear minimization problem with four variables:

$$x = (x_1, x_2, x_3, x_4)^T.$$

In the current iteration of the simplex algorithm, we obtain the reduced costs vector $\bar{c} = (-1, 0, 0, 0)^T$ and the current vector of basic variables is given by $x_B = B^{-1}b = (2, 4, 3)^T$. Which of the following propositions is certainly false?

(a) x_1 enters the basis in the following iteration of the simplex.

(b) x_3 is a basic variable.

(c) The current solution is not feasible.

Solution 02

Answer: (c).

$B^{-1}b$ has size 3, so there must be $m = 3$ basic variables. The size of x is $n = 4$, so we know that there is one non basic variable. The non basic variable is x_1, since the corresponding reduced cost is different from 0. As a consequence, x_2, x_3 and x_4 are all basic variables and answer (b) is true. Answer (a) is also true, since the reduced cost associated with x_1 is the only negative reduced cost in the current iteration. Answer (c) is false. Indeed, the current solution is feasible, since all basic variables are non negative.

Question 03

Consider the following optimization problem:

$$\min_{x \in \mathbb{R}^2} 3x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 &\geq 1, \\ x_1 + x_2 &\leq 2, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

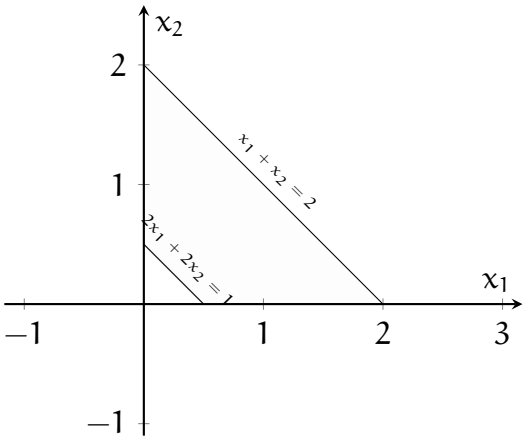
1. Draw the constraints of the problem.
2. Find all the basic solutions and, for each one, mention if it is feasible or not.
3. Find the optimal solution by vertex enumeration.

Solution 03

The constraint polyhedron is

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | 2x_1 + 2x_2 \geq 1, x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0 \right\},$$

and is represented in the following figure.



The problem in standard form is

$$\min_{x \in \mathbb{R}^4} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} -2 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \; c = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

with the variables x_3 and x_4 being slack variables. Each basic solution is obtained by selecting $m = 2$ variables out of $n = 4$ to be in the basis. There is total of 6 possible selections of basic variables.

1. Consider x_3 and x_4 are in the basis.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

The basic solution is not feasible since $B^{-1}b \not\geqslant 0$. It corresponds to the point $(0,0)$ in the figure, which is not a vertex.

2. Consider x_1 and x_3 are in the basis.

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; x = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the vertex $(2,0)$.

3. Consider x_1 and x_2 are in the basis.

$$B = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix};$$

The matrix B is not invertible. Indeed, $\det(B) = 0$. Geometrically, it happens because two constraints are parallel and, therefore, do not intersect.

4. Consider x_1 and x_4 are in the basis.

$$B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}; x = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the vertex $(1/2,0)$.

5. Consider x_2 and x_3 are in the basis.

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}; B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the vertex $(0,2)$.

6. Consider x_2 and x_4 are in the basis.

$$B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}; B^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1/2 & 1 \end{pmatrix}; x_B = B^{-1}b = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}; x = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 3/2 \end{pmatrix}.$$

The basic solution is feasible and corresponds to the vertex $(0, 1/2)$.

We now calculate the values of the objective function corresponding to the five basic solutions.

k	x_k	f_k
1	$\begin{pmatrix} 0 & 0 & -1 & 2 \end{pmatrix}^T$	$B^{-1}b \ngeq 0$
2	$\begin{pmatrix} 2 & 0 & 3 & 0 \end{pmatrix}^T$	6
3	$\begin{pmatrix} 1/2 & 0 & 0 & 3/2 \end{pmatrix}^T$	$3/2$
4	$\begin{pmatrix} 0 & 2 & 3 & 0 \end{pmatrix}^T$	2
5	$\begin{pmatrix} 0 & 1/2 & 0 & 3/2 \end{pmatrix}^T$	$1/2$

As we are solving a minimization problem, the optimal solution is

$$x^* = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 3/2 \end{pmatrix}.$$

The optimal value of the objective function is $1/2$.

Question 04

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -x_1 - \frac{3}{2}x_2$$

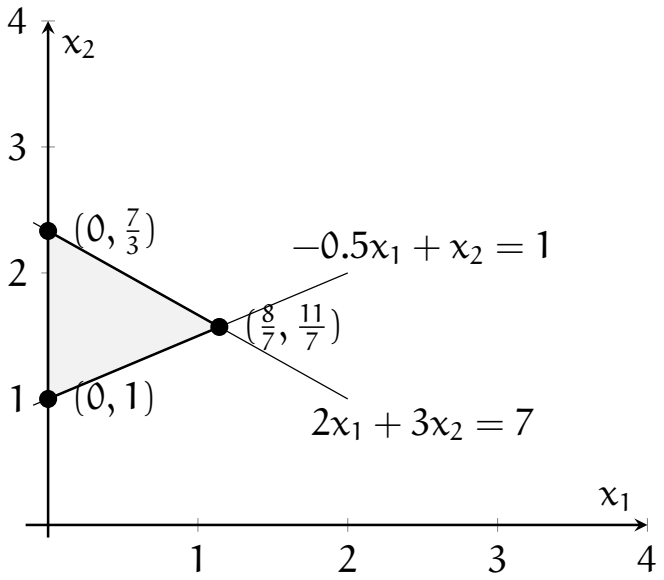
subject to

$$\begin{aligned} -x_1 + 2x_2 &\geq 2, \\ 2x_1 + 3x_2 &\leq 7, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve this problem using the graphical method.

Solution 04

The figure below represents the feasible domain in the space of variables (x_1, x_2).

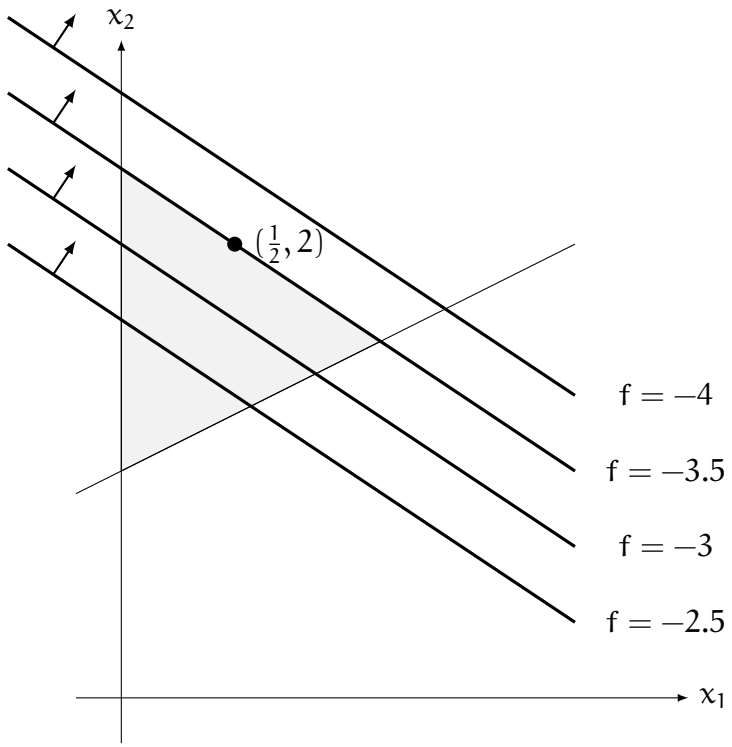


We notice that the constraint $x_2 \geq 0$ can be eliminated. The set of vertices of the polyhedron is $\left\{(0,1), (0, \frac{7}{3}), (\frac{8}{7}, \frac{11}{7})\right\}$.

We now have to add the level lines corresponding to different values of the objective function. We start by drawing an arbitrary level line intersecting the feasible domain, say $f = -2.5$ The gradient corresponds to the steepest ascent direction. As we minimize, we need to identify the steepest descent direction, that is

$$-\nabla f(x) = -c = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

That is displayed with an arrow on each level line in the figure below. Finally, we move this level line parallel to itself as far as possible in the direction of $-c$ as long as it intersects the feasible domain.



The minimum value that the objective function can take is $f = -3.5$. We observe that, in this example, the intersection of the level line with the polyhedron is not a vertex, but a face. It means that all the solutions on that face are optimal, with value -3.5.

For instance, the solution $(\frac{1}{2}, 2)$ is optimal, although it is not a vertex. Note that it does not contradict Theorem 16.2, that says "if there is an optimal solution, there exists an optimal vertex". Indeed, the vertices $(0, \frac{7}{3})$ and $(\frac{8}{7}, \frac{11}{7})$ are also optimal.

Question 05

Consider the following linear optimization problem

$$\max 4x_1 - 3x_2$$

subject to

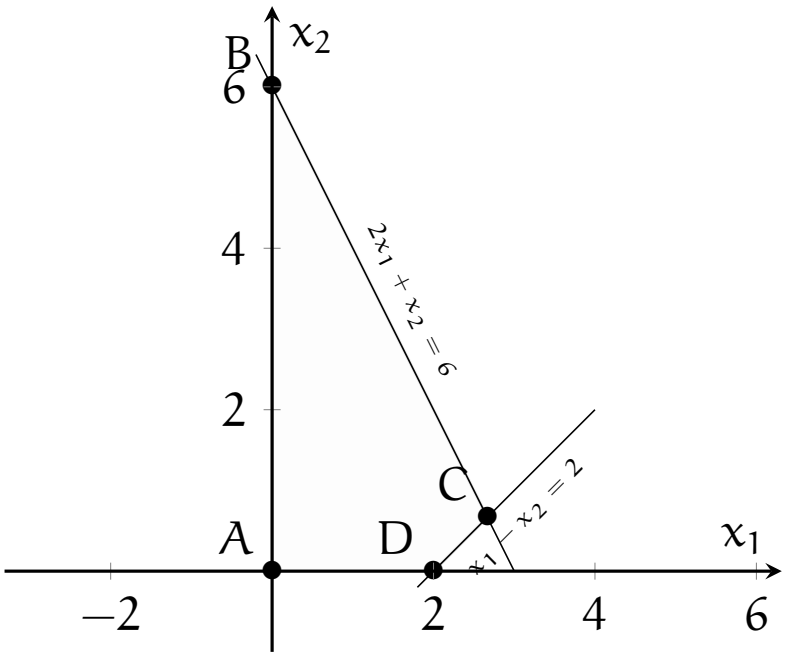
$$\begin{aligned} 2x_1 + x_2 &\leq 6, \\ x_1 - x_2 &\leq 2, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Represent the feasible region graphically.
2. Write the standard form of the problem.
3. Solve the problem with the algebraic simplex method. At each iteration, specify
   (a) the basic variables;
   (b) the current iteration (both analytically and graphically);
   (c) the basic direction imposed by the algorithm (both analytically and graphically);
   (d) the step length along the basic direction.

   Hint: Verify that the point (0, 0) is a basic feasible solution. If so, start the algorithm from this point.
4. Solve the problem using the graphical method, and compare the results.

Solution 05

1. The feasible region is represented in the following figure, where the vertices are labeled A, B, C and D.



2. We introduce two slack variables, x_3 and x_4. The problem in standard form is

$$\min -4x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 6, \\ x_1 - x_2 + x_4 &= 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Note that we had to change the sign of the objective function in order to obtain a minimization problem. It can be written as

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

3. The point $(0,0)^T$ is a feasible basic solution. Indeed, the constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 6, \\ x_1 - x_2 &\leq 2, \\ x_1, x_2 &\geq 0, \end{aligned}$$

are verified at this point. The corresponding constraints in standard form are

$$\begin{aligned} x_3 &= 6, \\ x_4 &= 2. \end{aligned}$$

Therefore, the solution $(0,0,6,2)^T$ is feasible. It is a basic feasible solution, where x_3 and x_4 are in the basis. It corresponds to vertex A in the figure above.

First iteration We first identify the nature of the variables:

- basic variables: x_3 and x_4,
- non-basic variables: x_1 and x_2.

The matrices B and B^{-1} are given by:

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The values of the basic variables are

$$x_B = B^{-1}b = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

so that

$$x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}.$$

The value of the objective function is

$$c_B^T x_B = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 0.$$

We now calculate the reduced costs associated with the non basic variables. If a reduced cost is negative, the corresponding variable is candidate to enter the basis.

$$\begin{aligned} \bar{c}_1 &= c_1 - c_B^T B^{-1} A_1 \\ &= -4 - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -4 - 0 = -4 \\ &< 0 \quad \Rightarrow \quad \underline{\text{candidate}}. \end{aligned}$$

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T B^{-1} A_2 \\ &= 3 - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 3 - 0 = 3 \\ &\geq 0 \quad \Rightarrow \quad \text{not a candidate.} \end{aligned}$$

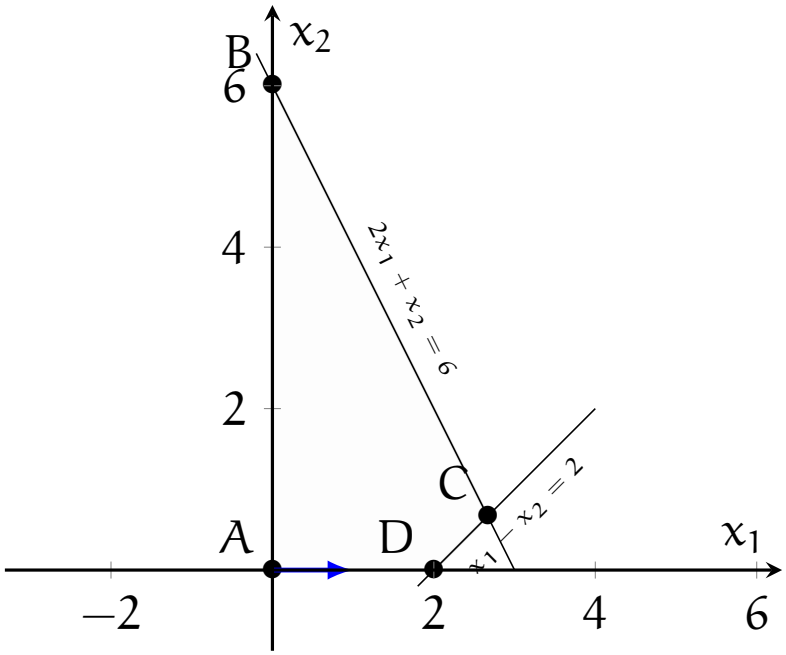
The sufficient optimality conditions are not verified, and x_1 is the only candidate to enter the basis. We calculate the corresponding basic direction. As

$$d_{B_1} = -B^{-1} A_1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix},$$

the direction is

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}.$$

This direction is shown in the following figure.



Therefore, the next iterate is obtained by performing a step of length α in this direction, that is

$$x(\alpha) = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 6 - 2\alpha \\ 2 - \alpha \end{pmatrix}.$$

To calculate the value of α, we need to consider all basic variables corresponding to a negative component of the direction. In this case, both basic variables correspond to a negative component. For each of them, we calculate

$$\alpha_i = \frac{x_i}{-d_i}$$

which gives:

$$\alpha_3 = \frac{x_3}{-d_3} = \frac{6}{-(-2)} = 3,$$
$$\alpha_4 = \frac{x_4}{-d_4} = \frac{2}{-(-1)} = 2.$$

The maximum step that can be made is $\alpha = 2$, so that the next iterate is

$$x(3) = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix},$$

which corresponds to vertex D on the figure. Indeed, D corresponds to the first constraint encountered when following the direction depicted in blue from A. Consequently, x_4 leaves the basis and x_1 becomes a basic variable for the next iteration. Moreover, we can predict that the objective function value decreases by:

$$\alpha \bar{c}_1 = 2 \cdot (-4) = -8,$$

so that the new value of the objective function is expected to be $0 - 8 = -8$.

Second iteration The nature of the variables:

- basic variables: x_1 and x_3,
- non-basic variables: x_2 and x_4.

The matrices B and B^{-1} are given by:

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

The values of the basic variables are

$$x_B = B^{-1}b = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

so that

$$x = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

as anticipated in the previous iteration. It corresponds to the vertex D in the figure.
The value of the objective function is

$$c_B^T x_B = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -8,$$

as anticipated at the end of the first iteration. We now calculate the reduced costs associated with the non basic variables. If a reduced cost is negative, the corresponding variable is candidate to enter the basis.

$$\begin{aligned} \bar{c}_2 &= c_2 - c_B^T B^{-1} A_2 \\ &= 3 - \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 3 - 4 = -1 \\ &< 0 \quad \Rightarrow \quad \underline{\text{candidate}}. \end{aligned}$$

$$\begin{aligned} \bar{c}_4 &= c_4 - c_B^T B^{-1} A_4 \\ &= 0 - \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 0 - (-4) = 4 \\ &\geq 0 \quad \Rightarrow \quad \text{not a candidate.} \end{aligned}$$

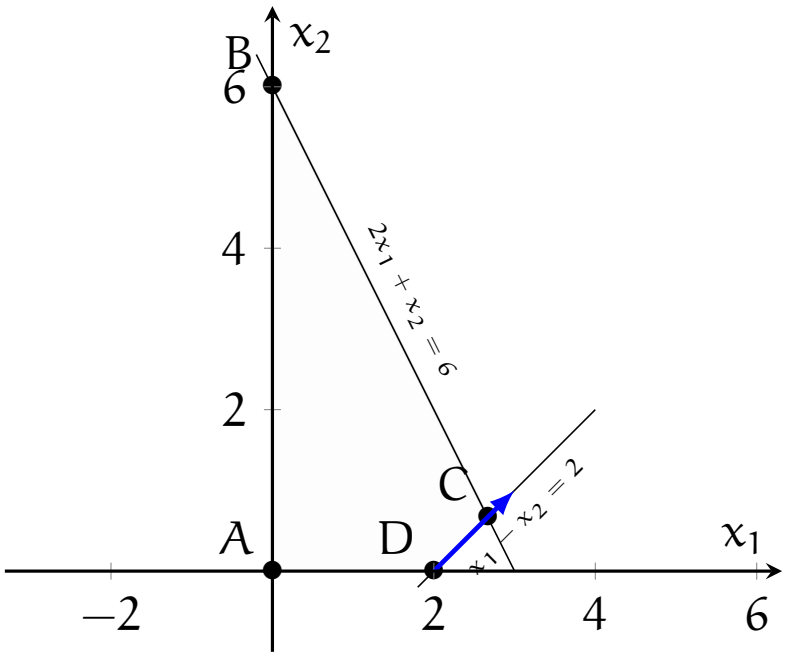
The sufficient optimality conditions are not verified, and x_2 is the only candidate to enter the basis. We calculate the corresponding basic direction. As

$$d_{B_2} = -B^{-1} A_2 = - \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_3 \end{pmatrix},$$

the direction is

$$d_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix}.$$

This direction is shown in the following figure.



Therefore, the next iterate is

$$x(\alpha) = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+\alpha \\ \alpha \\ 2-3\alpha \\ 0 \end{pmatrix}.$$

To calculate the value of α, we need to consider all basic variables corresponding to a negative component of the direction. In this case, only one basic variable corresponds to a negative component.

$$\alpha_3 = \frac{x_3}{-d_3} = \frac{2}{-(-3)} = \frac{2}{3}.$$

The maximum step that can be made is $\alpha = \frac{2}{3}$, so that the next iterate is

$$x(2/3) = \begin{pmatrix} 8/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix},$$

which corresponds to vertex C on the figure. Indeed, C corresponds to the first constraint encountered when following the direction depicted in blue from D.

Consequently, x_3 leaves the basis and x_2 becomes a basic variable for the next iteration. Moreover, we can predict that the objective function value decreases by:

$$\alpha_3 \bar{c}_2 = \frac{2}{3} \cdot (-1) = -\frac{2}{3},$$

so that the new value of the objective function is expected to be $-8 - 2/3 = -26/3$.

Third iteration The nature of the variables:

- basic variables: x_1 and x_2,
- non-basic variables: x_3 and x_4.

The matrices B and B^{-1} are given by:

$$B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

The values of the basic variables are

$$x_B = B^{-1} b = \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix},$$

so that

$$x = \begin{pmatrix} 8/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix}$$

as anticipated in the previous iteration. It corresponds to the vertex C in the figure.

We calculate the objective function value as follows:

$$c_B^T x_B = \begin{pmatrix} -4 & 3 \end{pmatrix} \begin{pmatrix} 8/3 \\ 2/3 \end{pmatrix} = -\frac{26}{3},$$

as anticipated at the end of the previous iteration.

We now calculate the reduced costs associated with the non basic variables. If a reduced cost is negative, the corresponding variable is candidate to enter the basis.

$$\begin{aligned}
\bar{c}_3 &= c_3 - c_B^T B^{-1} A_3 \\
&= 0 - \begin{pmatrix} -4 & 3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= 0 - \frac{1}{3}(-1) = \frac{1}{3} \\
&\geq 0 \quad \Rightarrow \quad \text{not a candidate.}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_4 &= c_4 - c_B^T B^{-1} A_4 \\
&= 0 - \begin{pmatrix} -4 & 3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= 0 - \frac{1}{3}(-10) = \frac{10}{3} \\
&\geq 0 \quad \Rightarrow \quad \text{not a candidate.}
\end{aligned}$$

Since the reduced costs of the two non-basic variables are positive, the sufficient optimality conditions are verified, and an optimum has been reached. Therefore, the solution to the problem

$$\min -4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + x_3 = 6 \tag{8}$$

$$x_1 - x_2 + x_4 = 2 \tag{9}$$

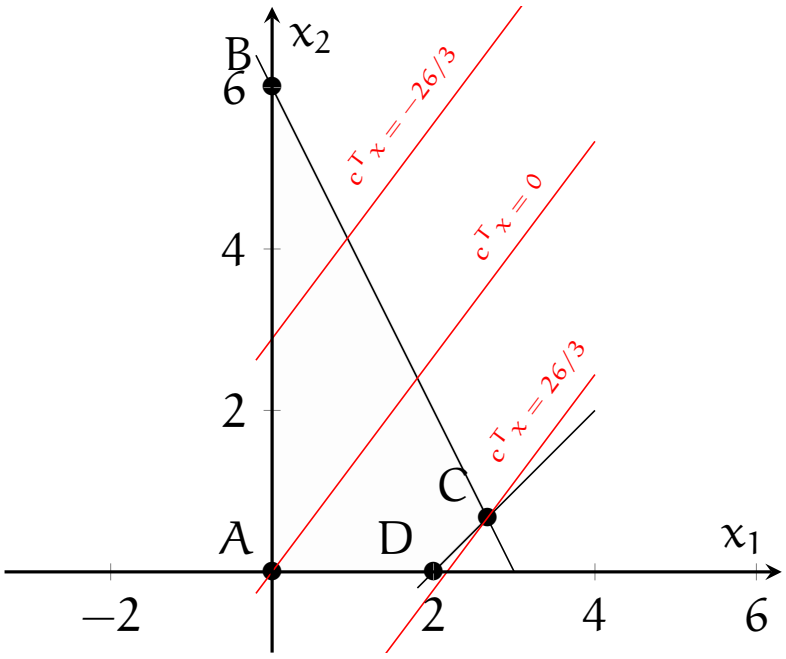
$$x_1, x_2, x_3, x_4 \geq 0$$

is given by

$$x^* = \begin{pmatrix} 8/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix}$$

and the objective function value is $-\frac{26}{3}$.

4. Finally, we verify that we obtain the same solution using the graphical method. The level lines are represented in the figure below. Note that the original problem is a maximization problem, so that we follow the level lines with increasing values, and the level line at the optimal solution is at value 26/3.



Question 06

Consider the optimization problem in standard form characterized by the following data:

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}; b = \begin{pmatrix} 2 \\ 6 \end{pmatrix}; c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Consider as well the following tableau:

x_1	x_2	x_3	x_4	
1	-1	1	0	2
0	2	-1	1	4
0	-3	2	0	4

1. What are the basic variables associated with this tableau?
2. What is the basic matrix B?
3. Verify that the tableau is valid, in the sense that it corresponds to its definition.
4. What is the variable corresponding to the second row of the tableau?
5. What is the vertex x corresponding to this tableau?
6. What is the value of the objective function at that vertex?
7. What are the reduced costs of the non basic variables?
8. What are the basic directions?
9. Verify that the values of the reduced costs in the tableau are consistent with the basic directions.

Solution 06

x_1	x_2	x_3	x_4	
1	-1	1	0	2
0	2	-1	1	4
0	-3	2	0	4

1. The tableau has two rows. Therefore, there are two basic variables. The columns corresponding to the basic variables are composed of zeros, except one "one". In this case, the basic variables are x_1 and x_4.

2. The basic matrix B contains columns 1 and 4 of the matrix A:

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

   Its inverse is

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

3. The definition of the tableau is

$B^{-1}A$	$B^{-1}b$
$c^T - c_B^T B^{-1} A$	$-c_B^T B^{-1} b$

   - For the top-left part, we have

$$B^{-1}A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix}.$$

   - For the top-right part, we have

$$B^{-1}b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

   - For the bottom-left, we need to identify c_B, the basic entries of the cost vector. Here, entries 1 and 4. Therefore,

$$c_B = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Therefore, we have

$$c^T - c_B^T B^{-1} A = (-2,\ -1,\ 0,\ 0) - (-2,\ 0) \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix} = (0,\ -3,\ 2,\ 0).$$

- Finally, for the bottom-right part, we have

$$-c_B^T B^{-1} b = -(-2\ 0) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4.$$

4. The position of the "one" in the column of a basic variable identifies the corresponding row. Therefore, x_1 corresponds to row 1, and x_4 to row 2.

5. The non basic variables are zero: $x_2 = x_3 = 0$. The values of the basic variables are found in the top-right part of the tableau: $x_1 = 2$ and $x_4 = 4$. Therefore,

$$x = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}.$$

6. The value of the objective function is found in the bottom-right part of the tableau, with a different sign. Therefore, it is $c^T x = -4$. It can actually be verified, as c=

$$\begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

we have $c^T x = -4 + 0 + 0 + 0 = -4$.

7. The reduced costs are found in the bottom-left part of the tableau. The reduced cost for x_2 is -3. It is negative, meaning that the objective function decreases along the corresponding basic direction. The reduced cost for x_3 is 2. It is positive, meaning that the objective function increases along the corresponding basic direction.

8. The basic entries of the basic directions are found in the top-left part of the tableau, with a different sign. Therefore,

$$(d_2)_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \; d_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix},$$

and

$$(d_3)_B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; d_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

9. If we calculate the inner product between the basic directions and the gradient of the objective function, that is the vector c, we obtain the directional derivatives of the objective function, that is the reduced costs:

$$c^T d_2 = -2 - 1 = -3,$$

and

$$c^T d_3 = 2 + 0 = 2.$$

Those are the values found in the tableau.

Question 07

Consider a linear optimization problem in standard form, with the following data:

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 20 \\ 20 \\ 20 \end{pmatrix}, \; c = \begin{pmatrix} -10 \\ -12 \\ -12 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

1. Calculate the tableau corresponding to the basis composed of x_4, x_1 and x_6 (in this order).
2. Pivot the tableau to include x_2 in the basis, and remove x_6.
3. Verify that the last row indeed corresponds to its definition, that is the reduced costs for the left part, and the value of the objective function (with the opposite sign) for the right part.

Remember that the tableau is defined as

$B^{-1}A$	$B^{-1}b$
$c^T - c_B^T B^{-1} A$	$-c_B^T B^{-1} b$

where c_B is the vector containing the entries of the cost vectors corresponding to the basic variables.

Remeber as well that the pivoting algorithm modifies the cell of the tableau ot row i and column k as follows:

$$\tilde{T}(i,k) := T(i,k) - T(i,p)\frac{T(q,k)}{T(q,p)},$$

where q is the row of the pivot, q is the column of the pivot, and $T(q,p)$ is the value of the pivot.

Solution 07

1. As x_4, x_1 and x_6 (in this order) are in the basis, the basic matrix is

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \text{ and } B^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

The tableau is

$B^{-1}A$	$B^{-1}b$
$c^T - c_B^T B^{-1} A$	$-c_B^T B^{-1} b$

As

$$c_B = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix},$$

the tableau is

x_1	x_2	x_3	x_4	x_5	x_6	
0	1.5	1	1	-0.5	0	10
1	0.5	1	0	0.5	0	10
0	1	-1	0	-1	1	0
0	-7	-2	0	5	0	100

2. We want to remove the variable x_6 from the basis. It corresponds to row 3 of the tableau. And we want to include variable x_2 in the basis, corresponding to column 2. Therefore, the pivot is on row 3 and column 2, as identified in the tableau below.

x_1	x_2	x_3	x_4	x_5	x_6	
0	1.5	1	1	-0.5	0	10
1	0.5	1	0	0.5	0	10
0	(1)	-1	0	-1	1	0
0	-7	-2	0	5	0	100

The input of the pivoting algorithm is:

- The simplex tableau T.

- The index $p = 2$ of the pivot column , i.e., the column corresponding to the non basic variable that enters the basis.
- The index $q = 3$ of the pivot row, i.e., the row corresponding to the basic variable $x_{j_3} = x_6$ that leaves the basis.

The output will be the updated simplex tableau $\tilde{T}$, corresponding to the new basis. The calculation of each term is given by:

$$\tilde{T}(i,k) := T(i,k) - T(i,p)\frac{T(q,k)}{T(q,p)}$$

for $k = 1, ..., n+1$.

Row 1 The formula for row 1 is

$$\tilde{T}(1,k) := T(1,k) - T(1,p)\frac{T(3,k)}{T(3,2)} = T(1,k) - T(1,2)\frac{T(3,k)}{1}.$$

Therefore, we have

Column 1

$$\tilde{T}(1,1) := T(1,1) - T(1,2)\frac{T(3,1)}{1} = 0 - 1.5\frac{0}{1} = 0.$$

Column 2

$$\tilde{T}(1,2) := T(1,2) - T(1,2)\frac{T(3,2)}{1} = 1.5 - 1.5\frac{1}{1} = 0.$$

Note that obtaining a zero here is an objective of the pivoting.

Column 3

$$\tilde{T}(1,3) := T(1,3) - T(1,2)\frac{T(3,3)}{1} = 1 - 1.5\frac{-1}{1} = 2.5.$$

Column 4

$$\tilde{T}(1,4) := T(1,4) - T(1,2)\frac{T(3,4)}{1} = 1 - 1.5\frac{0}{1} = 1.$$

Column 5

$$\tilde{T}(1,5) := T(1,5) - T(1,2)\frac{T(3,5)}{1} = -0.5 - 1.5\frac{-1}{1} = 1.$$

Column 6

$$\tilde{T}(1,6) := T(1,6) - T(1,2)\frac{T(3,6)}{1} = 0 - 1.5\frac{1}{1} = -1.5.$$

Column 7

$$\tilde{T}(1,7) := T(1,7) - T(1,2)\frac{T(3,7)}{1} = 10 - 1.5\frac{0}{1} = 10.$$

Row 2 The formula for row 2 is

$$\tilde{T}(2,k) := T(2,k) - T(2,2)\frac{T(3,k)}{1}.$$

Therefore, we have

Column 1

$$\tilde{T}(2,1) := T(2,1) - T(2,2)\frac{T(3,1)}{1} = 1 - 0.5\frac{0}{1} = 1.$$

Column 2

$$\tilde{T}(2,2) := T(2,2) - T(2,2)\frac{T(3,2)}{1} = 0.5 - 0.5\frac{1}{1} = 0.$$

Note that obtaining a zero here is an objective of the pivoting.

Column 3

$$\tilde{T}(2,3) := T(2,3) - T(2,2)\frac{T(3,3)}{1} = 1 - 0.5\frac{-1}{1} = 1.5.$$

Column 4

$$\tilde{T}(2,4) := T(2,4) - T(2,2)\frac{T(3,4)}{1} = 0 - 0.5\frac{0}{1} = 0.$$

Column 5

$$\tilde{T}(2,5) := T(2,5) - T(2,2)\frac{T(3,5)}{1} = 0.5 - 0.5\frac{-1}{1} = 1.$$

Column 6

$$\tilde{T}(2,6) := T(2,6) - T(2,2)\frac{T(3,6)}{1} = 0 - 0.5\frac{1}{1} = -0.5.$$

Column 7

$$\tilde{T}(2,7) := T(2,7) - T(2,2)\frac{T(3,7)}{1} = 10 - 0.5\frac{0}{1} = 10.$$

Row 3 This is the row of the pivot. We simply need to divide all entries by the value of the pivot:

$$\tilde{T}(3,k) := \frac{T(3,k)}{T(3,2)} = \frac{T(3,k)}{1}.$$

In this case, as the value of the pivot is 1, the row if the pivot is not modified.

Row 4 The formula for row 4 is

$$\tilde{T}(4,k) := T(4,k) - T(4,2)\frac{T(3,k)}{T(3,2)}.$$

Therefore, we have

Column 1

$$\tilde{T}(4,1) := T(4,1) - T(4,2)\frac{T(3,1)}{T(3,2)} = 0 - (-7)\frac{0}{1} = 0.$$

Column 2

$$\tilde{T}(4,2) := T(4,2) - T(4,2)\frac{T(3,2)}{T(3,2)} = -7 - (-7)\frac{1}{1} = 0.$$

Note that obtaining a zero here is an objective of the pivoting.

Column 3

$$\tilde{T}(4,3) := T(4,3) - T(4,2)\frac{T(3,3)}{T(3,2)} = -2 - (-7)\frac{-1}{1} = -9.$$

Column 4

$$\tilde{T}(4,4) := T(4,4) - T(4,2)\frac{T(3,4)}{T(3,2)} = 0 - (-7)\frac{0}{1} = 0.$$

Column 5

$$\tilde{T}(4,5) := T(4,5) - T(4,2)\frac{T(3,5)}{T(3,2)} = 5 - (-7)\frac{-1}{1} = -2.$$

Column 6

$$\tilde{T}(4,6) := T(4,6) - T(4,2)\frac{T(3,6)}{T(3,2)} = 0 - (-7)\frac{1}{1} = 7.$$

Column 7

$$\tilde{T}(4,7) := T(4,7) - T(4,2)\frac{T(3,7)}{T(3,2)} = 100 - (-7)\frac{0}{1} = 100.$$

We obtain the following tableau:

x_1	x_2	x_3	x_4	x_5	x_6	
0	0	2.5	1	1	-1.5	10
1	0	1.5	0	1	-0.5	10
0	1	-1	0	-1	1	0
0	0	-9	0	-2	7	100

3. For the new basis, we have

$$\bar{B} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \text{ and } \bar{B}^{-1} = \begin{pmatrix} 1 & 1 & -3/2 \\ 0 & 1 & -1/2 \\ 0 & -1 & 1 \end{pmatrix},$$

and

$$c_B = \begin{pmatrix} 0 \\ -10 \\ -12 \end{pmatrix}.$$

Therefore, the reduced costs are

$$c^T - c_{\bar{B}}^T \bar{B}^{-1} A = \begin{pmatrix} 0 & 0 & -9 & 0 & -2 & 7 \end{pmatrix}.$$

We observe that it corresponds to the left part of the last row of the new tableau. We also have

$$x_{\bar{B}} = \bar{B}^{-1} b = \begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix},$$

so that

$$c_{\bar{B}}^T x_B = -100,$$

which is the entry of the last row, last column, with the opposite sign.

Question 08

Consider the following optimization problem

$$\min_{x \in \mathbb{R}^2} -2x_1 - x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq 2, \\ x_1 + x_2 &\leq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Identify the initial tableau, where all the slack variables are in the basis.

2. Solve the problem using the tableau version of the simplex algorithm.

Solution 08

We first write the problem in standard form. We introduce the slack variables x_3 and x_4 to obtain

$$\min_{x \in \mathbb{R}^2} -2x_1 - x_2$$

subject to

$$\begin{aligned} x_1 - x_2 + x_3 &= 2, \\ x_1 + x_2 + x_4 &= 6, \\ x_1, x_2, x_3, x_4 &\geq 0, \end{aligned}$$

so that the data of the problem are

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \; c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

1. The slack variables are selected to be in the basis. In that case, B is the identity matrix and $c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore, the first tableau simplifies to

A	b
c^T	0

   that is

x_1	x_2	x_3	x_4	
1	-1	1	0	2
1	1	0	1	6
-2	-1	0	0	0

2. We now apply the simplex algorithm. For the first iteration, we observe that the smallest index corresponding to a negative reduced cost is $p = 1$: the variable x_1 enters the basis. Then, in order to find the variable which leaves the basis, we calculate the distances α_i to the constraints using the formula

$$\alpha_i = \begin{cases} \frac{T(i,n+1)}{T(i,p)} & \text{if } T(i,p) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

We obtain

$$\alpha_3 = 2/1 = 2,$$

and

$$\alpha_4 = 6/1 = 6.$$

Therefore, the smallest distance is 2 and the variable x_3 leaves the basis. The pivot is at column $p = 1$ and row $q = 1$. After pivoting, we obtain the following tableau, where x_1 and x_4 are in the basis:

x_1	x_2	x_3	x_4	
1	-1	1	0	2
0	2	-1	1	4
0	-3	2	0	4

The smallest index corresponding to a negative reduced cost is $p = 2$: the variable x_2 enters the basis. The distances α_i to the constraints are

$$\alpha_1 = +\infty,$$

as the corresponding entry in column 2 is negative (-1), and

$$\alpha_4 = 4/2 = 2.$$

Therefore, the variable which leaves the basis is x_4. The pivot is thus in column $p = 2$ and row $q = 2$. After pivoting, we obtain the following tableau, where x_1 and x_2 are in the basis:

x_1	x_2	x_3	x_4	
1	0	0.5	0.5	4
0	1	-0.5	0.5	2
0	0	0.5	1.5	10

This tableau is optimal as all reduced costs (in the last row) are non negative. The optimal value of the basic variables are available in the last column, so that

$$x^* = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$c^T x^* = -10.$$

Question 09

Consider the following linear optimization problem:

$$\min_{x \in \mathbb{R}^4} x_1 + x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3, \\ -x_1 + 2x_2 + 6x_3 &= 2, \\ +4x_2 + 9x_3 &= 5, \\ 3x_3 + x_4 &= 1, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

1. Find an initial tableau using an auxiliary problem.
2. Find an optimal tableau.

Solution 09

The problem is

$$\min c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 2 & 6 & 0 \\ 0 & 4 & 9 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 2 \\ 5 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

As there are four constraints, we introduce four auxiliary variables, to obtain the auxiliary problem:

$$\min x_1^a + x_2^a + x_3^a + x_4^a$$

subject to

$$\begin{aligned} x_1+2x_2+3x_3 \quad +x_1^a \qquad\qquad\quad &= 3, \\ -x_1+2x_2+6x_3 \qquad +x_2^a \qquad\quad &= 2, \\ +4x_2+9x_3 \qquad\quad +x_3^a \quad &= 5, \\ 3x_3+x_4 \qquad\qquad +x_4^a &= 1, \\ x_1, x_2, x_3, x_4, x_1^a, x_2^a, x_3^a, x_4^a &\geq 0, \end{aligned}$$

where x_1^a, x_2^a, x_3^a and x_4^a are the auxiliary variables. The initial tableau for the auxiliary problem is:

A	b
$-1^T A \vert 0$	$-1^T b$

where $c_B^T = 1^T$ is a row vector composed of all ones. We obtain

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
1	2	3	0	1	0	0	0	3
-1	2	6	0	0	1	0	0	2
0	4	9	0	0	0	1	0	5
0	0	3	1	0	0	0	1	1
0	-8	-21	-1	0	0	0	0	-11

We apply the simplex algorithm to solve the auxiliary problem. For the first iteration, we observe that the smallest index corresponding to a negative reduced cost is $p = 2$: the variable x_2 enters the basis. Then, in order to find the variable which leaves the basis, we calculate the distances α_i to the constraints using the formula

$$\alpha_i = \begin{cases} \frac{T(i,n+1)}{T(i,p)} & \text{if } T(i,p) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

We obtain

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ \frac{5}{4} \\ +\infty \end{pmatrix}.$$

Thus, the smallest distance is $\alpha_2 = 1$ and the variable x_2^a leaves the basis. The pivot is $(p, q) = (2, 2)$. After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
2	0	-3	0	1	-1	0	0	1
$-\frac{1}{2}$	1	3	0	0	$\frac{1}{2}$	0	0	1
2	0	-3	0	0	-2	1	0	1
0	0	3	1	0	0	0	1	1
-4	0	3	-1	0	4	0	0	-3

The smallest index corresponding to a negative reduced cost is $p = 1$: the variable x_1 enters the basis. The distances α_i to the constraints are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ +\infty \\ \frac{1}{2} \\ +\infty \end{pmatrix}.$$

The shortest distance being $\alpha_1 = 1/2$, the variable which leaves the basis is x_1^a (note that taking x_3^a out of the basis would have been equally valid, as $\alpha_3 = 1/2$ as well.) The pivot is thus $(p, q) = (1, 1)$. After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
1	0	$-\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
0	1	$\frac{9}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{5}{4}$
0	0	0	0	-1	-1	1	0	0
0	0	3	1	0	0	0	1	1
0	0	-3	-1	2	2	0	0	-1

The smallest index corresponding to a negative reduced cost is $p = 3$: the variable x_3 enters the basis. The distances α_i to the constraints are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} +\infty \\ \frac{5}{9} \\ +\infty \\ \frac{1}{3} \end{pmatrix}.$$

The shortest distance being $\alpha_4 = 1/3$, the variable which leaves the basis is x_4^a. The pivot is thus $(p, q) = (4, 3)$. After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	0	0	-1	-1	1	0	0
0	0	1	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	2	2	0	1	0

The last tableau is optimal and the optimal value is zero. We have identified a basic feasible solution for the original problem: $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$ and $x_4 = 0$.

We now need to generate the initial tableau for the original problem. But the basis corresponding to the optimal tableau of the auxiliary problem contains the auxiliary variable x_3^a. Therefore, it must be exchanged with a variable of the original problem that is non basic. Here, the only original variable not in the basis is x_4. However, it is not possible to perform the pivoting as the pivot $(3, 4)$ is zero. It means that the third constraint is redundant. It can indeed be checked that it is the sum of the first two constraints. The corresponding row of the matrix can simply be removed, as well as the column of the corresponding auxiliary variable. We obtain

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_4^a		
1	0	0	1/2	1/2	−1/2	1/2	1	x_1
0	1	0	−3/4	1/4	1/4	−3/4	1/2	x_2
0	0	1	1/3	0	0	1/3	1/3	x_3
0	0	0	0	2	2	1	0	

There are no longer any auxiliary variables in the basis, the corresponding columns can be removed to obtain a tableau.

x_1	x_2	x_3	x_4		
1	0	0	1/2	1	x_1
0	1	0	−3/4	1/2	x_2
0	0	1	1/3	1/3	x_3
−	−	−	−	−	

To obtain a feasible tableau from the initial problem, we need to calculate the elements of the last row. For the left part, these are reduced costs:

$$\bar{c}^T = c^T - c_B^T B^{-1} A,$$

and for the right part, the objective function with the opposite sign:

$$-c_B^T B^{-1} b,$$

where

$$c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } c_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that the quantities $B^{-1}A$ and $B^{-1}b$ are directly found in the upper part of the tableau, so that a simple inner product is needed. The reduced cost for x_4 is

$$(1\ 1\ 1\ 0) - (1\ 1\ 1) \begin{pmatrix} 1/2 \\ -3/4 \\ 1/3 \end{pmatrix} = -1/12,$$

and the value of the objective function is

$$(1\ 1\ 1) \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix} = 11/6.$$

We just need to change the sign before incorporating it in the tableau. Therefore, we obtain

x_1	x_2	x_3	x_4		
1	0	0	1/2	1	x_1
0	1	0	−3/4	1/2	x_2
0	0	1	1/3	1/3	x_3
0	0	0	−1/12	−11/6	

We can now start the second phase, to find an optimal tableau. The variable x_4 enters the basis, and the variable x_3 leaves it. After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4		
1	0	−3/2	0	1/2	x_1
0	1	9/4	0	5/4	x_2
0	0	3	1	1	x_4
0	0	1/4	0	−7/4	

This is an optimal tableau, and the solution is $x_1 = 1/2$, $x_2 = 5/4$, $x_3 = 0$ and $x_4 = 1$. The value of the objective function is $7/4$.

Question 10

Consider the following linear optimization problem:

$$\min_{x \in \mathbb{R}^2} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq -2, \\ -x_1 + x_2 &\leq -1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Find an initial tableau using an auxiliary problem.
2. Find an optimal tableau.

Solution 10

The problem is

$$\min_{x \in \mathbb{R}^2} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq -2, \\ -x_1 + x_2 &\leq -1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

We write it in standard form. We introduce two slack variables x_3, x_4:

$$\min_{x \in \mathbb{R}^4} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 + x_3 &= -2, \\ -x_1 + x_2 + x_4 &= -1, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

To obtain $b \geq 0$, we multiply each row by -1 and we obtain:

$$\min_{x \in \mathbb{R}^4} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 - x_3 &= 2, \\ x_1 - x_2 - x_4 &= 1, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

We introduce two auxiliary variables, and define the auxiliary problem as

$$\min_{x \in \mathbb{R}^6} x_1^a + x_2^a$$

subject to

$$\begin{aligned} -x_1 + x_2 - x_3 + x_1^a &= 2, \\ x_1 - x_2 - x_4 + x_2^a &= 1, \\ x_1, x_2, x_3, x_4, x_1^a, x_2^a &\geq 0. \end{aligned}$$

The polyhedron for this problem is

$$\mathcal{P} = \{x \in \mathbb{R}^6 | Ax = b, x \geq 0\}$$

where

$$A = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The initial tableau of the auxiliary problem is:

A	b
$-1^T A \vert 0$	$-1^T b$

where $c_B^T = 1^T$ is a row vector composed of all ones. We obtain

x_1	x_2	x_3	x_4	x_1^a	x_2^a	
-1	1	-1	0	1	0	2
1	-1	0	-1	0	1	1
0	0	1	1	0	0	-3

All the reduced costs are non negative. Therefore, we have an optimal tableau for the auxiliary problem. However, the optimal solution is not zero. It means that the original problem has no feasible solution. Considering the original problem, we can rewrite the constraints as

$$x_1 \leq x_2 - 2,$$
$$x_1 \geq 1 + x_2,$$

and observe that they are incompatible. That is why the optimal solution of the auxiliary problem is not zero. There is no way to set the auxiliary variables to zero without violating the problem constraints.

Question 11

For each of the following optimization problems, given the optimal tableau of Phase I of the simplex algorithm in two phases, clean it. Then, recalculate the last row.

1.

$$\min_{x \in \mathbb{R}^4} x_1 + x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3 \\ -x_1 + 2x_2 + 6x_3 &= 2 \\ +4x_2 + 9x_3 &= 5 \\ 3x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

The corresponding auxiliary problem has optimal tableau:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	0	0	-1	-1	1	0	0
0	0	1	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	2	2	0	1	0

2.

$$\min_{x \in \mathbb{R}^5} 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5$$

subject to

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + x_5 &= 2 \\ x_1 + 2x_2 - 3x_4 + x_5 &= 2 \\ -x_1 - 4x_2 + 3x_3 &= 1 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

The corresponding auxiliary problem has optimal tableau:

x_1	x_2	x_3	x_4	x_5	x_1^a	x_2^a	x_3^a	
1	3	0	4	1	1	0	0	2
0	-1	0	-7	0	-1	1	0	0
0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1
0	1	0	7	0	2	0	1	0

Solution 11

The optimal solution of the auxiliary problem identifies a feasible point of the original problem, if such a point exists. However, in order to apply the simplex algorithm, it is necessary to use the corresponding simplex tableau.

In the case where all the auxiliary variables are non-basic, it is sufficient to delete the corresponding columns and calculate the reduced costs to obtain an initial tableau for the original problem. It may be that the basis corresponding to the optimal tableau of the auxiliary problem contains auxiliary variables. Note that the feasible basic solution is necessarily degenerate because this basic (auxiliary) variable is zero. Therefore, it can be exchanged with a variable of the original problem that is non-basic and, therefore, equal to zero as well. To do this, we choose a column corresponding to a variable of the original problem so that the pivot is non-zero and we perform the pivoting that exchanges the two variables in the basis. Since the auxiliary basis variable is zero, the pivoting does not affect the values of the last column of the tableau. So the new tableau corresponds to exactly the same feasible basic solution as the previous one. Only the basis has changed.

1. The optimal tableau of the auxiliary problem is:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_3^a	x_4^a	
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	0	0	-1	-1	1	0	0
0	0	1	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	2	2	0	1	0

   The auxiliary variable x_3^a is in the basis. The only variable of the original problem that is not in the basis is x_4. Therefore, we should exchange x_3^a and x_4 in the basis. Therefore, the row of the pivot is row 3, and the column of the pivot is column 4.

   However, that pivot is zero: the procedure cannot be applied. The reason is that the matrix A of the initial problem is not of full rank. We observe that the sum of the two first constraints is the third one. Consequently, this constraint is redundant. Since such a constraint

can be ignored without modifying the problem, the corresponding row of the tableau (here the third one) can be removed, as well as the column of the corresponding auxiliary variable (here the seventh one).

Consequently, any time that an auxiliary variable cannot be pivoted out of the basis, and all potential pivots corresponding to original non basic variables are 0, the corresponding row and column of the tableau can be simply removed.

The tableau becomes:

x_1	x_2	x_3	x_4	x_1^a	x_2^a	x_4^a	
1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	2	2	1	0

There is no basic auxiliary variable anymore and we can remove the columns corresponding to auxiliary variables to obtain the tableau $\bar{T}_0^*$:

x_1	x_2	x_3	x_4	
1	0	0	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	1	$\frac{1}{3}$	$\frac{1}{3}$
-	-	-	-	-

Finally, we have to recalculate the last row. For each basic variable the reduced cost is zero. Thus we only need to calculate the reduced cost corresponding to x_4:

$$\begin{aligned}
\bar{c}_4 &= c_4 - c_B^T B^{-1} A_4 \\
&= 0 - \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ -3/4 \\ 1/3 \end{pmatrix} \\
&= -(\frac{1}{2} - \frac{3}{4} + \frac{1}{3}) \\
&= -\frac{1}{12},
\end{aligned}$$

using the fact that $B^{-1}A_4$ is directly obtained from column 4 of the tableau.

Furthermore the optimal value is

$$x_1^* + x_2^* + x_3^* = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$$

We change its sign and insert it as the entry in the last row and last column, to obtain:

x_1	x_2	x_3	x_4	
1	0	0	$\frac{1}{2}$	1
0	1	0	$-\frac{3}{4}$	$\frac{1}{2}$
0	0	1	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	$-\frac{1}{12}$	$-\frac{11}{6}$

2. The optimal tableau of the auxiliary problem is

x_1	x_2	x_3	x_4	x_5	x_1^a	x_2^a	x_3^a	
1	3	0	4	1	1	0	0	2
0	-1	0	-7	0	-1	1	0	0
0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1
0	1	0	7	0	2	0	1	0

The auxiliary variables x_2^a is basic, and corresponds to the second row of the tableau. The candidates to replace it in the basis are x_2 and x_4. Note that x_5 is also a variable of the original problem that is out of the basis, but the corresponding pivot (row 2, column 5) is zero, and exchanging x_2^a with x_5 is not possible.

We arbitrarily decide to exchange x_2^a with x_2. The pivot is on row 2, column 2. After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_5	x_1^a	x_2^a	x_3^a	
1	0	0	-17	1	-2	3	0	2
0	1	0	7	0	1	-1	0	0
0	0	1	3.67	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1
0	0	0	0	0	1	1	1	0

There is no auxiliary variable in the basis anymore and we can remove the columns corresponding to auxiliary variables to obtain the tableau $\bar{T}_0^*$:

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
0	0	1	3.67	$\frac{1}{3}$	1
-	-	-	-	-	-

We need to recalculate the last row. The reduced costs corresponding to basic variables are zero. For the others we get:

$$\bar{c}_4 = c_4 - c_B^T(B^{-1}A_4) = 1 - (2 \cdot (-17) + 3 \cdot 7 + 3 \cdot 3.67) = 3,$$
$$\bar{c}_5 = c_5 - c_B^T(B^{-1}A_5) = -2 - (2 \cdot 1 + 3 \cdot 0 + 3 \cdot 1/3) = -5.$$

The optimal value is

$$2x_1^* + 3x_2^* + 3x_3^* + x_4^* - 2x_5^* = 2 \cdot 2 + 3 \cdot 0 + 3 \cdot 1 + 0 - 2 \cdot 0 = 7.$$

Thus we get:

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
0	0	1	3.67	$\frac{1}{3}$	1
0	0	0	3	-5	-7

Question 12

Consider the optimization problem

$$\min_{x \in \mathbb{R}^5} 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5$$

subject to

$$\begin{aligned} x_1 + 3x_2 \quad\quad + 4x_4 + x_5 &= 2, \\ x_1 + 2x_2 \quad\quad - 3x_4 + x_5 &= 2, \\ -x_1 - 4x_2 + 3x_3 \quad\quad &= 1, \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

At the end of Phase I of the simplex algorithm, we obtain the following tableau $\bar{T}_0^*$:

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
0	0	1	3.67	$\frac{1}{3}$	1
-	-	-	-	-	-

Apply Phase II of the algorithm in order to solve this problem.

Solution 12

Starting from the tableau $\bar{T}_0^*$

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
0	0	1	3.67	$\frac{1}{3}$	1
0	0	0	0	0	0

we apply the Phase II of the simplex algorithm.

First, we have to recalculate the last row of the tableau.
We know that the reduced cost of the variables from the basis are 0. Thus, we only have to compute the reduced costs of x_4 and x_5. Using that

$$B = I_3, \quad c_B = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -17 \\ 7 \\ 3.67 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix},$$

we get

- $\bar{c_4} = c_4 - c_B^T(B^{-1}A_4) = 1 - (2 \cdot (-17) + 3 \cdot 7 + 3 \cdot 3.67) = 3,$
- $\bar{c_5} = c_5 - c_B^T(B^{-1}A_5) = -2 - (2 \cdot 1 + 3 \cdot 0 + 3 \cdot \frac{1}{3}) = -5.$

Furthermore, the value of the objective function is 7.
Thus the tableau becomes:

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
0	0	1	3.67	$\frac{1}{3}$	1
0	0	0	3	-5	-7

We are going to apply the iterations of the simplex algorithm to that tableau in order to solve the problem. The only index for which the corresponding reduced cost is negative is $p = 5$: the variable x_5 inters in the basis. Next, we calculate the distances α_i to the constraint $x_5 \geq 0$ and get:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ +\infty \\ 3 \end{pmatrix}.$$

Thus, the variable which leaves the basis is x_1. The pivot is thus $(p, q) = (1, 5)$.

After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_5	
1	0	0	-17	1	2
0	1	0	7	0	0
$-\frac{1}{3}$	0	1	9.33	0	$\frac{1}{3}$
5	0	0	-82	0	3

The only index for which the corresponding reduced cost is negative is $p = 4$: the variable x_4 enters in the basis. The distances α_i to the constraint $x_4 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} +\infty \\ 0 \\ 0.4 \end{pmatrix}.$$

The variable which leaves the basis is x_2. The pivot is thus $(p, q) = (2, 4)$. Note that the feasible basic solution is degenerate. The algorithm stays on the same vertex of the polyhedron, but characterized by another basis.

After pivoting, we obtain the following tableau:

x_1	x_2	x_3	x_4	x_5	
1	2.43	0	0	1	2
0	0.14	0	1	0	0
$-\frac{1}{3}$	-1.33	1	0	0	$\frac{1}{3}$
5	11.71	0	0	0	3

The last tableau is optimal since all the reduced costs are non negative. The optimal solution is

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = \frac{1}{3}, \quad x_4 = 0, \quad x_5 = 2,$$

and the optimal value is -3. Note that the optimal solution was already reached at the previous iteration, illustrating that the non negative reduced costs are sufficient but not necessary for optimality.

Question 13

A 90-year-old rich uncle decides to go to the notary to establish his inheritance. Among his heirs he has three grand-nephews, André, Blaise and Claude, with rather unusual habits. He wants to bequeath them a maximum of 100′000.- CHF. Each of them has declared that he will keep a share of the sum received to purchase various items, and invest the rest. Although he respects their choices, the uncle believes that the money should be invested at best, instead of being spent. Therefore, he would like to split the total amount in such a way that, by the end of the year, the sum that is left is maximal. Also, he wants that the sum expended by each of his three grand-nephews does not exceed $10,000$.- CHF.

According to their statements, André will spend 40% of his money during the year, Blaise 30% and Claude 20%. The money they do not spend will be invested in the following way: André invests in his new company, which is very profitable, at 50/3%, Blaise in shares at 300/7% and Claude puts it under his pillow, so it has no yield. Note that an investment with a value v at $x/y\%$ means that the interest earned for one year from that investment is $v \cdot \frac{x}{100 \cdot y}$.

1. Help the uncle to model the optimization problem. Then, transform the problem in standard form.
   Hint: It is assumed that grandnephews will receive the inheritance on January 1st and we will calculate the interest of the investment on December 31st, that is, when all the planned expenses have been made.

2. Solve the problem using the simplex tableau. Instead of using Bland's rule to choose the variable which enters the basis at each iteration of the simplex, pick the one with the most negative reduced cost.

Solution 13

1. We define the following decision variables: x_1, x_2 and x_3 the thousands CHF which will be bequeathed to André, Blaise and Claude respectively. We also introduce the following notation for the data of the problem:

   - d_1, d_2 and d_3 are the percents of André, Blaise and Claude's expenses respectively;
   - r_1, r_2 and r_3 are the interests from André, Blaise and Claude's investments.

   The data of the problem is

	André	Blaise	Claude
d	2/5	3/10	1/5
r	1/6	3/7	0

   For each nephew, we have:

expenses for one year:	$d_i x_i$,
remaining money on the 31th of December:	$x_i - d_i x_i$,
interests earned:	$(x_i - d_i x_i) r_i = (1 - d_i) r_i x_i$,
money left after interests calculation:	$(x_i - d_i x_i)(1 + r_i) = (1 - d_i)(1 + r_i) x_i$.

   Using those constraints we get the following linear optimization problem:

   $$\max (1 - d_1)(1 + r_1) x_1 + (1 - d_2)(1 + r_2) x_2 + (1 - d_3)(1 + r_3) x_3$$

   subject to

   $$\begin{aligned} d_1 x_1 &\leq 10, \\ d_2 x_2 &\leq 10, \\ d_3 x_3 &\leq 10, \\ x_1 + x_2 + x_3 &\leq 100, \\ x_1, x_2, x_3 &\geq 0, \end{aligned}$$

which leads, using the numerical values, to

$$\min -7/10x_1 - x_2 - 4/5x_3$$

subject to

$$\begin{aligned} 2/5x_1 &\leq 10, \\ 3/10x_2 &\leq 10, \\ 1/5x_3 &\leq 10, \\ x_1 + x_2 + x_3 &\leq 100, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

We write this problem in standard form as

$$\min -7/10x_1 - x_2 - 4/5x_3$$

subject to

$$\begin{aligned} 2/5x_1 + x_4 &= 10, \\ 3/10x_2 + x_5 &= 10, \\ 1/5x_3 + x_6 &= 10, \\ x_1 + x_2 + x_3 + x_7 &= 100, \\ x_i &\geq 0, \quad i \in \{1, ..., 7\}. \end{aligned}$$

Because $b \geq 0$, selecting the slack variables as basic variables leads to a feasible basic solution, so that there is no need to solve an auxiliary problem to find the initial tableau.

2. The first tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$\frac{2}{5}$	0	0	1	0	0	0	10
0	$\frac{3}{10}$	0	0	1	0	0	10
0	0	$\frac{1}{5}$	0	0	1	0	10
1	1	1	0	0	0	1	100
$-\frac{7}{10}$	-1	$-\frac{4}{5}$	0	0	0	0	0

The variable x_2 enters the basis (since it is the index corresponding to the most negative reduced cost) and x_5 leaves it, since it is the index corresponding to the smallest distance to the constraint $x_2 \geq 0$. Indeed,

$$\begin{aligned}
\alpha_4 &= +\infty, \\
\alpha_5 &= 10/(3/10) = 33.333, \\
\alpha_6 &= +\infty, \\
\alpha_7 &= 100/1 = 100.
\end{aligned}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$\frac{2}{5}$	0	0	1	0	0	0	10
0	1	0	0	$\frac{10}{3}$	0	0	$\frac{100}{3}$
0	0	$\frac{1}{5}$	0	0	1	0	10
1	0	1	0	$-\frac{10}{3}$	0	1	$\frac{200}{3}$
$-\frac{7}{10}$	0	$-\frac{4}{5}$	0	$\frac{10}{3}$	0	0	$\frac{100}{3}$

The variable x_3 enters in the basis and x_6 leaves it. Indeed,

$$\begin{aligned}
\alpha_4 &= +\infty, \\
\alpha_2 &= +\infty, \\
\alpha_6 &= 10/(1/5) = 50, \\
\alpha_7 &= (200/3)/1 = 60.666.
\end{aligned}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$\frac{2}{5}$	0	0	1	0	0	0	10
0	1	0	0	$\frac{10}{3}$	0	0	$\frac{100}{3}$
0	0	1	0	0	5	0	50
1	0	0	0	$-\frac{10}{3}$	-5	1	$\frac{50}{3}$
$-\frac{7}{10}$	0	0	0	$\frac{10}{3}$	4	0	$\frac{220}{3}$

The variable x_1 inters in the basis and x_7 leaves it. Indeed,

$$\begin{aligned}
\alpha_4 &= 10/(2/5) = 25, \\
\alpha_2 &= +\infty, \\
\alpha_3 &= +\infty, \\
\alpha_7 &= (50/3)/1 = 16.666.
\end{aligned}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	0	0	1	$\frac{4}{3}$	2	$-\frac{2}{5}$	$\frac{10}{3}$
0	1	0	0	$\frac{10}{3}$	0	0	$\frac{100}{3}$
0	0	1	0	0	5	0	50
1	0	0	0	$-\frac{10}{3}$	-5	1	$\frac{50}{3}$
0	0	0	0	1	$\frac{1}{2}$	$\frac{7}{10}$	85

The final tableau is optimal and the optimal solution is $x_1 = 50/3$, $x_2 = 100/3$ and $x_3 = 50$ with optimal value -85.

Therefore, the greater-uncle should give

- $\sim$ 16′666.- CHF to André,
- $\sim$ 33′333.- CHF to Blaise and
- 50′000.- CHF to Claude.

He bequeaths $\sim$ 100′000.- CHF and, at the end of the year, it will remain 85′000.- CHF.

Question 14

Consider the following optimization problem:

$$\min -x_1 - x_2$$

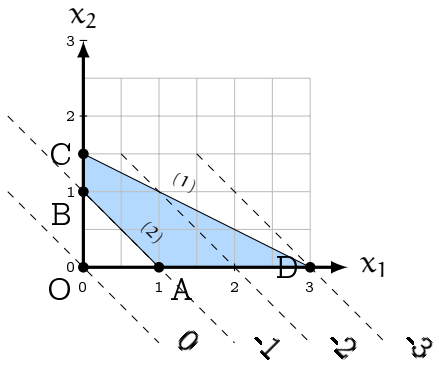
subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 3, \\ x_1 + x_2 &\geq 1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Represent graphically the feasible set and solve the problem using the graphical method.
2. Solve the problem using the simplex algorithm in two phases. For each iteration of the algorithm, show graphically the updates.

Solution 14

The feasible set is shown on the Figure below, as well as several level lines.



The optimal solution, which is the vertex which minimize the objective function, is $(x_1, x_2) = (3, 0)$ with optimal value -3.

In order to write the problem in standard form, such that $b \geq 0$, we introduce two slack variables e_1 and e_2:

$$\min -x_1 - x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 + e_1 &= 3 \\ x_1 + x_2 - e_2 &= 1 \\ x_1, x_2, e_1, e_2 &\geq 0. \end{aligned}$$

We introduce the auxiliary variables a_1 and a_2 to get the following auxiliary problem:

$$\min a_1 + a_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 + e_1 + a_1 &= 3 \\ x_1 + x_2 - e_2 + a_2 &= 1 \\ x_1, x_2, e_1, e_2, a_1, a_2 &\geq 0. \end{aligned}$$

The first tableau is:

x_1	x_2	e_1	e_2	a_1	a_2	
1	2	1	0	1	0	3
1	1	0	-1	0	1	1
-2	-3	-1	1	0	0	-4

Since the smallest index corresponding to a negative reduced cost is $p = 1$, the variable x_1 enters in the basis. The distances to the constraint $x_1 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

and a_2 leaves the basis. The pivot is $(p, q) = (1, 2)$.

After pivoting we get the following tableau:

x_1	x_2	e_1	e_2	a_1	a_2	
0	1	1	1	1	-1	2
1	1	0	-1	0	1	1
0	-1	-1	-1	0	2	-2

Since the smallest index corresponding to a negative reduced cost is $p = 2$, the variable x_2 enters in the basis. The distances to the constraint $x_2 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

and x_1 leaves the basis. The pivot is $(p, q) = (2, 2)$.

After pivoting we get the following tableau:

x_1	x_2	e_1	e_2	a_1	a_2	
-1	0	1	2	1	-2	1
1	1	0	-1	0	1	1
1	0	-1	-2	0	3	-1

Since the smallest index corresponding to a negative reduced cost is $p = 3$, the variable e_1 enters in the basis. The distances to the constraint $e_1 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ +\infty \end{pmatrix},$$

and a_1 leaves the basis. The pivot is $(p, q) = (1, 3)$.

After pivoting we get the following tableau:

x_1	x_2	e_1	e_2	a_1	a_2	
-1	0	1	2	1	-2	1
1	1	0	-1	0	1	1
0	0	0	0	1	1	0

The tableau is optimal as all reduced costs are non negative. The optimal value of the first phase is $x_1 = 0, x_2 = 1, e_1 = 1, e_2 = 0, a_1 = 0, a_2 = 0$.

No auxiliary variable is left in the basis. The tableau is ready to be cleaned. We can remove the columns corresponding to the auxiliary variables.

The initial solution of the second phase is $x_1 = 0, x_2 = 1, e_1 = 1, e_2 = 0$.

To obtain a feasible tableau from the initial problem, we recalculate the elements of the last row by computing the reduced costs.
Since e_1 and x_2 are basic variables, the corresponding reduced costs are zero.

For the other variables, using that $c_B = (0 - 1)^T$, we get

- $\bar{c_1} = c_1 - c_B^T B^{-1} A_1 = -1 - ((0 \cdot -1) + (-1 \cdot 1)) = 0$,
- $\bar{c_{e_2}} = c_{e_2} - c_B^T B^{-1} A_{e_2} = 0 - ((0 \cdot 2) + (-1 \cdot -1)) = -1$.

Note that $c_{e_2} = 0$ since the variable e_2 does not take part to the objective function.

For the last column, we compute the opposite of the optimal value. Since $x_1 = 0$ and $x_2 = 1$, the objective function is -1, and the value in the cell is 1.

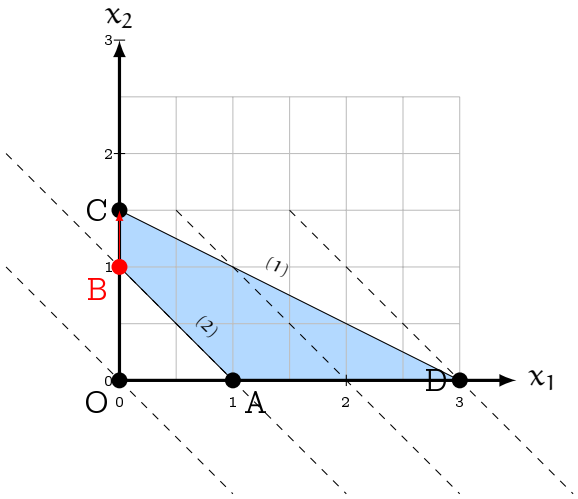
x_1	x_2	e_1	e_2	
-1	0	1	2	1
1	1	0	-1	1
0	0	0	-1	1

The initial solution corresponds to the point B in the graph since $B = (x_1, x_2, e_1, e_2) = (0, 1, 1, 0)$.

Since the smallest index corresponding to a negative reduced cost is $p = 4$, the variable e_2 enters in the basis. The distances to the constraint $e_2 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ +\infty \end{pmatrix},$$

and e_1 leaves the basis. The pivot is $(p, q) = (1, 4)$.



After pivoting we get the following tableau:

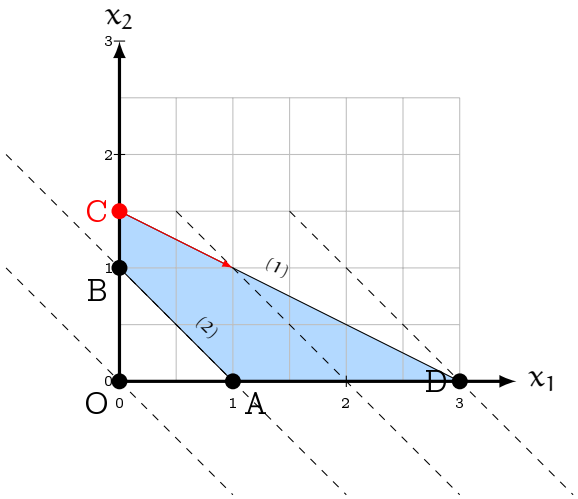
x_1	x_2	e_1	e_2	
-1/2	0	1/2	1	1/2
1/2	1	1/2	0	3/2
-1/2	0	1/2	0	3/2

The solution corresponds to the point C in the graph since $C = (x_1, x_2, e_1, e_2) = (0, 3/2, 0, 1/2)$.

Since the smallest index corresponding to a negative reduced cost is $p = 1$, the variable x_1 enters in the basis. The distances to the constraint $x_1 \geq 0$ are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} +\infty \\ 3 \end{pmatrix},$$

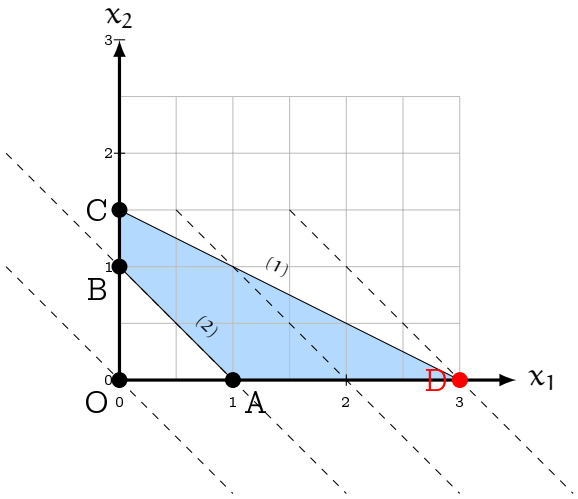
and x_2 leaves the basis. The pivot is $(p, q) = (2, 1)$.



After pivoting we get the following tableau:

x_1	x_2	e_1	e_2	
0	1	1	1	2
1	2	1	0	3
0	1	1	0	3

The final tableau is optimal since all reduced costs are non negative. The optimal solution of the problem is the point D in the graph since $D = (x_1, x_2, e_1, e_2) = (3, 0, 0, 2)$ and the optimal value is -3.



Question 15

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -x_1 - \frac{3}{2}x_2$$

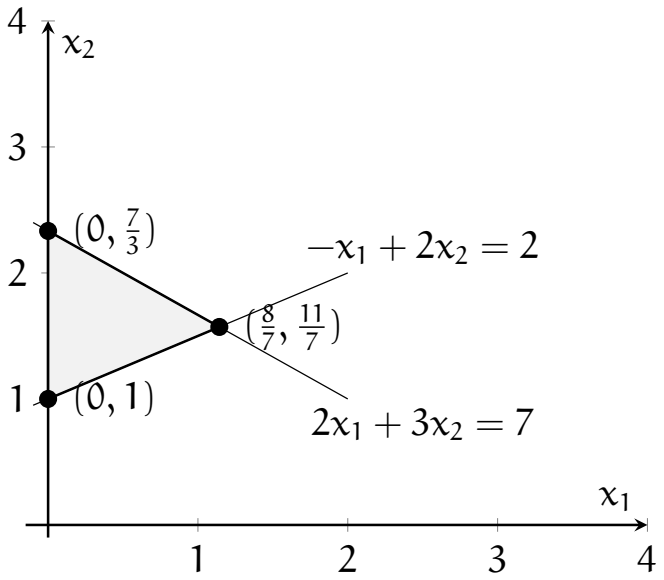
subject to

$$\begin{aligned} -x_1 + 2x_2 &\geq 2, \\ 2x_1 + 3x_2 &\leq 7, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve this problem using the graphical method.

Solution 15

The figure below represents the feasible domain in the space of variables (x_1, x_2).

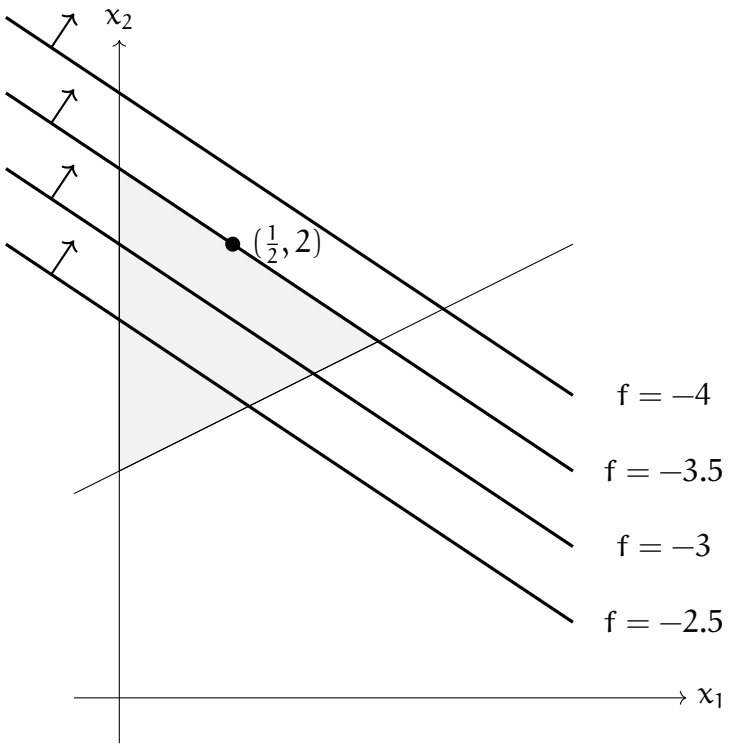


We notice that the constraint $x_1 \geq 0$ could be eliminated. The set of vertices of the polyhedron is $\{(0,1), (0, \frac{7}{3}), (\frac{8}{7}, \frac{11}{7})\}$.

We now have to add the level lines corresponding to different values of the objective function. We start by drawing an arbitrary level line intersecting the feasible domain, say $f = -x_1 - \frac{3}{2}x_2 = -2.5$. The gradient corresponds to the steepest ascent direction. As we minimize, we need to identify the steepest descent direction, that is

$$-\nabla f(x) = -c = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

That is displayed with an arrow on each level line in the figure below. Finally, we move this level line parallel to itself as far as possible in the direction of $-c$ as long as it intersects the feasible domain.



The minimum value that the objective function can take is $f = -3.5$. We observe that, in this example, the intersection of the level line with the polyhedron is not a vertex, but a face. It means that all the solutions on that face are optimal, with value -3.5.

For instance, the solution $(\frac{1}{2}, 2)$ is optimal, although it is not a vertex. Note that it does not contradict Theorem 16.2, that says "if there is an optimal solution, there exists an optimal vertex". Indeed, the vertices $(0, \frac{7}{3})$ and $(\frac{8}{7}, \frac{11}{7})$ are also optimal.

Question 16

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^2} -3x_1 - 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq -2, \\ 2x_1 + x_2 &\leq 8, \\ x_1 + x_2 &\leq 5, \\ x_1 + 2x_2 &\leq 10, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Write the problem in standard form. Consider first the vertex where all slack variables are in the basis, and solve the problem using the simplex algorithm.

Solution 16

The problem in standard form is

$$\min_{x\in\mathbb{R}^2} -3x_1 - 2x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 + x_3 &= 2,\\ 2x_1 + x_2 + x_4 &= 8,\\ x_1 + x_2 + x_5 &= 5,\\ x_1 + 2x_2 + x_6 &= 10,\\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

It means that the data of the problem are:

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0\\ 2 & 1 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 6},\ b = \begin{pmatrix} 2\\ 8\\ 5\\ 10 \end{pmatrix} \in \mathbb{R}^4,\ c = \begin{pmatrix} -3\\ -2\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \in \mathbb{R}^6,$$

as well as the indices of the basic variables

$$J^0 = (3, 4, 5, 6).$$

Iteration 1

The basic matrix is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is the identity matrix. Therefore, $B^{-1} = I$ as well. As the basic variables are slack variables, they do not appear in the objective function, and $c_B = 0$. Consequently, the objective function is $c_B^T x_B = 0$. The reduced costs for the non basic variables are

$$\bar{c}_1 = c_1 - c_B^T B^{-1} A_1 = -3,$$

and

$$\bar{c}_2 = c_2 - c_B^T B^{-1} A_2 = -2.$$

We select variable 1 to enter the basis. We have

$$x_B = B^{-1} b = \begin{pmatrix} 2 \\ 8 \\ 5 \\ 10 \end{pmatrix}, \; x = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 8 \\ 5 \\ 10 \end{pmatrix}.$$

It corresponds to point A in the figure below The basic entries of the basic direction are

$$d_B = -B^{-1} A_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix},$$

so that the basic direction is

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}.$$

It is represented by a thick arrow starting from A in the figure below. We calculate the distance to the constraint:

$$\begin{aligned} \alpha_3 &= +\infty \\ \alpha_4 &= -8/(-2) = 4 \\ \alpha_5 &= -5/(-1) = 5 \\ \alpha_6 &= -10/(-1) = 10. \end{aligned}$$

The smallest value is 4, and the variable that exits the basis is x_4. The new set of indices is therefore

$$J^1 = (3, 1, 5, 6).$$

Iteration 2

The basic matrix is

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; B^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

To calculate the reduced costs, we realize that

$$c_B = \begin{pmatrix} c_3 \\ c_1 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$\bar{c}_2 = c_2 - c_B^T B^{-1} A_2 = -\frac{1}{2},$$

and

$$\bar{c}_4 = c_4 - c_B^T B^{-1} A_4 = \frac{3}{2}.$$

Note that the basic direction 4 is pointing toward the vertex that we just left at the previous iteration. It makes sense therefore that it is an ascent direction, and the reduced cost is positive. Therefore, the only candidate to enter the basis is x_2. We have

$$x_B = B^{-1} b = \begin{pmatrix} 6 \\ 4 \\ 1 \\ 6 \end{pmatrix}, \; x = \begin{pmatrix} 4 \\ 0 \\ 6 \\ 0 \\ 1 \\ 6 \end{pmatrix}$$

It is represented by point B in the figure below.

$$c_B^T x_B = -12.$$

Note that, during the last iteration, the reduced cost along the basic direction was -3, and the step along that direction was $\alpha = 4$. Therefore, the objective function achieved a reduction of 12 units, moving from 0 to -12.

The basic entries of the basic direction are:

$$d_B = -B^{-1}A_1 = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix},$$

so that the basic direction is

$$d_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}.$$

It is represented by a thick arrow starting from B in the figure below. We calculate the distance to the constraint:

$$\alpha_3 = 6\,\frac{2}{3} = 4,$$
$$\alpha_1 = 4\,\frac{2}{1} = 8,$$
$$\alpha_5 = 1\,\frac{2}{1} = 2,$$
$$\alpha_6 = 6\,\frac{2}{3} = 4.$$

The smallest value is 2, and the variable that exits the basis is x_5. The new set of indices is therefore

$$J^2 = (3, 1, 2, 6).$$

Iteration 3

The basic matrix is

$$B = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix}.$$

To calculate the reduced costs, we realize that

$$c_B = \begin{pmatrix} c_3 \\ c_1 \\ c_2 \\ c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -2 \\ 0 \end{pmatrix}.$$

Therefore,

$$\bar{c}_4 = c_4 - c_B^T B^{-1} A_4 = 1,$$

and

$$\bar{c}_5 = c_5 - c_B^T B^{-1} A_5 = 1.$$

Both reduced costs are non negative. We have therefore identified an optimal vertex. We have

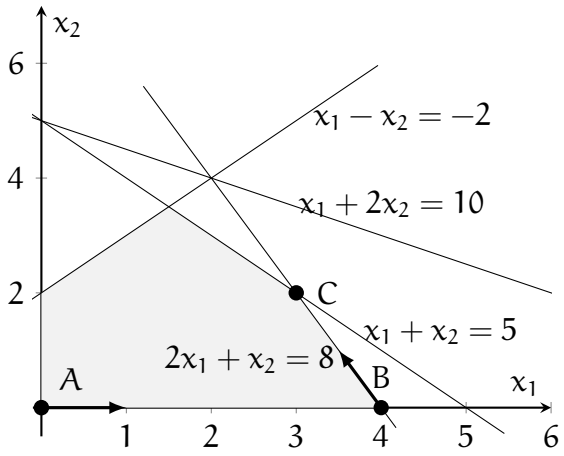
$$x_B = B^{-1} b = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 3 \end{pmatrix}, \; x = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

$$c_B^T x_B = -9 - 4 = -13.$$

Note that, during the last iteration, the reduced cost along the basic direction was $-1/2$, and the step along that direction was $\alpha = 2$. Therefore, the objective function achieved a reduction of 1 units, moving from -12 to -13.

The optimal set of indices is $J^* = (3, 1, 2, 6)$, and the optimal solution is

$$x^* = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 0 \\ 0 \\ 3 \end{pmatrix}.$$


x_2
6
4
2
$x_1 - x_2 = -2$
$x_1 + 2x_2 = 10$
C
$x_1 + x_2 = 5$
$2x_1 + x_2 = 8$
A
B
x_1
1
2
3
4
5
6

Question 17

Consider the following optimization problem:

$$\min -2x_1 - x_2$$

subject to

$$\begin{aligned} x_1 &\leq 5, \\ 4x_1 + x_2 &\leq 25, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Identify the initial tableau, where all the slack variables are in the basis.

2. Solve the problem using the tableau version of the simplex algorithm.

Solution 17

We first write the problem in standard form. We introduce the slack variables x_3 and x_4 to obtain

$$\min_{x \in \mathbb{R}^2} -2x_1 - x_2$$

subject to

$$\begin{aligned} x_1 \quad + x_3 \quad &= 5, \\ 4x_1 + x_2 \quad + x_4 \quad &= 25, \\ x_1, x_2, x_3, x_4 &\geq 0, \end{aligned}$$

so that the data of the problem are

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 5 \\ 25 \end{pmatrix}, \; c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

1. The slack variables are selected to be in the basis. As B is the identity matrix and $c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the first tableau simplifies to

A	b
c^T	0

   that is

x_1	x_2	x_3	x_4	
1	0	1	0	5
4	1	0	1	25
-2	-1	0	0	0

2. We now apply the simplex algorithm. For the first iteration, we observe that the smallest index corresponding to a negative reduced cost is in column $p = 1$. Indeed, the reduced cost, found in the last row, is -2. Therefore, the variable x_1 enters the basis. Then, in order to

find the variable which leaves the basis, we calculate the distances α_i to the constraints using the formula

$$\alpha_i = \begin{cases} \frac{T(i,n+1)}{T(i,p)} & \text{if } T(i,p) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

The first row corresponds to variables x_3 in the basis. Therefore, we obtain

$$\alpha_3 = 5/1 = 5,$$

which is the ratio of the two numbers circled below

x_1	x_2	x_3	x_4	
(1)	0	1	0	(5)
4	1	0	1	25
-2	-1	0	0	0

The second row corresponds to the variable x_4, and

$$\alpha_4 = 25/4 = 6.25,$$

which is the ratio of the two numbers circled below

x_1	x_2	x_3	x_4	
1	0	1	0	5
(4)	1	0	1	(25)
-2	-1	0	0	0

Therefore, the smallest distance is 5 and the variable x_3 leaves the basis. The pivot is located on row 1 and column 1, that is $p = 1$, $q = 1$, and its value is $T(q,p) = 1$:

x_1	x_2	x_3	x_4	
(1)	0	1	0	5
4	1	0	1	25
-2	-1	0	0	0

The objective of the pivoting is to apply elementary row operations in order to replace the column of the pivot from

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \text{ into } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We proceed row by row.

Row 1 This is the row of the pivot. The value of the pivot is already 1. Nothing has to be done for that row.

Row 2 The value of the entry in the column of the pivot is 4:

x_1	x_2	x_3	x_4	
1	0	1	0	5
(4)	1	0	1	25
-2	-1	0	0	0

In order to transform this entry into 0, we need to subtract the row of the pivot multiplied by 4:

Original row:	4	1	0	1	25
4 times the row of the pivot:	4	0	4	0	20
New row, difference of the two:	0	1	-4	1	5.

Row 3 The value of the entry in the column of the pivot is -2:

x_1	x_2	x_3	x_4	
1	0	1	0	5
4	1	0	1	25
(-2)	-1	0	0	0

In order to transform this entry into 0, we need to add the row of the pivot multiplied by 2:

Original row:	-2	-1	0	0	0
2 times the row of the pivot:	2	0	2	0	10
New row, sum of the two:	0	-1	2	0	10.

After pivoting, we obtain the following tableau, where x_1 and x_4 are in the basis:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	-4	1	5
0	-1	2	0	10

From an algorithmic point of view (statement 13 in Algorithm 16.3), pivoting consists in replacing each entry $T(i,k)$ of each row $i \neq q$ (that is, except for the row of the pivot itself) by

$$T(i,k) - T(i,p)\frac{T(q,k)}{T(q,p)}.$$

For instance, as $p = 1$ and $q = 1$, the element in row $i = 2$ and column $k = 3$, which is 0 before pivoting, is replaced by

$$T(2,3) - T(2,1)\frac{T(1,3)}{T(1,1)} = 0 - 4\frac{1}{1} = -4.$$

$T(2,3)$

x_1	x_2	x_3	x_4	
1	0	1	0	5
4	1	(0)	1	25
-2	-1	0	0	0

$T(2,1)$

x_1	x_2	x_3	x_4	
1	0	1	0	5
(4)	1	0	1	25
-2	-1	0	0	0

$T(1,3)$

x_1	x_2	x_3	x_4	
1	0	(1)	0	5
4	1	0	1	25
-2	-1	0	0	0

$T(1,1)$

x_1	x_2	x_3	x_4	
(1)	0	1	0	5
4	1	0	1	25
-2	-1	0	0	0

The row of the pivot must simply be divided by the value of the pivot. It means that we replace $T(q,k)$ by

$$\frac{T(q,k)}{T(q,p)}.$$

Here, as $T(q,p) = 1$, the row of the pivot is not modified.

At this point, the current tableau is

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	-4	1	5
0	-1	2	0	10

As the bottom left part of the tableau contains negative entries, the sufficient conditions for optimality are not verified and the iterations must continue.

The smallest index[1] corresponding to a negative reduced cost is $p = 2$. Indeed, the reduced cost, read in the last row, is -2. Consequently, the variable x_2 enters the basis. As the variable x_1 corresponds to the first row, the distances α_1 to the constraint is

$$\alpha_1 = +\infty,$$

as the corresponding entry in column 2 is zero:

x_1	x_2	x_3	x_4	
1	(0)	1	0	5
0	1	-4	1	5
0	-1	2	0	10

For variable x_4, corresponding to the second row, we have

$$\alpha_4 = 5/1 = 5,$$

that is the ratio of the two values circled below:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	(1)	-4	1	(5)
0	-1	2	0	10

Therefore, the variable which leaves the basis is x_4. The pivot is thus at column $p = 2$ and row $q = 2$:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	(1)	-4	1	5
0	-1	2	0	10

Pivoting consists in applying elementary row operations to transform the column of the pivot from

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ into } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

[1] Actually, here, the only index.

As the first two entries already contain the requested values, we only need to transform row 3. In order to obtain 0, we must add row 3 and the row of the pivot:

Original row:	0	-1	2	0	10
Row of the pivot:	0	1	-4	1	5
New row, sum of the two:	0	0	-2	1	15.

After pivoting, we obtain the following tableau, where x_1 and x_2 are in the basis:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	-4	1	5
0	0	-2	1	15

As the bottom left part of the tableau contains a negative entry, the sufficient conditions for optimality are not verified and the iterations must continue.

The smallest index corresponding to a negative reduced cost is $p = 3$: the variable x_3 enters the basis. As variable x_1 corresponds to the first row, the distance α_1 to the constraint for that variable is

$$\alpha_1 = 5/1 = 5,$$

that is the ratio of the two circled value below:

x_1	x_2	x_3	x_4	
1	0	(1)	0	(5)
0	1	-4	1	5
0	0	-2	1	15

As variable x_2 corresponds to the second row, the distance α_2 to the constraint for that variable is

$$\alpha_2 = +\infty,$$

because the corresponding entry in column 3 is negative (-4):

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	(-4)	1	5
0	0	-2	1	15

Therefore, the variable which leaves the basis is x_1. The pivot is thus at row $q = 1$ and column $p = 3$. Pivoting consists in applying elementary row operations to transform the column of the pivot from

$$\begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} \text{ into } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We proceed row by row.

Row 1 This is the row of the pivot. The value of the pivot is already 1. Nothing has to be done for that row.

Row 2 The value of the entry in the column of the pivot is -4:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	(-4)	1	5
0	0	-2	1	15

In order to transform this entry into 0, we need to add the row of the pivot multiplied by 4:

Original row:	0	1	-4	1	5
4 times the row of the pivot:	4	0	4	0	20
New row, sum of the two:	4	1	0	1	25.

Row 3 The value of the entry in the column of the pivot is -2:

x_1	x_2	x_3	x_4	
1	0	1	0	5
0	1	-4	1	5
0	0	(-2)	1	15

In order to transform this entry into 0, we need to add the row of the pivot multiplied by 2:

Original row:	0	0	-2	1	15
2 times the row of the pivot:	2	0	2	0	10
New row, sum of the two:	2	0	0	1	25.

After pivoting, we obtain the following tableau, where x_2 and x_3 are in the basis:

x_1	x_2	x_3	x_4	
1	0	1	0	5
4	1	0	1	25
2	0	0	1	25

This tableau is optimal as all reduced costs (in the last row) are non negative. The optimal value of the basic variables are available in the last column, so that

$$x^* = \begin{pmatrix} 0 \\ 25 \\ 5 \\ 0 \end{pmatrix}$$

and

$$c^T x^* = -25.$$

Question 18

For the following optimization problem, write the polyhedron induced by the constraints in standard form, specify the data of the problem (matrix of constraints A, right hand side vector b and cost vector c), and then apply the phase I of the simplex algorithm to identify a feasible tableau for phase II. Finally, identify the optimal tableau and the optimal solution of the optimization problem.

$$\min_{x \in \mathbb{R}^6} -x_1 - 2x_2$$

subject to

$$\begin{aligned}
x_2 \qquad\qquad -x_6 &= 1/2,\\
2x_1 + x_3 + x_4 &= 2,\\
x_1 + x_2 + x_3 &= 1,\\
x_1 - x_2 + x_4 &= 1,\\
-x_1 \qquad + x_5 &= -1/2,\\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0.
\end{aligned}$$

Solution 18

We first multiply the last constraint by -1 to obtain $b \geq 0$.

$$\min_{x \in \mathbb{R}^6} -x_1 - 2x_2$$

subject to

$$\begin{aligned}
x_2 \qquad\qquad -x_6 &= 1/2,\\
2x_1 + x_3 + x_4 &= 2,\\
x_1 + x_2 + x_3 &= 1,\\
x_1 - x_2 \quad + x_4 &= 1,\\
x_1 \qquad\quad - x_5 &= 1/2,\\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0.
\end{aligned}$$

The data of the problem is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 1/2 \\ 2 \\ 1 \\ 1 \\ 1/2 \end{pmatrix}, \; c = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We introduce the auxiliary variables x_1^a, x_2^a, x_3^a, x_4^a and x_5^a, and we replace the objective function.

$$\min_{x \in \mathbb{R}^5} x_1^a + x_2^a + x_3^a + x_4^a + x_5^a,$$

subject to

$$\begin{aligned}
x_2 \qquad\qquad -x_6 + x_1^a \qquad\qquad &= 1/2,\\
2x_1 + x_3 + x_4 \qquad\quad + x_2^a \qquad\quad &= 2,\\
x_1 + x_2 + x_3 \qquad\qquad\quad + x_3^a \qquad &= 1,\\
x_1 - x_2 \quad + x_4 \qquad\qquad\quad + x_4^a \qquad &= 1,\\
x_1 \qquad\quad - x_5 \qquad\qquad\qquad + x_5^a &= 1/2,\\
x_1, x_2, x_3, x_4, x_5, x_6, x_1^a, x_2^a, x_3^a, x_4^a, x_5^a &\geq 0.
\end{aligned}$$

The initial tableau for phase I is

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	1/2
2	0	1	1	0	0	0	1	0	0	0	2
1	1	1	0	0	0	0	0	1	0	0	1
1	-1	0	1	0	0	0	0	0	1	0	1
1	0	0	0	-1	0	0	0	0	0	1	1/2
-5	-1	-2	-2	1	1	0	0	0	0	0	-5

Note that the last row is obtained by adding all the elements of the column in the upper part of the tableau, and changing the sign (see p. 388 in the book).

The leftmost negative reduced cost is -5, and corresponds to x_1, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_1^a &= +\infty, \\
\alpha_2^a &= 2/2 = 1, \\
\alpha_3^a &= 1/1 = 1, \\
\alpha_4^a &= 1/1 = 1, \\
\alpha_5^a &= \frac{1}{2}/1 = \frac{1}{2}.
\end{aligned}$$

Therefore, the variable x_5^a exits the basis. The pivot is at row 5 and column 1:

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	1/2
2	0	1	1	0	0	0	1	0	0	0	2
1	1	1	0	0	0	0	0	1	0	0	1
1	-1	0	1	0	0	0	0	0	1	0	1
(1)	0	0	0	-1	0	0	0	0	0	1	1/2
-5	-1	-2	-2	1	1	0	0	0	0	0	-5

The objective of the pivoting is to apply elementary row operations in order

to replace the column of the pivot from

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 1 \\ -5 \end{bmatrix} \text{ into } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We proceed row by row.

Row 1 The entry is already zero. Nothing has to be done for that row.

Row 2 The entry is 2. In order to transform the entry into 0, we need to subtract the row of the pivot, multiplied by 2.

Orig.:	2	0	1	1	0	0	0	1	0	0	0	2
$2 \times$ pivot:	2	0	0	0	-2	0	0	0	0	0	2	1
New (diff):	0	0	1	1	2	0	0	1	0	0	-2	1

Row 3 The entry is 1. In order to transform the entry into 0, we need to subtract the row of the pivot:

Orig.:	1	1	1	0	0	0	0	0	1	0	0	1
Pivot:	1	0	0	0	-1	0	0	0	0	0	1	1/2
New (diff):	0	1	1	0	1	0	0	0	1	0	-1	1/2

Row 4 The entry is 1. In order to transform the entry into 0, we need to subtract the row of the pivot:

Orig.:	1	-1	0	1	0	0	0	0	0	1	0	1
Pivot:	1	0	0	0	-1	0	0	0	0	0	1	1/2
New (diff):	0	-1	0	1	1	0	0	0	0	1	-1	1/2

Row 5 This is the row of the pivot. The entry is already 1, as requested. Nothing has to be done.

Row 6 The entry is -5. In order to transform the entry into 0, we need to add the row of the pivot, multiplied by 5:

Orig.:	-5	-1	-2	-2	1	1	0	0	0	0	0	-5
$5 \times$ pivot:	5	0	0	0	-5	0	0	0	0	0	5	5/2
New (sum):	0	-1	-2	-2	-4	1	0	0	0	0	5	-5/2

After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	1/2
0	0	1	1	2	0	0	1	0	0	-2	1
0	1	1	0	1	0	0	0	1	0	-1	1/2
0	-1	0	1	1	0	0	0	0	1	-1	1/2
1	0	0	0	-1	0	0	0	0	0	1	1/2
0	-1	-2	-2	-4	1	0	0	0	0	5	-5/2

The leftmost negative reduced cost is -1, and corresponds to x_2, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_1^a &= \frac{1}{2}/1 = \frac{1}{2}, \\
\alpha_2^a &= +\infty, \\
\alpha_3^a &= \frac{1}{2}/1 = \frac{1}{2}, \\
\alpha_4^a &= +\infty, \\
\alpha_1 &= +\infty.
\end{aligned}$$

Therefore, the variable x_1^a exits the basis. Note that both x_1^a and x_3^a are candidate. We apply Bland's rule and select the first one. The pivot is at row 1 and column 2. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	1/2
0	0	1	1	2	0	0	1	0	0	-2	1
0	0	1	0	1	1	-1	0	1	0	-1	0
0	0	0	1	1	-1	1	0	0	1	-1	1
1	0	0	0	-1	0	0	0	0	0	1	1/2
0	0	-2	-2	-4	0	1	0	0	0	5	-2

The leftmost negative reduced cost is -2, and corresponds to x_3, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_2 &= +\infty, \\
\alpha_2^a &= 1/1 = 1, \\
\alpha_3^a &= 0, \\
\alpha_4^a &= +\infty, \\
\alpha_1 &= +\infty.
\end{aligned}$$

Therefore, the variable x_3^a exits the basis. It is a degenerate tableau. The step is 0. The pivot is at row 3 and column 3. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	$1/2$
0	0	0	1	1	-1	1	1	-1	0	-1	1
0	0	1	0	1	1	-1	0	1	0	-1	0
0	0	0	1	1	-1	1	0	0	1	-1	1
1	0	0	0	-1	0	0	0	0	0	1	$1/2$
0	0	0	-2	-2	2	-1	0	2	0	3	-2

The leftmost negative reduced cost is -2, and corresponds to x_4, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_2 &= \infty, \\
\alpha_2^a &= 1/1 = 1, \\
\alpha_3 &= \infty, \\
\alpha_4^a &= 1/1 = 1, \\
\alpha_1 &= +\infty.
\end{aligned}$$

Therefore, the variable x_2^a exits the basis. The pivot is at row 2 and column 4. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	x_1^a	x_2^a	x_3^a	x_4^a	x_5^a	
0	1	0	0	0	-1	1	0	0	0	0	1/2
0	0	0	1	1	-1	1	1	-1	0	-1	1
0	0	1	0	1	1	-1	0	1	0	-1	0
0	0	0	0	0	0	0	-1	1	1	0	0
1	0	0	0	-1	0	0	0	0	0	1	1/2
0	0	0	0	0	0	1	2	0	0	1	0

This is an optimal tableau, as all reduced costs are non negative. It corresponds to a feasible solution of the original problem, as the value of the objective function is zero. But it cannot be used directly for phase II, as the auxiliary variable x_4^a is still in the basis. We need to replace it wither with x_5 or x_6, that are the original variables currently not in the basis. However, none of the pivoting is possible, as the pivots are zero (row 4, columns 5 and 6). It means that the constraint corresponding to row 4 is redundant and can be removed. Therefore, we can clean the tableau:

- we remove row 4, that is a redundant constraint,
- we remove all columns corresponding to the auxiliary variables,
- we calculate the last row using the data from the original problem.

We obtain the initial tableau for phase II, where the last row

$$|c^T - c_B^T B^{-1} A| - c_B^T B^{-1} b|$$

must be calculated using the vector c of the original problem, that is

$$c = (-1,\ -2,\ 0,\ 0,\ 0,\ 0)^T.$$

For example, the reduced cost for variable x_6 is

$$\begin{aligned}
\bar{c}_6 &= c_6 - c_B^T B^{-1} A_6 \\
&= 0 - c_B^T \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= 0 - (-2\ 0\ 0\ -1) \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= -2.
\end{aligned}$$

Note that the arrangement of the elements of c_B must be consistent with the tableau. The first row corresponds to x_2, the second to x_4, the third to x_3 and the fourth to x_1. Therefore,

$$c_B = (c_2, c_4, c_3, c_1)^T.$$

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	0	0	-1	1/2
0	0	0	1	1	-1	1
0	0	1	0	1	1	0
1	0	0	0	-1	0	1/2
0	0	0	0	-1	-2	3/2

The leftmost negative reduced cost is -1, and corresponds to x_5, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_2 &= +\infty, \\
\alpha_4 &= 1/1 = 1, \\
\alpha_3 &= 0/1 = 0, \\
\alpha_1 &= +\infty.
\end{aligned}$$

Therefore, the variable x_3 exits the basis. The pivot is at row 3 and column 5. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	0	0	-1	1/2
0	0	-1	1	0	-2	1
0	0	1	0	1	1	0
1	0	1	0	0	1	1/2
0	0	1	0	0	-1	3/2

The leftmost negative reduced cost is -1, and corresponds to x_6, that enters the basis. The distances to the constraints are

$$\begin{aligned}
\alpha_2 &= +\infty, \\
\alpha_4 &= +\infty, \\
\alpha_5 &= 0/1 = 0, \\
\alpha_1 &= \frac{1}{2}/1 = \frac{1}{2}.
\end{aligned}$$

Therefore, the variable x_5 exits the basis. The pivot is at row 3 and column 6. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	1	0	1	0	1/2
0	0	1	1	2	0	1
0	0	1	0	1	1	0
1	0	0	0	-1	0	1/2
0	0	2	0	1	0	3/2

This is an optimal tableau as all reduced costs are non negative. The optimal solution is $x_1 = 1/2$, $x_2 = 1/2$, $x_3 = 0$, $x_4 = 1$, $x_5 = 0$, $x_6 = 0$. Note that this solution was already found at the end of phase I. But the solution is degenerate, and the tableau was not optimal. We needed another two iterations to obtain an optimal tableau, verifying the sufficient optimality conditions.

Question 19

1. Consider the optimization problem in standard form characterized by the following data:

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \; c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

and the following simplex tableau representing a basic solution of the problem:

	x_1	x_2	x_3	x_4	
	1	-1	1	0	2
	0	2	-1	1	4
	0	-3	2	0	4

What is the basic matrix B?

- $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.
- $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$.
- $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. Consider the following simplex tableau representing a feasible basic solution of a linear minimization problem:

	x_1	x_2	x_3	x_4	x_5	
	1	0	-2	-1	0	2
	0	1	1	1	0	1
	0	0	-3	-2	1	0
	0	0	7	3	0	-9

Which of the following statements is true?

- The problem is unbounded.
- The feasible basic solution is degenerate.
- The feasible basic solution is not optimal.
- There are two basic variables and three non basic variables.

3. The following tableau represents a basic solution for phase I of the simplex algorithm. a_1 and a_2 are the auxiliary variables associated with the two constraints of the linear optimization problem.

	x_1	x_2	x_3	x_4	a_1	a_2	
	2/3	1	0	1/3	0	1/3	6
	-1/3	0	1	1/3	-1	1/3	2
	0	0	0	0	1	1	0

One of the following statements is true. Identify that statement.

- The variable a_1 is a basic variable.
- There is a feasible solution for the original problem and there are no auxiliary variables in the basis.
- The variable x_1 must enter the basis in the next iteration of the algorithm.
- The corresponding solution is optimal for the auxiliary problem, but the original problem is infeasible.

4. Consider the following valid simplex tableau:

	x_1	x_2	x_3	x_4	x_5	
	1	-7	δ	0	α	10
	0	32	-3	1	0	γ
	β	-6	1	0	ε	10
	0	λ	4	θ	ζ	π

In the tableau, α, β, γ, δ, ε, ζ, θ, λ and π are parameters. The objective function of the minimization problem is $c^T x$, where $c^T = \begin{pmatrix} \mu & \rho & 0 & 0 & 0 \end{pmatrix}$. μ and ρ are also parameters.

One of the following statements is false. Which one?

- We must have $\beta = 0$, $\alpha = 0$ and $\varepsilon = 1$.
- The parameter γ must be non negative.
- $\pi = 10\mu$.
- If the tableau is not optimal, then x_2 enters the basis in the next iteration.

Solution 19

1. The two variables in the basis are x_1 and x_4, as the corresponding columns in the upper part of the tableau are the columns of the identity matrix. Therefore, the basic matrix is composed of columns 1 and 4 of matrix A, that is

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

2. First, we observe that the tableau is optimal, as all reduced costs are non negative. We comment on each statement. The one that should have been identified is displayed in bold.

   - The problem is unbounded. Incorrect, as there is an optimal solution.
   - **The feasible basic solution is degenerate.** This is correct, as one of the basic variables (x_5) is equal to zero.
   - The feasible basic solution is not optimal. This is incorrect, as the tableau is optimal.
   - There are two basic variables and three non basic variables. This is incorrect. There are three basic variables: x_1, x_2 and x_5, and two non basic variables: x_3 and x_4.

3. We comment on each statement. The one that should have been identified is displayed in bold.

   - The variable a_1 is a basic variable. This is wrong. Indeed, basic variables correspond to the columns of the $m \times m$ identity matrix in the upper part of the tableau, and are associated with zero reduced costs. This is not the case for a_1.
   - **There is a feasible solution for the original problem and there are no auxiliary variables in the basis.** This is correct. The tableau is optimal as all reduced costs are non negative. The optimal value of the auxiliary problem is zero, which means that it corresponds to a feasible solution of the original problem. Both auxiliary variables, a_1 and a_2 are out of the basis.

- The variable x_1 must enter the basis in the next iteration of the algorithm. This is wrong. The tableau is optimal and all auxiliary variables are out of the basis. There is no need to enter x_1 in the basis.
- The corresponding solution is optimal for the auxiliary problem, but the original problem is infeasible. This is wrong. On the contrary, as the corresponding solution is optimal for the auxiliary problem, with optimal value zero, we have found a feasible solution for the original problem, which is therefore feasible.

4. We comment on each statement. The one that should have been identified is displayed in bold.

   - We must have $\beta = 0$, $\alpha = 0$ and $\varepsilon = 1$. This is correct. The only variables that can be in the basis are x_1, x_4 and x_5. The corresponding columns of the upper part of the tableau should contain only zeros, except for one entry which is one.
   - The parameter γ must be non negative. This is correct. The last column of the upper part of the tableau contains $B^{-1}b$, the value of the basic variables of the current solution. As the tableau corresponds to a feasible basic solution, each entry must be non negative.
   - **$\pi = 10\mu$.** This is incorrect. π is the value of the objective function, *with the opposite sign*:

     $$\pi = -c^T x.$$

     To calculate the value of the objective function, we need to know the value of x_1 and x_2. x_1 is in the basis, and corresponds to the first row. It value is 10. x_2 is not in the basis, and its value is zero. Therefore, the value of the objective function is 10μ. The correct statement would have been $\pi = -10\mu$.
   - If the tableau is not optimal, then x_2 enters the basis in the next iteration. This is correct. There are only two variables out of the basis: x_2 and x_3. As the tableau is not optimal, one of them is associated with a negative reduced cost. It can only be x_2, as the reduced cost for x_3 is 4. Therefore, $\lambda < 0$. This is also the only candidate to enter the basis in the next iteration.

4 Duality

Question 01

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

The optimal solution is $x^* = (2, 0)^T$ with an optimal value 2.

1. Write a new optimization problem by relaxing the first constraint, and penalizing its violation with a parameter λ.
2. Solve the new problem for $\lambda = 0$, 1 and 2, and, for each case, discuss about the value of the objective function at the optimal solution.

Solution 01

To relax the constraint $x_1 + x_2 = 2$, we first write it as $2 - x_1 - x_2 = 0$. Indeed, the quantity $2 - x_1 - x_2$ quantifies the amount of violation of the constraint. If it is zero, the constraint is not violated. If it is different from zero, it is. We introduce a penalty that is proportional to the violation of the constraint, with a proportionality factor λ. The new optimization problem is

$$\min_{x \in \mathbb{R}^2} x_1 + 4x_2 + \lambda(2 - x_1 - x_2)$$

subject to

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

We examine the different cases corresponding to the different values of λ.

- If $\lambda = 0$, the problem is

  $$\min_{x \in \mathbb{R}^2} x_1 + 4x_2$$

  subject to

  $$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

  Its optimal solution is $x^* = (0, 0)^T$ with a value of 0. This solution violates the constraint of the original problem, and the optimal value is lower.

- If $\lambda = 1$, the problem is

  $$\min_{x \in \mathbb{R}^2} 2 + 3x_2$$

  subject to

  $$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

Any feasible x such that $x_2 = 0$ is an optimal solution to the problem with a value of 2, which is the same optimal value as the original problem, with the constraint. Because of the penalty, there is no incentive to violate the constraint, as it does not allow to improve the solution of the original problem.

- If $\lambda = 2$, the problem is

$$\min_{x \in \mathbb{R}^2} 4 - x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

The problem is not bounded, as the value of the objective function goes to $-\infty$ when x_1 goes to $+\infty$.The value of the penalty parameter is not appropriate, as it transforms a well-defined problem into an unbounded problem.

Question 02

Consider the following optimization problem.

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

- Assess if it has an optimal solution, if it is unbounded, or if it is infeasible.
- Write the Lagrangian, the dual function and the dual problem.
- Assess if the dual problem has an optimal solution, if it is unbounded, or if it is infeasible.

Solution 02

The primal problem is

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

Rewriting the equality constraint as $x_1 = 1 + x_2$, the problem becomes

$$\min_{x \in \mathbb{R}^2} 4 + 6x_2$$

subject to

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

The optimal solution is $x_2^* = 0$ and $x_1^* = 1 + x_2^* = 1$. The optimal value is 4. Consequently, we expect the dual problem to have an optimal solution as well.

To write the Lagrangian, we first write the problem as follows:

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} h_1(x) &= 1 - x_1 + x_2 = 0 \quad (\lambda), \\ g_1(x) &= \quad -x_1 \qquad \leq 0 \quad (\mu_1), \\ g_2(x) &= \qquad\quad -x_2 \leq 0 \quad (\mu_2), \end{aligned}$$

where λ, μ_1 and μ_2 are the penalty parameters associated with each constraint.

The Lagrangian function is

$$\begin{aligned} L(x_1, x_2, \lambda, \mu_1, \mu_2) &= 4x_1 + 2x_2 + \lambda(1 - x_1 + x_2) - \mu_1 x_1 - \mu_2 x_2 \\ &= (4 - \lambda - \mu_1)x_1 + (2 + \lambda - \mu_2)x_2 + \lambda. \end{aligned}$$

The dual function is

$$q(\lambda, \mu_1, \mu_2) = \min_{x_1, x_2} L(x_1, x_2, \lambda, \mu_1, \mu_2).$$

In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero, since we removed all the constraints on x_1 and x_2. Therefore, we have to impose the following constraints on the penalty parameters:

$$\mu_1 = 4 - \lambda, \tag{10}$$

$$\mu_2 = 2 + \lambda. \tag{11}$$

In that case, the dual function becomes

$$q(\lambda, \mu_1, \mu_2) = \min_{x_1, x_2} \lambda = \lambda.$$

Moreover, the parameters associated with the inequality constraints must be non negative. That is, $\mu_1 \geq 0$ and $\mu_2 \geq 0$. The dual problem is therefore

$$\max_{\lambda, \mu_1, \mu_2} \lambda$$

subject to

$$\begin{aligned} \mu_1 &= 4 - \lambda, \\ \mu_2 &= 2 + \lambda, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0. \end{aligned}$$

As the parameters μ_1 and μ_2 do not appear in the objective function, we rewrite (10) and (11) as

$$\begin{aligned} 4 - \lambda &\geq 0, \\ 2 + \lambda &\geq 0, \end{aligned}$$

and eliminate the μ's, so that the dual problem becomes

$$\max_{\lambda} \lambda$$

subject to

$$\begin{aligned} 4 - \lambda &\geq 0, \\ 2 + \lambda &\geq 0. \end{aligned}$$

The optimal solution is $\lambda^* = 4$. Note that $\mu_1^* = 4 - \lambda^* = 0$ and $\mu_2^* = 2 + \lambda^* = 6$. The optimal value is 4. It is the same value as the primal problem.

Question 03

Consider the following optimization problem.

$$\min_{x \in \mathbb{R}^2} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &= 2, \\ x_1 - x_2 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

- Assess if it has an optimal solution, if it is unbounded, or if it is infeasible.
- Write the Lagrangian, the dual function and the dual problem.
- Assess if the dual problem has an optimal solution, if it is unbounded, or if it is infeasible.

Solution 03

The primal problem is

$$\min_{x \in \mathbb{R}^2} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &= 2, \\ x_1 - x_2 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

The two equality constraints are incompatible, and the optimization problem is infeasible. Consequently, we expect the dual problem to be either infeasible or unbounded.

To write the Lagrangian, we first write the problem as follows:

$$\min_{x \in \mathbb{R}^2} -4x_1 + 2x_2$$

subject to

$$\begin{aligned} h_1(x) &= 2 + x_1 - x_2 = 0 \quad (\lambda_1), \\ h_2(x) &= 1 - x_1 + x_2 = 0 \quad (\lambda_2), \\ g_1(x) &= \quad -x_1 \quad\quad \leq 0 \quad (\mu_1), \\ g_2(x) &= \quad\quad\quad -x_2 \leq 0 \quad (\mu_2), \end{aligned}$$

where λ_1, λ_2, μ_1 and μ_2 are the penalty parameters associated with each constraint.

The Lagrangian function is

$$\begin{aligned} L(x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2) &= -4x_1 + 2x_2 + \lambda_1(2 + x_1 - x_2) \\ &\quad + \lambda_2(1 - x_1 + x_2) - \mu_1 x_1 - \mu_2 x_2 \\ &= (-4 + \lambda_1 - \lambda_2 - \mu_1)x_1 \\ &\quad + (2 - \lambda_1 + \lambda_2 - \mu_2)x_2 + 2\lambda_1 + \lambda_2. \end{aligned}$$

The dual function is

$$q(\lambda_1, \lambda_2, \mu_1, \mu_2) = \min_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2).$$

In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero. Therefore, we have to impose the following constraints on the penalty parameters:

$$\mu_1 = -4 + \lambda_1 - \lambda_2, \tag{12}$$
$$\mu_2 = 2 - \lambda_1 + \lambda_2. \tag{13}$$

In that case, the dual function becomes

$$q(\lambda_1, \lambda_2, \mu_1, \mu_2) = 2\lambda_1 + \lambda_2.$$

Moreover, the parameters associated with the inequality constraints must be non negative. That is, $\mu_1 \geq 0$ and $\mu_2 \geq 0$. The dual problem is therefore

$$\max_{\lambda_1, \lambda_2, \mu_1, \mu_2} 2\lambda_1 + \lambda_2$$

subject to

$$\begin{aligned}
\mu_1 &= -4 + \lambda_1 - \lambda_2, \\
\mu_2 &= 2 - \lambda_1 + \lambda_2, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0.
\end{aligned}$$

As the parameters μ_1 and μ_2 do not appear in the objective function, we rewrite (12) and (13) as

$$\begin{aligned}
-4 + \lambda_1 - \lambda_2 &\geq 0, \\
2 - \lambda_1 + \lambda_2 &\geq 0,
\end{aligned}$$

and eliminate the μ's, so that the dual problem becomes

$$\max_{\lambda_1, \lambda_2} 2\lambda_1 + \lambda_2$$

subject to

$$\begin{aligned}
\lambda_1 - \lambda_2 &\geq 4, \\
\lambda_1 - \lambda_2 &\leq 2.
\end{aligned}$$

The quantity $\lambda_1 - \lambda_2$ cannot be both larger or equal to 4, and lesser or equal to 2. Therefore, the dual problem is also infeasible.

Question 04

Consider the following optimization problem.

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 2, \\ -x_1 - x_2 &\geq 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

- Assess if it has an optimal solution, if it is unbounded, or if it is infeasible.
- Write the Lagrangian, the dual function and the dual problem.
- Assess if the dual problem has an optimal solution, if it is unbounded, or if it is infeasible.

Remember that the Lagrangian is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{j=1}^{m} \lambda_j h_i(x) + \sum_{j=1}^{p} \mu_j g_i(x),$$

where f, g and h define the optimization problem, as follows:

$$\min f(x) \quad [f : \mathbb{R}^n \to \mathbb{R}]$$

subject to

$$\begin{aligned} h(x) &= 0, \quad [h : \mathbb{R}^n \to \mathbb{R}^m] \\ g(x) &\leq 0. \quad [g : \mathbb{R}^n \to \mathbb{R}^p] \end{aligned}$$

Solution 04

The primal problem is

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 2, \\ -x_1 - x_2 &\geq 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

The quantity $x_1 + x_2$ must be both larger or equal to 2, and lesser or equal to -1. This is not possible, and the problem is infeasible. Consequently, we expect the dual problem to be either infeasible or unbounded.

To write the Lagrangian, we first write the problem as follows:

$$\min_{x \in \mathbb{R}^2} 4x_1 + 2x_2$$

subject to

$$\begin{aligned} g_1(x) &= 2 - x_1 - x_2 \leq 0 \quad (\mu_1), \\ g_2(x) &= 1 + x_1 + x_2 \leq 0 \quad (\mu_2), \\ g_3(x) &= \quad -x_1 \quad\quad \leq 0 \quad (\mu_3), \\ g_4(x) &= \quad\quad\quad -x_2 \leq 0 \quad (\mu_4), \end{aligned}$$

where μ_1, μ_2, μ_3 and μ_3 are the penalty parameters associated with each constraint.

The Lagrangian function is

$$\begin{aligned} L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4) &= 4x_1 + 2x_2 + \mu_1(2 - x_1 - x_2) \\ &\quad + \mu_2(1 + x_1 + x_2) - \mu_3 x_1 - \mu_4 x_2 \\ &= (4 - \mu_1 + \mu_2 - \mu_3)x_1 \\ &\quad + (2 - \mu_1 + \mu_2 - \mu_4)x_2 + 2\mu_1 + \mu_2. \end{aligned}$$

The dual function is

$$q(\mu_1, \mu_2, \mu_3, \mu_4) = \min_{x_1, x_2} L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4).$$

In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero. Therefore, we have to impose the following constraints on the penalty parameters:

$$\mu_3 = 4 - \mu_1 + \mu_2, \tag{14}$$

$$\mu_4 = 2 - \mu_1 + \mu_2. \tag{15}$$

In that case, the dual function becomes

$$q(\mu_1, \mu_2, \mu_3, \mu_4) = 2\mu_1 + \mu_2.$$

Moreover, the parameters associated with the inequality constraints must be non negative. That is, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$. The dual problem is therefore

$$\max_{\mu_1,\mu_2,\mu_3,\mu_4} 2\mu_1 + \mu_2$$

subject to

$$\begin{aligned}
\mu_3 &= 4 - \mu_1 + \mu_2, \\
\mu_4 &= 2 - \mu_1 + \mu_2, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0, \\
\mu_3 &\geq 0, \\
\mu_4 &\geq 0.
\end{aligned}$$

As the parameters μ_3 and μ_4 do not appear in the objective function, we rewrite (14) and (15) as

$$\begin{aligned}
4 - \mu_1 + \mu_2 &\geq 0, \\
2 - \mu_1 + \mu_2 &\geq 0,
\end{aligned}$$

and eliminate μ_3 and μ_4, so that the dual problem becomes

$$\max_{\mu_1,\mu_2} 2\mu_1 + \mu_2$$

subject to

$$\begin{aligned}
4 - \mu_1 + \mu_2 &\geq 0, \\
2 - \mu_1 + \mu_2 &\geq 0, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0.
\end{aligned}$$

The problem happens to be unbounded. To see it, first fix the value of μ_1 to 0. We obtain

$$\max_{\mu_2} \mu_2$$

subject to

$$\begin{aligned}
\mu_2 &\geq -4, \\
\mu_2 &\geq -2, \\
\mu_2 &\geq 0.
\end{aligned}$$

As μ_2 is not constrained from above, its value and, therefore, the value of the objective function, can always be made larger than any arbitrary value.

Question 05

If the primal is a minimization problem, and the dual is written as a maximization problem, then the following rules apply.

- For each constraint of the primal there is a dual variable. The sign of the dual variable depends on the sign of the primal constraint, according to the following table:

Constraint of the primal	Dual variable
$=$	free
$\leq$	≤ 0
$\geq$	≥ 0

- For each primal variable there is a dual constraint. The sign of the constraint depends on the sign of the primal variable, according to the table below.

Primal variable	Dual constraint
≥ 0	$\leq$
≤ 0	$\geq$
free	$=$

Write the dual of the following linear optimization problem in two ways: first use the summary tables above, and then use the Lagrangian method in order to see the correctness of those summary tables.

$$\min x_1 - 4x_2 + 2x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 10, \\ x_1 + 4x_3 &= 16, \\ x_2 - x_3 &\leq 5, \\ x_1 &\geq 0, \\ x_2 &\in \mathbb{R}, \\ x_3 &\leq 0. \end{aligned}$$

Solution 05

There are three constraints (different from the non negativity constraints), so that there are three dual variables. Their sign is defined by the summary table.

Constraint of the primal	Dual variable
$2x_1 + x_2 \geq 10,$	$\gamma_1 \geq 0,$
$x_1 + 4x_3 = 16,$	$\gamma_2 \in \mathbb{R},$
$x_2 - x_3 \leq 5,$	$\gamma_3 \leq 0.$

There is a dual constraint for each primal variable. As $x_1 \geq 0$, we have

$$2\gamma_1 + \gamma_2 + 0\gamma_3 \leq 1,$$

where 2, 1 and 0 are the coefficients of x_1 in the primal constraints, and the 1 at the right hand side is the coefficient of x_1 in the objective function. The sign of the constraint is determined by the summary table. As x_1 must be non negative in the primal, the corresponding constraint in the dual is $\leq$. Similarly, as $x_2 \in \mathbb{R}$, we have

$$\gamma_1 + \gamma_3 = -4,$$

and as $x_3 \leq 0$, we have

$$4\gamma_2 - \gamma_3 \geq 2.$$

Therefore, according to the summary tables, the dual problem is:

$$\max 10\gamma_1 + 16\gamma_2 + 5\gamma_3 \tag{16}$$

subject to

$$\begin{aligned} 2\gamma_1 + \gamma_2 &\leq 1, \\ \gamma_1 + \gamma_3 &= -4, \\ 4\gamma_2 - \gamma_3 &\geq 2, \\ \gamma_1 &\geq 0, \\ \gamma_2 &\in \mathbb{R}, \\ \gamma_3 &\leq 0. \end{aligned} \tag{17}$$

In order to calculate the Lagrangian, we write the problem as

$$\min x_1 - 4x_2 + 2x_3$$

subject to

$$\begin{aligned}
h_1(x) &= 16 - x_1 - 4x_3 = 0, && (\lambda)\\
g_1(x) &= 10 - 2x_1 - x_2 \leq 0, && (\mu_1)\\
g_2(x) &= -5 + x_2 - x_3 \leq 0, && (\mu_2)\\
g_3(x) &= -x_1 \leq 0, && (\mu_3)\\
g_4(x) &= x_3 \leq 0. && (\mu_4)
\end{aligned}$$

The Lagrangian function of this problem is

$$\begin{aligned}
L(x_1, x_2, x_3, \lambda, \mu_1, \mu_2, \mu_3, \mu_4) &= x_1 - 4x_2 + 2x_3 + \lambda(16 - x_1 - 4x_3)\\
&\quad + \mu_1(10 - 2x_1 - x_2) + \mu_2(-5 + x_2 - x_3)\\
&\quad - \mu_3 x_1 + \mu_4 x_3\\
&= x_1(1 - \lambda - 2\mu_1 - \mu_3) + x_2(-4 - \mu_1 + \mu_2)\\
&\quad + x_3(2 - 4\lambda - \mu_2 + \mu_4) + 16\lambda + 10\mu_1 - 5\mu_2.
\end{aligned}$$

In order for the dual function to be bounded, the coefficients of x_1, x_2, x_3 have to be zero. Therefore,

$$\mu_3 = 1 - \lambda - 2\mu_1, \tag{18}$$
$$\mu_2 = 4 + \mu_1, \tag{19}$$
$$\mu_4 = -2 + 4\lambda + \mu_2, \tag{20}$$

and the dual function is

$$q(\lambda, \mu_1, \mu_2, \mu_3, \mu_4) = 16\lambda + 10\mu_1 - 5\mu_2.$$

Moreover, we must impose $\mu_1, \mu_2, \mu_3, \mu_4 \geq 0$. As μ_3 and μ_4 do not appear in the dual function, they can be eliminated.

- Combining (18) and $\mu_3 \geq 0$, we obtain $\lambda \leq 1 - 2\mu_1$.
- Combining (20) and $\mu_4 \geq 0$, we obtain $\lambda \geq \frac{1}{2} - \frac{1}{4}\mu_2$.

Therefore, the constraints on the dual variables are

$$\begin{aligned}
\lambda &\leq 1 - 2\mu_1, \\
\lambda &\geq \frac{1}{2} - \frac{1}{4}\mu_2, \\
\mu_2 &= 4 + \mu_1, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0.
\end{aligned}$$

Therefore, the dual problem is written as

$$\max 16\lambda + 10\mu_1 - 5\mu_2$$

subject to

$$\begin{aligned}
\lambda + 2\mu_1 &\leq 1, \\
\lambda + \frac{1}{4}\mu_2 &\geq \frac{1}{2}, \\
-\mu_1 + \mu_2 &= 4, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0.
\end{aligned}$$

If we define

$$\gamma_1 = \mu_1, \quad \gamma_2 = \lambda, \quad \gamma_3 = -\mu_2,$$

we obtain

$$\max 10\gamma_1 + 16\gamma_2 + 5\gamma_3$$

subject to

$$\begin{aligned}
\gamma_2 + 2\gamma_1 &\leq 1, \\
\gamma_2 - \frac{1}{4}\gamma_3 &\geq \frac{1}{2}, \\
-\gamma_1 - \gamma_3 &= 4, \\
\gamma_1 &\geq 0, \\
-\gamma_3 &\geq 0,
\end{aligned}$$

which corresponds to the optimization problem (16)–(17) obtained using the summary tables.

Question 06

If the primal is a minimization problem, and the dual is written as a maximization problem, then the following rules apply.

- For each constraint of the primal there is a dual variable. The sign of the dual variable depends on the sign of the primal constraint, according to the following table:

Constraint of the primal	Dual variable
$=$	free
$\leq$	≤ 0
$\geq$	≥ 0

- For each primal variable there is a dual constraint. The sign of the constraint depends on the sign of the primal variable, according to the table below.

Primal variable	Dual constraint
≥ 0	$\leq$
≤ 0	$\geq$
free	$=$

Write the dual of the following linear optimization problem in two ways: first use the summary tables above, and then use the Lagrangian method in order to see the correctness of those summary tables.

$$\min -4x_1 - 7x_2$$

subject to

$$\begin{aligned} -5x_1 + 4x_2 &\leq 16 \\ x_1 + 3x_2 &\leq 31 \\ x_1 + 2x_2 &\leq 24 \\ 3x_1 + 5x_2 &\leq 68 \\ x_1 + x_2 &\leq 122 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

Solution 06

There are five constraints (different from the non negativity constraints), so that there are five dual variables. As all constraints are $\leq$, the corresponding dual variables are non negative, according to the summary table.

$$\begin{aligned} -5x_1 + 4x_2 &\leq 16, & \gamma_1 &\leq 0, \\ x_1 + 3x_2 &\leq 31, & \gamma_2 &\leq 0, \\ x_1 + 2x_2 &\leq 24, & \gamma_3 &\leq 0, \\ 3x_1 + 5x_2 &\leq 68, & \gamma_4 &\leq 0, \\ x_1 + x_2 &\leq 122, & \gamma_5 &\leq 0. \end{aligned}$$

As the two variables of the primal are non negative, the constraints of the dual are $\leq$, according to the summary table. Therefore, the dual problem is:

$$\max 16\gamma_1 + 31\gamma_2 + 24\gamma_3 + 68\gamma_4 + 122\gamma_5 \tag{21}$$

subject to

$$\begin{aligned} -5\gamma_1 + \gamma_2 + \gamma_3 + 3\gamma_4 + \gamma_5 &\leq -4, \\ 4\gamma_1 + 3\gamma_2 + 2\gamma_3 + 5\gamma_4 + \gamma_5 &\leq -7, \\ \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 &\leq 0. \end{aligned} \tag{22}$$

In order to calculate the Lagrangian, we write the problem as

$$\min -4x_1 - 7x_2$$

subject to

$$\begin{aligned} g_1(x) &= -16 \; -5x_1 + 4x_2 \leq 0 \quad (\mu_1) \\ g_2(x) &= -31 \quad +x_1 + 3x_2 \leq 0 \quad (\mu_2) \\ g_3(x) &= -24 \quad +x_1 + 2x_2 \leq 0 \quad (\mu_3) \\ g_4(x) &= -68 \; +3x_1 + 5x_2 \leq 0 \quad (\mu_4) \\ g_5(x) &= -122 \; +x_1 + x_2 \leq 0 \quad (\mu_5) \\ g_6(x) &= \quad -x_1 \leq 0 \quad (\mu_6) \\ g_7(x) &= \quad -x_2 \leq 0 \quad (\mu_7). \end{aligned}$$

The Lagrangian function of this problem is

$$\begin{aligned}
L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7) &= -4x_1 - 7x_2 + \mu_1(-16 - 5x_1 + 4x_2) \\
&\quad + \mu_2(-31 + x_1 + 3x_2) + \mu_3(-24 + x_1 + 2x_2) \\
&\quad + \mu_4(-68 + 3x_1 + 5x_2) + \mu_5(-122 + x_1 + x_2) \\
&\quad - \mu_6 x_1 - \mu_7 x_2 \\
&= x_1(-4 - 5\mu_1 + \mu_2 + \mu_3 + 3\mu_4 + \mu_5 - \mu_6) \\
&\quad + x_2(-7 + 4\mu_1 + 3\mu_2 + 2\mu_3 + 5\mu_4 + \mu_5 - \mu_7) \\
&\quad - 16\mu_1 - 31\mu_2 - 24\mu_3 - 68\mu_4 - 122\mu_5.
\end{aligned}$$

In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero:

$$\mu_6 = -4 - 5\mu_1 + \mu_2 + \mu_3 + 3\mu_4 + \mu_5, \tag{23}$$

$$\mu_7 = -7 + 4\mu_1 + 3\mu_2 + 2\mu_3 + 5\mu_4 + \mu_5, \tag{24}$$

and the dual function is

$$q(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7) = -16\mu_1 - 31\mu_2 - 24\mu_3 - 68\mu_4 - 122\mu_5.$$

Moreover, we must impose $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7 \geq 0$. Thus, As μ_6 and μ_7 do not appear in the dual function, they can be eliminated.

- Combining (23) and $\mu_6 \geq 0$, we obtain

$$-4 - 5\mu_1 + \mu_2 + \mu_3 + 3\mu_4 + \mu_5 \geq 0.$$

- Combining (24) and $\mu_7 \geq 0$, we obtain

$$-7 + 4\mu_1 + 3\mu_2 + 2\mu_3 + 5\mu_4 + \mu_5 \geq 0.$$

Therefore, the constraints on the dual variables are

$$\begin{aligned}
-5\mu_1 + \mu_2 + \mu_3 + 3\mu_4 + \mu_5 &\geq 4, \\
4\mu_1 + 3\mu_2 + 2\mu_3 + 5\mu_4 + \mu_5 &\geq 7, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0, \\
\mu_3 &\geq 0, \\
\mu_4 &\geq 0, \\
\mu_5 &\geq 0.
\end{aligned}$$

The dual problem is written as

$$\max_{\mu \in \mathbb{R}^5} -16\mu_1 - 31\mu_2 - 24\mu_3 - 68\mu_4 - 122\mu_5$$

subject to

$$\begin{aligned} -5\mu_1 + \mu_2 + \mu_3 + 3\mu_4 + \mu_5 &\geq 4, \\ 4\mu_1 + 3\mu_2 + 2\mu_3 + 5\mu_4 + \mu_5 &\geq 7, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0, \\ \mu_3 &\geq 0, \\ \mu_4 &\geq 0, \\ \mu_5 &\geq 0. \end{aligned}$$

If we define $\gamma_i = -\mu_i$, $i = 1, \ldots, 5$, we obtain

$$\max_{\mu \in \mathbb{R}^5} 16\gamma_1 + 31\gamma_2 + 24\gamma_3 + 68\gamma_4 + 122\gamma_5$$

subject to

$$\begin{aligned} 5\gamma_1 - \gamma_2 - \gamma_3 - 3\gamma_4 - \gamma_5 &\geq 4, \\ -4\gamma_1 - 3\gamma_2 - 2\gamma_3 - 5\gamma_4 - \gamma_5 &\geq 7, \\ -\gamma_1 &\geq 0, \\ -\gamma_2 &\geq 0, \\ -\gamma_3 &\geq 0, \\ -\gamma_4 &\geq 0, \\ -\gamma_5 &\geq 0, \end{aligned}$$

which corresponds to the optimization problem (21)–(22) obtained using the summary tables.

Question 07

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -3x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq 2, \\ -x_1 + x_2 &\leq -3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Write the Lagrangian, the dual function and the dual problem.
2. Represent graphically the feasible set of the primal problem.
3. Represent graphically the feasible set of the dual problem.

Solution 07

We derive the dual problem.

1. We write the optimization problem as

$$\min_{x \in \mathbb{R}^2} -3x_1 + 2x_2$$

subject to

$$\begin{aligned} g_1(x) &= -2 + x_1 - x_2 \leq 0, \quad (\mu_1) \\ g_2(x) &= \ \ 3 - x_1 + x_2 \leq 0, \quad (\mu_2) \\ g_3(x) &= \qquad -x_1 \leq 0, \quad (\mu_3) \\ g_4(x) &= \qquad -x_2 \leq 0. \quad (\mu_4) \end{aligned}$$

The Lagrangian function of this problem is

$$\begin{aligned} &L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4) = \\ &\quad -3x_1 + 2x_2 + \mu_1(-2 + x_1 - x_2) + \mu_2(3 - x_1 + x_2) - \mu_3 x_1 - \mu_4 x_2 = \\ &\quad (-3 + \mu_1 - \mu_2 - \mu_3)x_1 + (2 - \mu_1 + \mu_2 - \mu_4)x_2 - 2\mu_1 + 3\mu_2. \end{aligned}$$

In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero and we get

$$-3 + \mu_1 - \mu_2 - \mu_3 = 0,$$

$$2 - \mu_1 + \mu_2 - \mu_4 = 0.$$

The dual function becomes

$$q(\mu_1, \mu_2, \mu_3, \mu_4) = -2\mu_1 + 3\mu_2.$$

The dual problem is written as

$$\max -2\mu_1 + 3\mu_2$$

subject to

$$\begin{aligned} -3 + \mu_1 - \mu_2 - \mu_3 &= 0, \\ 2 - \mu_1 + \mu_2 - \mu_4 &= 0, \\ \mu_1 &\geq 0, \\ \mu_2 &\geq 0, \\ \mu_3 &\geq 0, \\ \mu_4 &\geq 0. \end{aligned}$$

2. Figure 8 shows that the feasible set of the primal problem is empty since $x_1 - x_2$ cannot be both less than 2 and greater than 3.

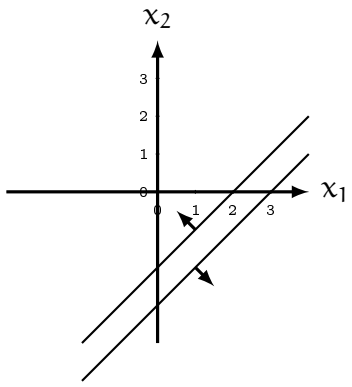


Figure 8: Feasible set of the primal problem.

3. Figure 9 shows that the feasible set is exactly the same than the primal problem's one. It is empty too.

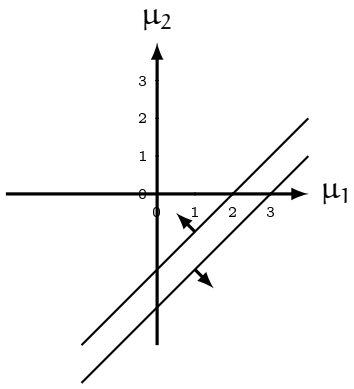


Figure 9: Feasible set of the dual problem.

Question 08

Consider a matrix $A \in \mathbb{R}^{n \times n}$ such that $A^T = -A$, and the vector $c \in \mathbb{R}^n$. Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax \geq -c,$$
$$x \geq 0.$$

Demonstrate that this problem is self-dual, i.e., that the dual problem is equivalent to the primal problem.

Solution 08

We write the problem as

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$\begin{aligned} g_1(x) &= -c - Ax \leq 0, \\ g_2(x) &= -x \leq 0, \end{aligned}$$

where $g_1 : \mathbb{R}^n \to \mathbb{R}^n$, and $g_2 : \mathbb{R}^n \to \mathbb{R}^n$. We introduce the dual variables $\mu_1 \in \mathbb{R}^n$ and $\mu_2 \in \mathbb{R}^n$, and we write the Lagrangian

$$\begin{aligned} L(x, \mu_1, \mu_2) &= c^T x - \mu_1^T (Ax + c) - \mu_2^T x \\ &= (c - A^T \mu_1 - \mu_2)^T x - c^T \mu_1. \end{aligned}$$

Therefore, the dual problem is

$$\max_{\mu_1} -c^T \mu_1$$

subject to

$$\begin{aligned} \mu_2 = c - A^T \mu_1 &\geq 0, \\ \mu_1 &\geq 0. \end{aligned}$$

Writing it as a minimization problem, and renaming μ_1 as x, we obtain the following problem:

$$\min_{x} c^T x$$

subject to

$$\begin{aligned} c - A^T x &\geq 0, \\ x &\geq 0. \end{aligned}$$

We now exploit the fact that $A^T = -A$ to obtain

$$\min_{x} c^T x$$

subject to

$$\begin{aligned} c + Ax &\geq 0, \\ x &\geq 0, \end{aligned}$$

which is the original problem.

Question 09

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2^2$$

subject to

$$x_1 = 1.$$

1. What is the optimal solution of this problem?
2. Write the Lagrangian of this problem.
3. Write the dual function.
4. Write and solve the dual problem.

Solution 09

1. The optimal solution of the problem is $x^* = (1, 0)$, and its optimal value is 1. Indeed, x_1 is constrained to be 1. And x_2 achieves its minimum at $x_2 = 0$, irrespectively of the value of x_1.

2. In order to write the Lagrangian, we first express the problem as

$$\min x_1^2 + x_2^2$$

subject to

$$h_1(x) = x_1 - 1 = 0. \quad (\lambda)$$

The Lagrangian function of this problem is

$$\begin{aligned} L(x_1, x_2, \lambda) &= x_1^2 + x_2^2 + \lambda(x_1 - 1) \\ &= x_1^2 + x_2^2 + \lambda x_1 - \lambda. \end{aligned}$$

If is a quadratic function, such that the first derivatives are

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 2x_1 + \lambda, \quad \frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 2x_2,$$

and the second derivatives matrix is

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

As the second derivatives matrix is positive definite, the quadratic function is convex and, therefore, bounded for any value of λ. Consequently, there is no need to impose constraints on the dual variables in this context.

3. We are looking for

$$q(\lambda) = \min_{x \in \mathbb{R}^2} L(x_1, x_2, \lambda).$$

The first order optimality conditions lead to

$$2x_1 + \lambda = 0, \qquad 2x_2 = 0,$$

and the Lagrangian is minimized at the point $(x_1, x_2) = (-\frac{\lambda}{2}, 0)$. Thus, the dual function is

$$q(\lambda) = \frac{\lambda^2}{4} - \frac{\lambda}{2}\lambda - \lambda = -\frac{\lambda^2}{4} - \lambda.$$

4. Since there are no constraints on λ, the dual problem is

$$\max_{\lambda \in \mathbb{R}} -\frac{\lambda^2}{4} - \lambda.$$

To solve it, we use the first order optimality conditions, which lead to

$$-\frac{\lambda^*}{2} - 1 = 0 \Leftrightarrow \lambda^* = -2.$$

As the second derivative is negative $(-1/2)$, the function is concave, and λ^* is indeed a maximum, with optimal value 1. It happens to be the same as the optimal value of the primal. Note that we have seen that strong duality applies to linear optimization, while this optimization problem is nonlinear. Strong duality does not in general apply to non linear optimization problems. It applies only in some specific cases. For instance, when the constraints are linear and the objective function convex, like in this example.

Question 10

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -x_1 - x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq 1, \\ x_1 - \frac{1}{2} x_2 &\leq 0, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Write the Lagrangian of this problem.
2. Write the dual function.
3. Write the dual problem.
4. Represent graphically the feasible set of the primal problem.
5. Represent graphically the feasible set of the dual problem.

Solution 10

1. The problem can be written as

$$\min -x_1 - x_2$$

subject to

$$\begin{aligned} g_1(x) &= -1 - x_1 + x_2 \leq 0 \quad (\mu_1) \\ g_2(x) &= x_1 - \frac{1}{2}x_2 \leq 0 \quad (\mu_2) \\ g_3(x) &= -x_1 \leq 0 \quad (\mu_3) \\ g_4(x) &= -x_2 \leq 0 \quad (\mu_4). \end{aligned}$$

The Lagrangian function of this problem is

$$\begin{aligned} L(\mu_1, \mu_2, \mu_3, \mu_4) &= -x_1 - x_2 + \mu_1(-1 - x_1 + x_2) + \mu_2(x_1 - \frac{1}{2}x_2) - \mu_3 x_1 - \mu_4 x_2 \\ &= x_1(-1 - \mu_1 + \mu_2 - \mu_3) + x_2(-1 + \mu_1 - \frac{1}{2}\mu_2 - \mu_4) - \mu_1. \end{aligned}$$

2. In order for the dual function to be bounded, the coefficients of x_1 and x_2 have to be zero, and we impose $\mu_3, \mu_4 \geq 0$. Thus,

$$\mu_3 = -1 - \mu_1 + \mu_2 \geq 0, \qquad \mu_4 = -1 + \mu_1 - \frac{1}{2}\mu_2 \geq 0,$$

or equivalently

$$\mu_1 \leq -1 + \mu_2, \qquad \mu_1 \geq 1 + \frac{1}{2}\mu_2.$$

Therefore

$$X_q = \left\{ \mu_1, \mu_2 | \mu_1 \leq -1 + \mu_2, \mu_1 \geq 1 + \frac{1}{2}\mu_2, \mu_1 \geq 0, \mu_2 \geq 0 \right\},$$

and the Lagrangian function becomes

$$q(\mu_1, \mu_2) = -\mu_1.$$

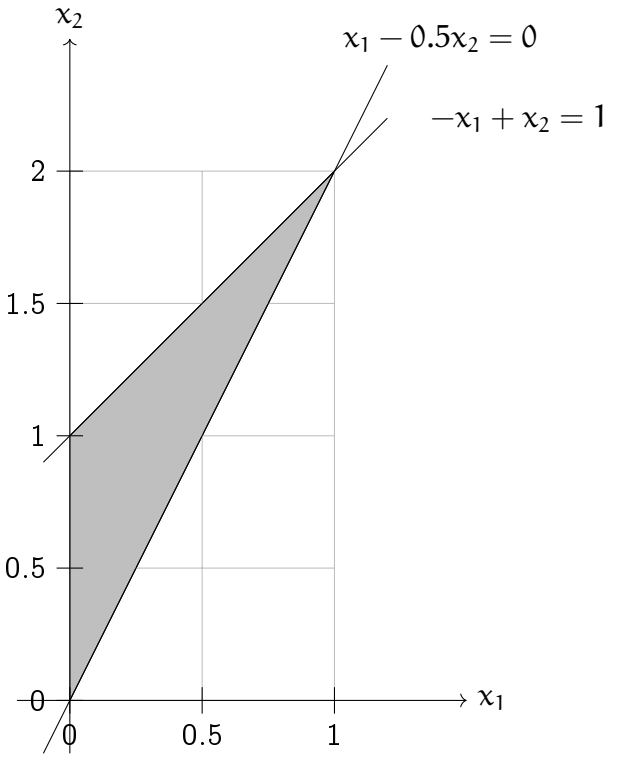
3. Thus, the dual problem is written as

$$\max -\mu_1$$

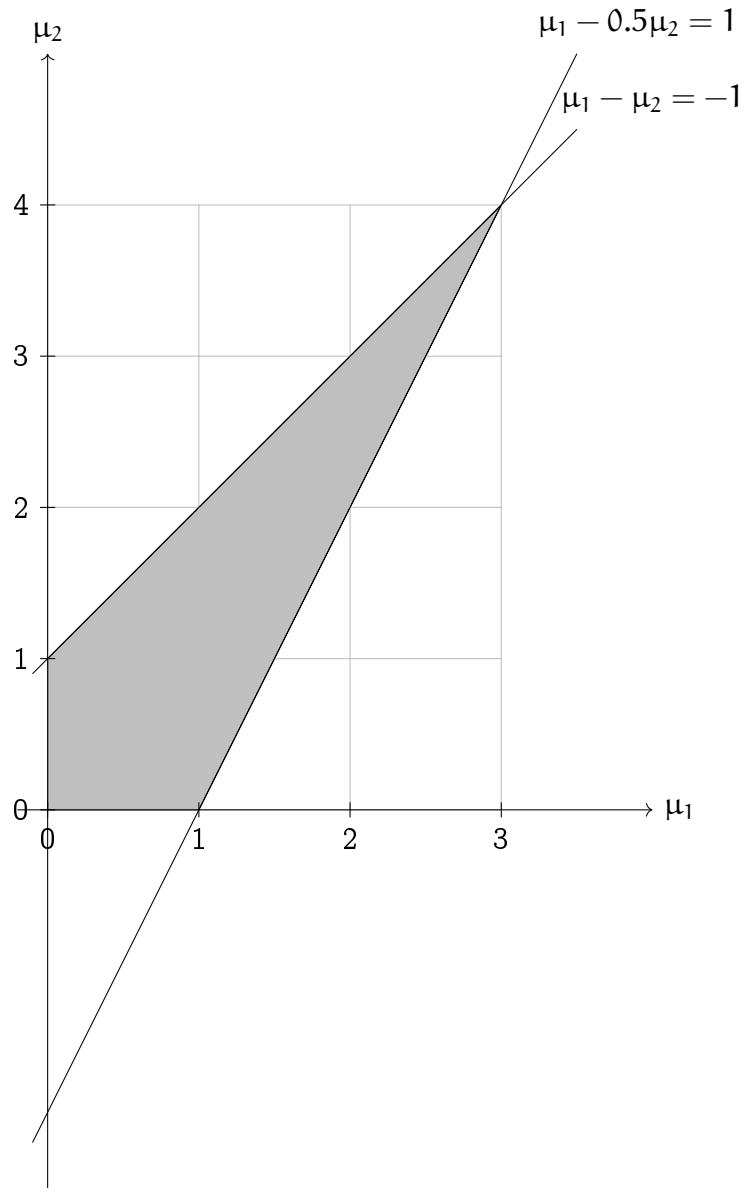
subject to

$$\begin{aligned} \mu_1 - \mu_2 &\leq -1 \\ \mu_1 - \frac{1}{2}\mu_2 &\geq 1 \\ \mu_1, \mu_2 &\geq 0. \end{aligned}$$

4. The feasible set of the primal problem is shown in the figure below.



5. The feasible set of the dual problem is shown in the figure below.


μ_2
$\mu_1 - 0.5\mu_2 = 1$
$\mu_1 - \mu_2 = -1$
4
3
2
1
0
0
1
2
3
μ_1

Question 11

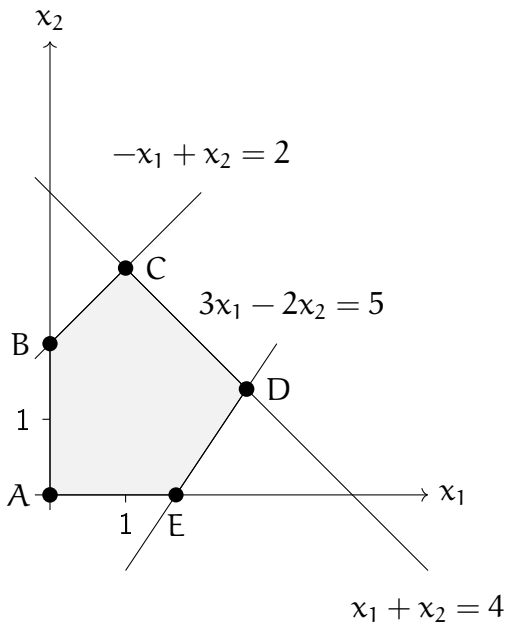
Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} -3x_1 - 4x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq 2, \\ -3x_1 + 2x_2 &\geq -5, \\ x_1 + x_2 &\leq 4, \\ x_1, x_2 &\geq 0. \end{aligned}$$

The feasible set is shown on the Figure below.



1. Solve the problem by calculating the value of the objective function at each vertex.
2. Write the dual of this problem.
3. Use the complementarity slackness conditions to solve the dual.

Reminder: if x^* is primal feasible and λ^* is dual feasible, the complemetarity slackness conditions state that x^* is optimal for the primal and λ^* is optimal for the dual if and only if

$$(c_i - \sum_{j=1}^{m} a_{ji}\lambda_j^*)x_i^* = 0\,, \qquad i = 1, \ldots, n.$$

If we exchange the role of the primal and the dual, we have

$$(b_j - \sum_{i=1}^{n} a_{ji}x_i^*)\lambda_j^* = 0\,, \qquad j = 1, \ldots, m.$$

It means that if a variable is non zero, the corresponding constraint in the dual problem must be active. And if a constraint is not active, the corresponding variable must be zero.

Solution 11

1. The set of vertices is

	(x_1, x_2)	$-3x_1 - 4x_2$
A	$(0, 0)$	0
B	$(0, 2)$	-8
C	$(1, 3)$	-15
D	$(13/5, 7/5)$	-13.4
E	$(5/3, 0)$	$-5.$

   Calculating the objective function at each vertex, we obtain that $C(1, 3)$ is optimal with optimal value -15. Note that, at the optimal solution the constraints

$$\begin{aligned} -x_1 + x_2 &\leq 2, \\ x_1 + x_2 &\leq 4, \end{aligned}$$

   are active, while the constraint

$$-3x_1 + 2x_2 \geq -5$$

   is not.

2. Using the summary tables, we obtain the dual problem:

$$\max_{\gamma \in \mathbb{R}^3} 2\gamma_1 - 5\gamma_2 + 4\gamma_3$$

   subject to

$$\begin{aligned} -\gamma_1 - 3\gamma_2 + \gamma_3 &\leq -3, \\ \gamma_1 + 2\gamma_2 + \gamma_3 &\leq -4, \\ \gamma_1 &\leq 0, \\ \gamma_2 &\geq 0, \\ \gamma_3 &\leq 0. \end{aligned}$$

3. If a variable is non zero, the corresponding constraint in the dual problem must be active. And if a constraint is not active, the corresponding variable must be zero. Here, as the second constraint of the

primal is not active, the corresponding dual variable must be zero: $\gamma_2^* = 0$. Also, as the two primal variables x_1^* and x_2^* are non zero, the corresponding dual constraints must be active, and we have the following equations:

$$\begin{aligned} -\gamma_1^* - 3\gamma_2^* + \gamma_3^* &= -3, \\ \gamma_1^* + 2\gamma_2^* + \gamma_3^* &= -4. \end{aligned}$$

Putting the two together, we have

$$\begin{aligned} -\gamma_1^* + \gamma_3^* &= -3, \\ \gamma_1^* + \gamma_3^* &= -4, \end{aligned}$$

so that the optimal solution of the dual is

$$\begin{aligned} \gamma_1^* &= -1/2, \\ \gamma_2^* &= 0, \\ \gamma_3^* &= -7/2. \end{aligned}$$

The optimal value is

$$2\gamma_1^* - 5\gamma_2^* + 4\gamma_3^* = -1 - 14 = -15,$$

as expected from the strong duality theorem.

Question 12

In linear optimization, the dual can be constructed "mechanically" from the primal, using the following procedure (see Theorem 4.14, p. 103 and the tables on p. 105):

- If the primal has m constraints and n variables, the dual has n constraints and m variables.
- If the primal is a minimization problem, the dual is a maximization problem, and the other way around. The coefficients of the objective function of the dual are the right-hand side of the constraints of the primal.
- There is a dual variable for each constraint of the primal, except the sign constraints. The nature of a constraint of the primal determines the sign constraints on the corresponding dual variable in the following way:

Constraint of the primal	Dual variable
$=$	free
$\leq$	≤ 0
$\geq$	≥ 0

- There is a dual constraint for each variable of the primal. The nature of the sign constraint of the primal variable determines the type of the corresponding dual constraint in the following way:

Primal variable	Dual constraint
≥ 0	$\leq$
≤ 0	$\geq$
free	$=$

- The entries at the right-hand side of the dual constraints are the coefficients of the objective function of the primal.

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax \leq b,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Calculate its dual once using the procedure above, and once with the Lagrangian.

Solution 12

Using the procedure, the dual is constructed as follows:

- As the primal has m constraints and n variables, the dual has n constraints and m variables.
- As the primal is a minimization problem, the dual is a maximization problem. The coefficients of the objective function of the dual are the right-hand side of the constraints of the primal:

$$\max_{\mu \in \mathbb{R}^m} b^T \mu.$$

- As the constraints of the primal are $\leq$, the corresponding dual variables must be non positive:

$$\mu \leq 0.$$

- As the variables of the primal are free to take any sign, the corresponding constraints of the dual are equality constraints. The right-hand side of the constraints of the dual are the coefficients of the ojbjective function of the primal:

$$A^T \mu = c.$$

Therefore, we obtain

$$\max b^T \mu \tag{25}$$

subject to

$$\begin{aligned} \mu &\leq 0, \\ A^T \mu &= c, \end{aligned} \tag{26}$$

where $\mu \in \mathbb{R}^m$.

We can also derive the dual using the Lagrangian. Here we use the notation γ instead of μ to avoid the confusion with the μ in (25)–(26).

$$\begin{aligned} L(x, \gamma) &= c^T x + \gamma^T (Ax - b) \\ &= (c + A^T \gamma)^T x - \gamma^T b, \end{aligned}$$

where $\gamma \geq 0$. For the function to be bounded, we need to impose that

$$c + A^T \gamma = 0.$$

In that case, the Lagrangian is constant, and the dual function is

$$q(\gamma) = -\gamma^T b.$$

The dual problem is therefore

$$\max -\gamma^T b \tag{27}$$

subject to

$$\begin{aligned} \gamma &\geq 0, \\ A^T\gamma &= -c. \end{aligned} \tag{28}$$

It can be seen that the problem (27)–(28) is equivalent to (25)–(26) by setting $\gamma = -\mu$.

For those who are not comfortable with the matrix notation, we now perform the same derivations without it. The problem is written

$$\min c_1x_1 + c_2x_2 + \ldots + c_nx_n,$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_3 + \cdots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_3 + \cdots + a_{2n}x_n &\leq b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_3 + \cdots + a_{mn}x_n &\leq b_m. \end{aligned}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Using the table page 105, we obtain its dual:

$$\max b_1\mu_1 + b_2\mu_2 + \cdots b_m\mu_m \tag{29}$$

subject to

$$\begin{aligned} \mu_1 &\leq 0, \\ \mu_2 &\leq 0, \\ &\vdots \\ \mu_m &\leq 0, \\ a_{11}\mu_1 + a_{21}\mu_2 + \cdots + a_{m1}\mu_m &= c_1, \\ a_{12}\mu_1 + a_{22}\mu_2 + \cdots + a_{m2}\mu_m &= c_2, \\ &\vdots \\ a_{1n}\mu_1 + a_{2n}\mu_2 + \cdots + a_{mn}\mu_m &= c_n, \end{aligned} \tag{30}$$

where $\mu \in \mathbb{R}^m$.

We can also derive the dual using the Lagrangian. Here we use the notation γ instead of μ to avoid the confusion with the μ in (25)–(26).

$$\begin{aligned} L(x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_m) =& c_1 x_1 + \ldots + c_n x_n \\ &+ \gamma_1 (a_{11} x_1 + a_{12} x_3 + \cdots + a_{1n} x_n - b_1) + \ldots \\ &+ \gamma_m (a_{m1} x_1 + a_{m2} x_3 + \cdots + a_{mn} x_n - b_m) \\ =& (c_1 + a_{11}\gamma_1 + a_{21}\gamma_2 + \cdots + a_{m1}\gamma_m) x_1 + \ldots \\ &+ (c_n + a_{1n}\gamma_1 + a_{2n}\gamma_2 + \cdots + a_{mn}\gamma_m) x_n \\ &- b_1\gamma_1 - \cdots - b_m\gamma_m, \end{aligned}$$

where $\gamma \geq 0$. For the function to be bounded, we need to impose that

$$\begin{aligned} c_1 + a_{11}\gamma_1 + a_{21}\gamma_2 + \cdots + a_{m1}\gamma_m &= 0, \\ &\vdots \\ c_n + a_{1n}\gamma_1 + a_{2n}\gamma_2 + \cdots + a_{mn} &= 0. \end{aligned}$$

In that case, the Lagrangian is constant, and the dual function is

$$q(\gamma) = -b_1\gamma_1 - \cdots - b_m\gamma_m.$$

The dual problem is therefore

$$\max -b_1\gamma_1 - \cdots - b_m\gamma_m \tag{31}$$

subject to

$$\begin{aligned} \gamma_1 &\geq 0, \\ &\vdots \\ \gamma_m &\geq 0, \\ a_{11}\gamma_1 + a_{21}\gamma_2 + \cdots + a_{m1} &= -c_1, \\ &\vdots \\ a_{1n}\gamma_1 + a_{2n}\gamma_2 + \cdots + a_{mn} &= -c_n. \end{aligned} \tag{32}$$

It can be seen that the problem (31)–(32) is equivalent to (29)–(30) by setting $\gamma = -\mu$.

Question 13

In linear optimization, the dual can be constructed "mechanically" from the primal, using the following procedure (see Theorem 4.14, p. 103 and the tables on p. 105):

- If the primal has m constraints and n variables, the dual has n constraints and m variables.
- If the primal is a minimization problem, the dual is a maximization problem, and the other way around. The coefficients of the objective function of the dual are the right-hand side of the constraints of the primal.
- There is a dual variable for each constraint of the primal, except the sign constraints. The nature of a constraint of the primal determines the sign constraints on the corresponding dual variable in the following way:

Constraint of the primal	Dual variable
$=$	free
$\leq$	≤ 0
$\geq$	≥ 0

- There is a dual constraint for each variable of the primal. The nature of the sign constraint of the primal variable determines the type of the corresponding dual constraint in the following way:

Primal variable	Dual constraint
≥ 0	$\leq$
≤ 0	$\geq$
free	$=$

- The coefficient of variable i in constraint j of the primal is the coefficient of variable j in constraint i of the dual, and the entries at the right-hand side of the dual constraints are the coefficients of the objective function of the primal.

Consider the linear optimization problem

$$\min_{x \in \mathbb{R}^2} -3x_1 + 2x_2$$

subject to

$$\begin{aligned} -x_1 \quad -x_2 \quad &\leq 4, \\ x_1 \quad\quad\; &\geq -5, \\ 7x_1 \quad -2x_2 \quad &= 6, \\ x_1 \quad\quad\; &\leq 0, \\ \quad\quad x_2 \quad &\geq 0. \end{aligned}$$

Calculate its dual once using the procedure above, and once with the Lagrangian.

Solution 13

Using the procedure, the dual is constructed as follows:

- As the primal has $m = 3$ constraints and $n = 2$ variables, the dual has $n = 2$ constraints and $m = 3$ variables. We denote the dual variables γ to avoid confusion with the Lagrangian-based derivation below.
- As the primal is a minimization problem, the dual is a maximization problem. The coefficients of the objective function of the dual are the right-hand side of the constraints of the primal:

$$\max_{\gamma \in \mathbb{R}^3} 4\gamma_1 - 5\gamma_2 + 6\gamma_3.$$

- Considering the type of constraints of the primal, the sign constraints on the dual variables are:

$$\gamma_1 \leq 0,\ \gamma_2 \geq 0,\ \text{and}\ \gamma_3 \in \mathbb{R}.$$

- Considering the sign constraints on the variables of the primal, we determine the corresponding constraints of the dual. The coefficient of variable i in constraint j of the primal is the coefficient of variable j in constraint i of the dual, and the entries at the right-hand side of the dual constraints are the coefficients of the objective function of the primal:

$$\begin{array}{llll} -\gamma_1 & +\gamma_2 & +7\gamma_3 & \geq -3, \\ -\gamma_1 & & -2\gamma_3 & \leq 2. \end{array}$$

Therefore, we obtain

$$\max_{\gamma \in \mathbb{R}^3} 4\gamma_1 - 5\gamma_2 + 6\gamma_3 \tag{33}$$

subject to

$$\begin{array}{llll} \gamma_1 & & & \leq 0, \\ & \gamma_2 & & \geq 0, \\ & & \gamma_3 & \in \mathbb{R}, \\ -\gamma_1 & +\gamma_2 & +7\gamma_3 & \geq -3, \\ -\gamma_1 & & -2\gamma_3 & \leq 2. \end{array} \tag{34}$$

We can also derive the dual using the Lagrangian. First, we write the problem such that all inequality constraints are $\leq$, and the right-hand side of each constraint is zero:

$$\min_{x \in \mathbb{R}^2} -3x_1 + 2x_2$$

subject to

$$\begin{aligned} 7x_1 \quad -2x_2 \quad -6 \quad &= 0, \\ -x_1 \quad -x_2 \quad -4 \quad &\leq 0, \\ -x_1 \quad\quad\quad -5 \quad &\leq 0, \\ x_1 \quad\quad\quad &\leq 0, \\ -x_2 \quad &\leq 0. \end{aligned}$$

The equality constraint is associated with a dual variable λ, and each inequality constraint is associated with a non negative dual variable μ_i. The Lagrangian is

$$\begin{aligned} L(x, \lambda, \mu) = &-3x_1 + 2x_2 \\ &+ \lambda(7x_1 - 2x_2 - 6) \\ &+ \mu_1(-x_1 - x_2 - 4) \\ &+ \mu_2(-x_1 - 5) \\ &+ \mu_3 x_1 \\ &- \mu_4 x_2. \end{aligned}$$

If we rearrange the terms, we obtain

$$\begin{aligned} L(x, \lambda, \mu) = &(-3 + 7\lambda - \mu_1 - \mu_2 + \mu_3)x_1 \\ &+ (2 - 2\lambda - \mu_1 - \mu_4)x_2 \\ &- 6\lambda - 4\mu_1 - 5\mu_2. \end{aligned}$$

As a function of x, L must be bounded from below. Therefore, the coefficients of x must be zero:

$$\begin{aligned} -3 + 7\lambda - \mu_1 - \mu_2 + \mu_3 &= 0, \\ 2 - 2\lambda - \mu_1 - \mu_4 &= 0. \end{aligned}$$

If these constraints are verified, for any x, we have

$$L(x, \lambda, \mu) = -6\lambda - 4\mu_1 - 5\mu_2.$$

And the dual problem is written as

$$\max_{\lambda \in \mathbb{R}, \mu \in \mathbb{R}^4} -6\lambda - 4\mu_1 - 5\mu_2$$

subject to

$$\begin{aligned}
\mu_3 &= 3 - 7\lambda + \mu_1 + \mu_2, \\
\mu_4 &= 2 - 2\lambda - \mu_1, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0, \\
\mu_3 &\geq 0, \\
\mu_4 &\geq 0.
\end{aligned}$$

We can eliminate μ_3 and μ_4 and rewrite the constraints as follows:

$$\begin{aligned}
3 - 7\lambda + \mu_1 + \mu_2 &\geq 0, \\
2 - 2\lambda - \mu_1 &\geq 0, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0,
\end{aligned}$$

or, equivalently

$$\begin{aligned}
-7\lambda + \mu_1 + \mu_2 &\geq -3, \\
2\lambda + \mu_1 &\leq 2, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0,
\end{aligned}$$

so that we obtain the dual problem

$$\max_{\lambda \in \mathbb{R}, \mu \in \mathbb{R}^4} \; -6\lambda - 4\mu_1 - 5\mu_2 \tag{35}$$

subject to

$$\begin{aligned}
-7\lambda + \mu_1 + \mu_2 &\geq -3, \\
2\lambda + \mu_1 &\leq 2, \\
\mu_1 &\geq 0, \\
\mu_2 &\geq 0.
\end{aligned} \tag{36}$$

It can be seen that the problem (35)–(36) is equivalent to (33)–(34) by setting

$$\begin{aligned}
\gamma_1 &= -\mu_1, \\
\gamma_2 &= \mu_2, \\
\gamma_3 &= -\lambda.
\end{aligned}$$

Question 14

Consider the optimization problem

$$\min_{x \in \mathbb{R}} x^2$$

subject to

$$x \geq 1.$$

Let μ be the dual variable of the constraint, with $\mu \geq 0$.

1. Write the Lagrangian of this problem.
2. Write the dual function?
3. What are the optimal solutions of the dual and the primal?

Solution 14

We write the constraint

$$g(x) = 1 - x \leq 0,$$

and the Lagrangian is

$$L(x, \mu) = x^2 + \mu(1 - x) = x^2 - \mu x + \mu.$$

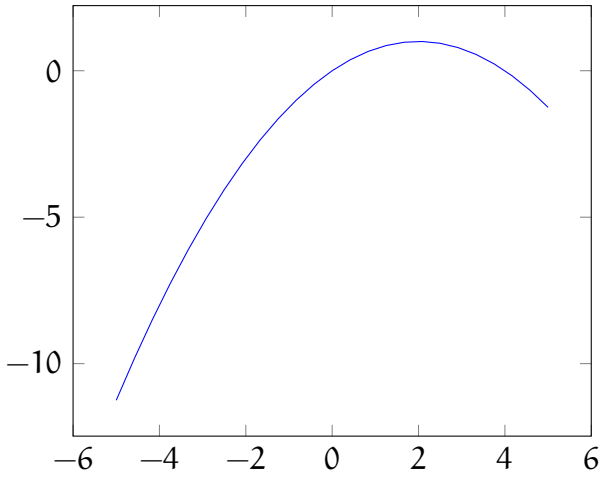
In order to find its minimum, we calculate the derivatives:

$$L'(x, \mu) = 2x - \mu$$
$$L''(x, \mu) = 2.$$

As the second derivative is positive, the Lagrangian is always convex and has a minimum. The domain is therefore $X_q = \mathbb{R}$, and the minimum is the zero of the first derivative, that is $x^* = \mu/2$. The dual function is

$$q(\mu) = L(x^*, \mu) = \left(\frac{\mu}{2}\right)^2 - \mu\left(\frac{\mu}{2}\right) + \mu = \mu - \frac{\mu^2}{4},$$

and is plotted below



The dual problem is

$$\max_{\mu \in \mathbb{R}} q(\mu) = \mu - \frac{\mu^2}{4},$$

subject to

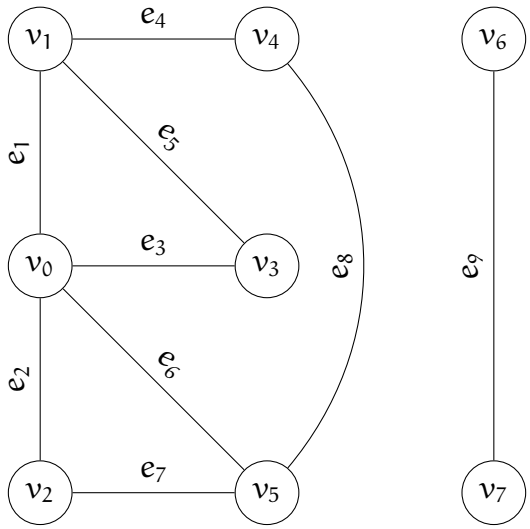
$$\mu \geq 0.$$

The optimal solution of the dual is $\mu^* = 2$, with an optimal value 1. The optimal solution of the primal is $x^* = \mu^*/2 = 1$. The optimal value of the primal is also 1.

5 Networks

Question 01

Consider the following undirected graph:



1. What is the set $\mathcal{V}$ of vertices? What is the set $\mathcal{E}$ of edges?
2. What is the incidence function?
3. What is the degree of each vertex?

Solution 01

1. $\mathcal{V} = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is the set of vertices.
   $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ is the set of edges.

2. The incidence function is defined as

$$\Phi : \mathcal{E} \to \mathcal{P}_2(\mathcal{V}),$$

   and maps the set of edges to the set of pair of vertices.

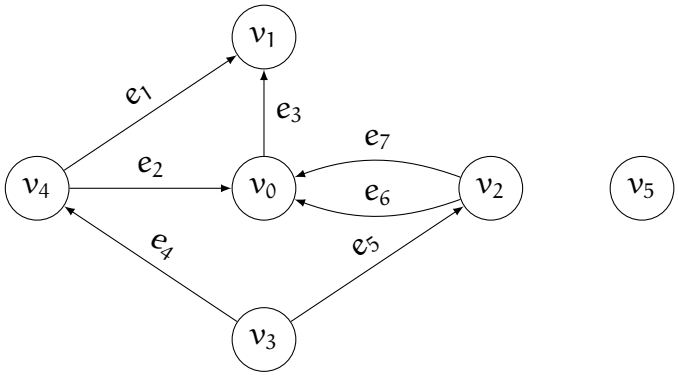
$$\begin{aligned}
&\Phi(e_1) = \{v_0, v_1\}, \quad \Phi(e_2) = \{v_0, v_2\},\\
&\Phi(e_3) = \{v_0, v_3\}, \quad \Phi(e_4) = \{v_1, v_4\},\\
&\Phi(e_5) = \{v_1, v_3\}, \quad \Phi(e_6) = \{v_0, v_5\},\\
&\Phi(e_7) = \{v_2, v_5\}, \quad \Phi(e_8) = \{v_4, v_5\},\\
&\Phi(e_9) = \{v_6, v_7\}.
\end{aligned}$$

3. The degree d_i of the vertex v_i is the number of incident edges. We have:

$$\begin{aligned}
&d_0 = 4, \quad d_1 = 3, \quad d_2 = 2, \quad d_3 = 2,\\
&d_4 = 2, \quad d_5 = 3, \quad d_6 = 1, \quad d_7 = 1.
\end{aligned}$$

Question 02

Consider the following directed graph:



1. What is the set $\mathcal{N}$ of nodes? What is the set $\mathcal{A}$ of arcs?
2. What is the incidence function? Is the incidence function injective?
3. What is the degree, the indegree and the outdegree of each node?

Solution 02

1. $\mathcal{N} = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ is the set of nodes.
   $\mathcal{A} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is the set of arcs.

2. The incidence function is defined as

$$\Phi : \mathcal{A} \to \mathcal{N} \times \mathcal{N},$$

   and maps the set of arcs to the set of pair of nodes.

$$\begin{aligned} \Phi(e_1) &= (v_4, v_1), \quad \Phi(e_2) = (v_4, v_0), \\ \Phi(e_3) &= (v_0, v_1), \quad \Phi(e_4) = (v_3, v_4), \\ \Phi(e_5) &= (v_3, v_2), \quad \Phi(e_6) = (v_2, v_0), \\ \Phi(e_7) &= (v_2, v_0). \end{aligned}$$

   Note that, in this example, the incidence function is not injective because there are two arcs links nodes v_2 and v_0 (e_6 and e_7). In the rest of the course, we will only deal with graphs with an injective incidence function.

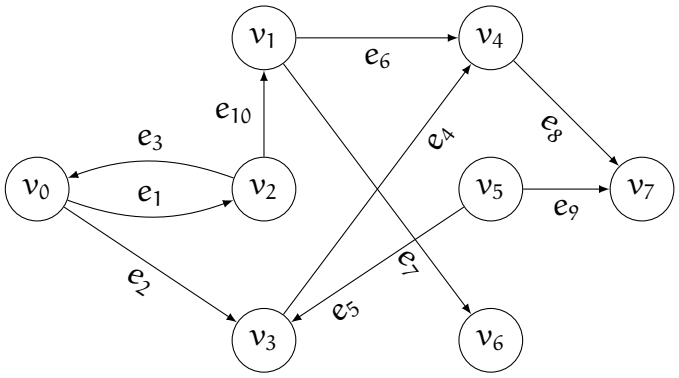
3. The degree d_i of the node v_i is the number of incident arcs. The outdegree d_i^+ of node i represents the number of incident arcs of the form (i, j) (we say that i is an *upstream node*), and the indegree d_i^- of node i is the number of incident arcs of the form (j, i) (we say that i is a *downstream node*).

$$\begin{aligned} &d_0 = 4, \quad d_1 = 2, \quad d_2 = 3, \quad d_3 = 2, \quad d_4 = 3, \quad d_5 = 0, \\ &d_0^- = 3, \quad d_1^- = 2, \quad d_2^- = 1, \quad d_3^- = 0, \quad d_4^- = 1, \quad d_5^- = 0, \\ &d_0^+ = 1, \quad d_1^+ = 0, \quad d_2^+ = 2, \quad d_3^+ = 2, \quad d_4^+ = 2, \quad d_5^+ = 0. \end{aligned}$$

   We notice that for each node i we have $d_i = d_i^- + d_i^+$. Also, we have that $\sum_{i \in \mathcal{N}} d_i^-$ and $\sum_{i \in \mathcal{N}} d_i^+$ are equal to the total number of arcs (7 here).

Question 03

Consider the following directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Phi)$:



First, write the incidence function Φ of the directed graph, and check whether it is injective[2]. Then, for each of the following graphs, say whether it is a subgraph of $\mathcal{G}$.

1. First graph: $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \Phi_1)$, where

$$\begin{aligned} \mathcal{V}_1 &= \{v_0, v_2, v_3, v_5\}, \\ \mathcal{E}_1 &= \{a_1, a_2, a_3\}, \\ \Phi_1(a_1) &= (v_0, v_2), \\ \Phi_1(a_2) &= (v_3, v_5), \\ \Phi_1(a_3) &= (v_0, v_3). \end{aligned}$$

2. Second graph: $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \Phi_2)$, where

$$\begin{aligned} \mathcal{V}_2 &= \{v_1, v_4, v_6, v_7\}, \\ \mathcal{E}_2 &= \{a_1, a_2, a_3\}, \\ \Phi_2(a_1) &= (v_1, v_4), \\ \Phi_2(a_2) &= (v_1, v_6), \\ \Phi_2(a_3) &= (v_4, v_7). \end{aligned}$$

[2] A function f is injective if it maps distinct elements to distinct elements: $f(x_1) = f(x_2) \implies x_1 = x_2$.

3. Third graph: : $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3, \Phi_3)$, where

$$\begin{aligned}
\mathcal{V}_3 &= \{v_1, v_3, v_4, v_6\}, \\
\mathcal{E}_3 &= \{a_1, a_2, a_3, a_4\}, \\
\Phi_3(a_1) &= (v_1, v_4), \\
\Phi_3(a_2) &= (v_4, v_7), \\
\Phi_3(a_3) &= (v_3, v_4), \\
\Phi_3(a_4) &= (v_1, v_6).
\end{aligned}$$

Solution 03

The incidence function is defined as

$$\begin{aligned}
&\Phi(e_1) = (v_0, v_2), && \Phi(e_2) = (v_0, v_3), && \Phi(e_3) = (v_2, v_0),\\
&\Phi(e_4) = (v_3, v_4), && \Phi(e_5) = (v_5, v_3), && \Phi(e_6) = (v_1, v_4),\\
&\Phi(e_7) = (v_1, v_6), && \Phi(e_8) = (v_4, v_7), && \Phi(e_9) = (v_5, v_7),\\
&\Phi(e_{10}) = (v_2, v_1). &&
\end{aligned}$$

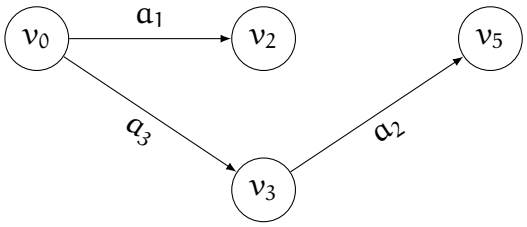
The incidence function is injective, as there is at most one arc linking any pair of nodes.

Recall that given a graph $(\mathcal{V}, \mathcal{E}, \Phi)$, we define $(\mathcal{V}', \mathcal{E}', \Phi')$ as a subgraph if

- $\mathcal{V}' \subseteq \mathcal{V}$,
- $\mathcal{E}' \subseteq \mathcal{E}$,
- $\Phi'(e) = \Phi(e)$, for each $e \in \mathcal{E}'$,
- for each $e \in \mathcal{E}'$, if $\Phi'(e) = (i, j)$, then i and j belong to $\mathcal{V}'$.

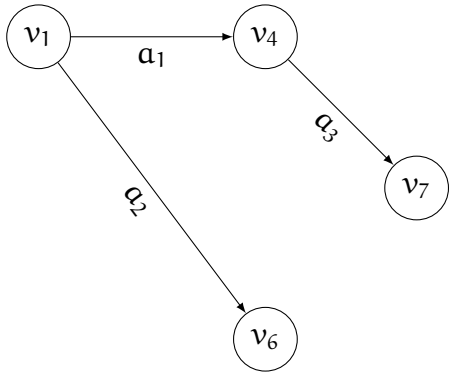
We analyze now the three cases.

1. First graph.



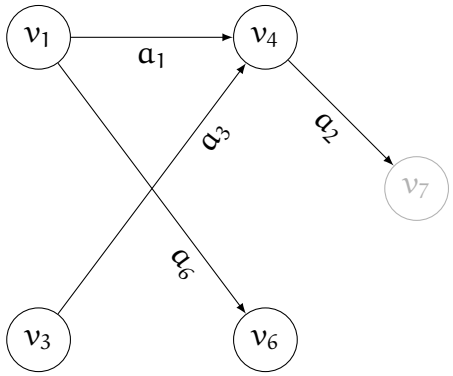
We have $\mathcal{V}_1 \subseteq \mathcal{V}$. Moreover, it can easily be verified that for each $a \in \mathcal{E}_1$, if $\Phi_1(a) = (i, j)$, then i and j belong to $\mathcal{V}_1$. As we need $\Phi_1(a_1) = \Phi(a_1) = (v_0, v_2)$, we define $a_1 = e_1$. Similarly, we define $a_3 = e_2$. Therefore, both a_1 and a_3 belong to $\mathcal{E}$. But we cannot find a corresponding arc for a_2 in the original graph, as there is no arc linking nodes v_3 and v_5. Therefore, $\mathcal{E}_1 \nsubseteq \mathcal{E}$, and $(\mathcal{V}_1, \mathcal{E}_1, \Phi_1)$ is not a subgraph of $\mathcal{G}$.

2. Second graph.



We have $\mathcal{V}_1 \subseteq \mathcal{V}$. Moreover, it can easily be verified that for each $a \in \mathcal{E}_2$, if $\Phi_2(a) = (i, j)$, then i and j belong to $\mathcal{V}_2$. As we need $\Phi_2(a_1) = \Phi(a_1) = (v_1, v_4)$, we define $a_1 = e_6$. Similarly, we define $a_2 = e_7$ so that $\Phi_1(a_2) = \Phi(a_2) = \Phi(e_7) = (v_1, v_6)$. Finally, we define $a_3 = e_8$ so that $\Phi_1(a_3) = \Phi(a_3) = \Phi(e_8) = (v_4, v_7)$. Therefore, all arcs in $\mathcal{E}_2$ belong to $\mathcal{E}$, the two incidence functions are consistent, and $(\mathcal{V}_2, \mathcal{E}_2, \Phi_2)$ is a subgraph of $\mathcal{G}$.

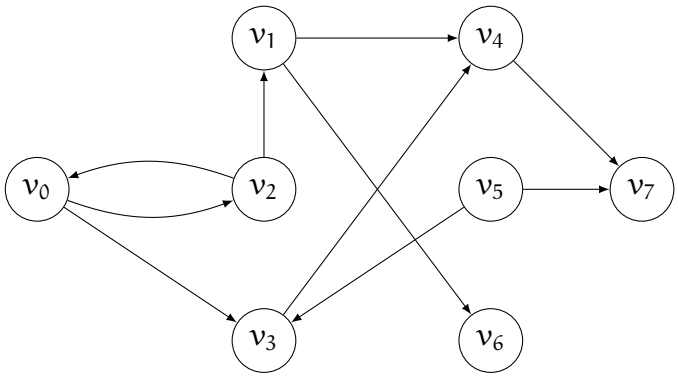
3. Third graph.



We have $\mathcal{V}_3 \subseteq \mathcal{V}$ and $\mathcal{E}_3 \subseteq \mathcal{E}$. The arcs are also incident to the same nodes as in the original graph, so that the third property is verified. But the last property is not satisfied: $(v_4, v_7) \in \mathcal{E}_3$ but $v_7 \notin \mathcal{V}_3$. Thus, $(\mathcal{V}_3, \mathcal{E}_3, \Phi_3)$ is not a subgraph of $\mathcal{G}$.

Question 04

Consider the following graph.

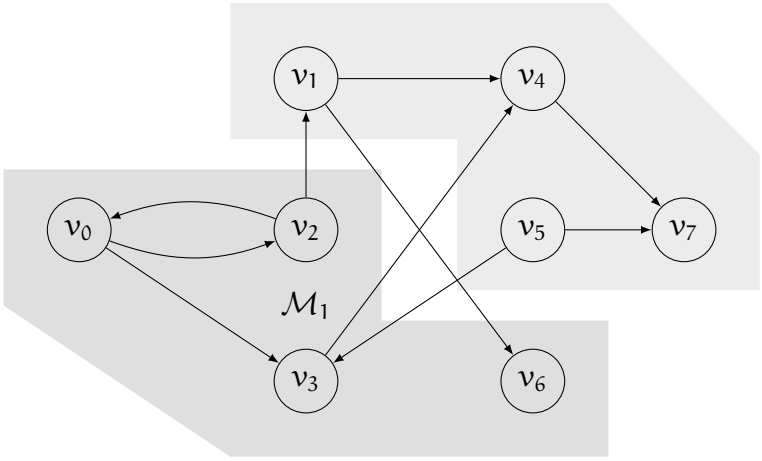


For each of the following cuts, give the set of forward arcs and the set of backward arcs.

1. $\Gamma_1 = (\mathcal{M}_1, \mathcal{N} \setminus \mathcal{M}_1)$ with $\mathcal{M}_1 = \{v_0, v_2, v_3, v_6\}$.
2. $\Gamma_2 = (\mathcal{M}_2, \mathcal{N} \setminus \mathcal{M}_2)$ with $\mathcal{M}_2 = \{v_4, v_5, v_6, v_7\}$.
3. $\Gamma_3 = (\mathcal{M}_3, \mathcal{N} \setminus \mathcal{M}_3)$ with $\mathcal{M}_3 = \{v_1, v_2, v_7\}$.

Solution 04

1. The first cut Γ_1 is shown in the figure below.



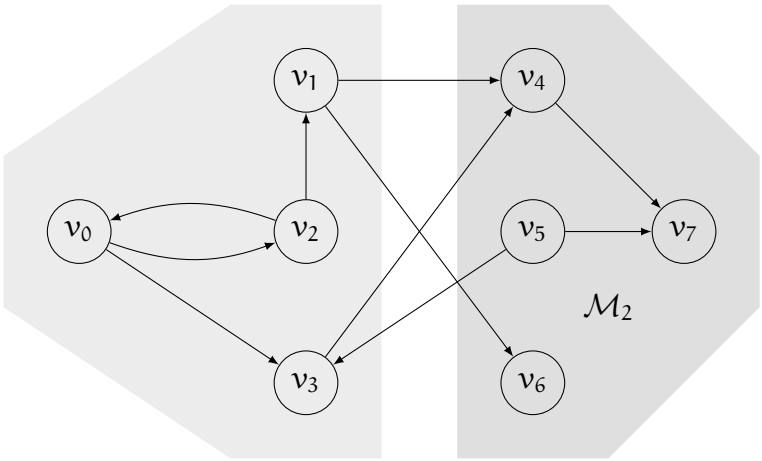
   The set of forward arcs is

$$\Gamma_1^{\rightarrow} = \{(v_2, v_1), (v_3, v_4)\},$$

   and the set of backward arcs is

$$\Gamma_1^{\leftarrow} = \{(v_1, v_6), (v_5, v_3)\}.$$

2. The second cut Γ_2 is shown in the figure below.



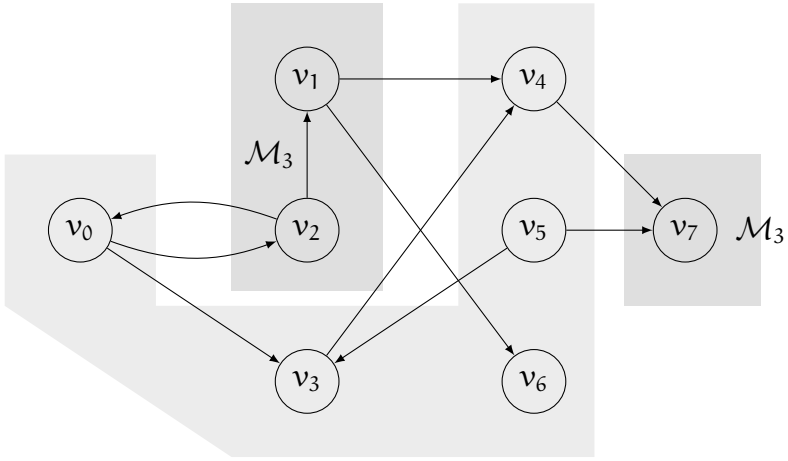
The set of forward arcs is

$$\Gamma_2^{\rightarrow} = \{(v_5, v_3)\},$$

and the set of backward arcs is

$$\Gamma_2^{\leftarrow} = \{(v_1, v_4), (v_1, v_6), (v_3, v_4)\}.$$

3. The third cut Γ_3 is shown in the figure below.



The set of forward arcs is

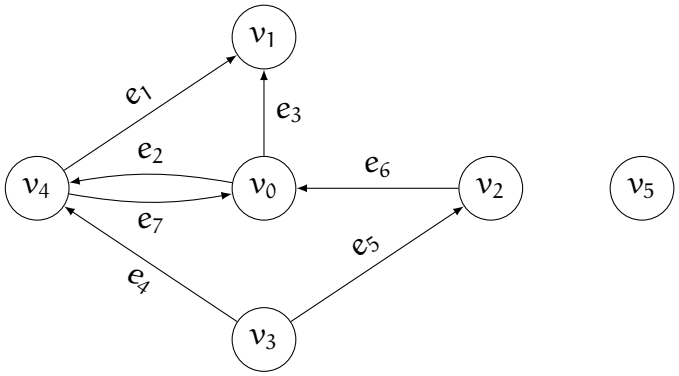
$$\Gamma_3^{\rightarrow} = \{(v_1, v_4), (v_1, v_6), (v_2, v_0)\},$$

and the set of backward arcs is

$$\Gamma_3^{\leftarrow} = \{(v_0, v_2), (v_5, v_7), (v_4, v_7)\}.$$

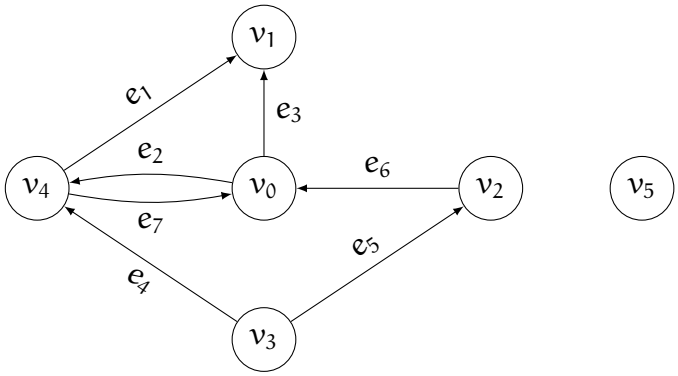
Question 05

Consider the following directed graph.



1. Give a path from v_1 to v_2.
2. Is the path a forward path? If so, mention it.
3. If there is a forward path from v_1 to v_2, mention it.
4. Enumerate the simple paths from v_3 to v_1. Identify those that are forward paths.
5. Give an example of a path from v_3 to v_1 which is not simple.
6. Give an example of a simple cycle.
7. Give an example of a cycle which is not simple.

Solution 05



1. An example of a path from v_1 to v_2 could be $v_1 \leftarrow v_0 \leftarrow v_2$. Another example would be $v_1 \leftarrow v_4 \rightarrow v_0 \rightarrow v_4 \leftarrow v_3 \rightarrow v_2$.

2. None of the examples above is a forward path.

3. Since the outdegree of v_1 is $d_1^+ = 0$, there is no forward path starting at v_1.

4. There are 6 simple paths from v_3 to v_1:

   - $\mathcal{P}_1 : v_3 \rightarrow v_4 \rightarrow v_1$, which is a simple forward path:

     $$[\mathcal{P}_1^{\rightarrow} = \{e_4, e_1\}, \mathcal{P}_1^{\leftarrow} = \emptyset],$$

     where $\mathcal{P}_1^{\rightarrow}$ is the set of forward arcs of the path $\mathcal{P}_1$ and $\mathcal{P}_1^{\leftarrow}$ is the set of backward arcs.

   - $\mathcal{P}_2 : v_3 \rightarrow v_4 \rightarrow v_0 \rightarrow v_1$, which is a simple forward path:

     $$[\mathcal{P}_2^{\rightarrow} = \{e_4, e_7, e_3\}, \mathcal{P}_2^{\leftarrow} = \emptyset].$$

   - $\mathcal{P}_3 : v_3 \rightarrow v_4 \leftarrow v_0 \rightarrow v_1$, which is a simple path:

     $$[\mathcal{P}_3^{\rightarrow} = \{e_4, e_3\}, \mathcal{P}_3^{\leftarrow} = \{e_2\}].$$

   - $\mathcal{P}_4 =: v_3 \rightarrow v_2 \rightarrow v_0 \rightarrow v_1$, which is a simple forward path:

     $$[\mathcal{P}_4^{\rightarrow} = \{e_5, e_6, e_3\}, \mathcal{P}_4^{\leftarrow} = \emptyset].$$

- $\mathcal{P}_5 =: v_3 \to v_2 \to v_0 \to v_4 \to v_1$, which is a simple forward path:

$$[\mathcal{P}_5^{\rightarrow} = \{e_5, e_6, e_2, e_1\}, \mathcal{P}_5^{\leftarrow} = \emptyset] .$$

- $\mathcal{P}_6 =: v_3 \to v_2 \to v_0 \leftarrow v_4 \to v_1$, which is a simple path:

$$[\mathcal{P}_6^{\rightarrow} = \{e_5, e_6, e_1\}, \mathcal{P}_6^{\leftarrow} = \{e_7\}] .$$

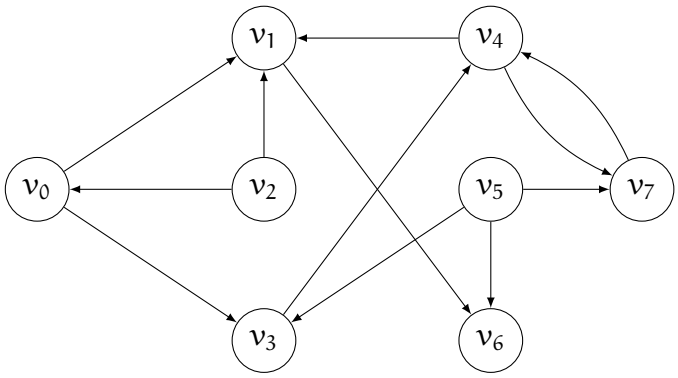
5. The path $v_3 \to v_4 \to v_0 \to v_4 \to v_1$ is not simple since the node v_4 is repeated.

6. The path $v_1 \leftarrow v_0 \leftarrow v_2 \leftarrow v_3 \to v_4 \to v_1$ is an example of a simple cycle since the origin and the destination coincide, and no other node is repeated. Another one would be $v_0 \to v_4 \to v_0$.

7. The path $v_1 \leftarrow v_0 \to v_4 \to v_0 \to v_1$ is a cycle since the origin and the destination coincide. It is not a simple cycle as node v_0 appears twice in the path.

Question 06

Consider the following graph:

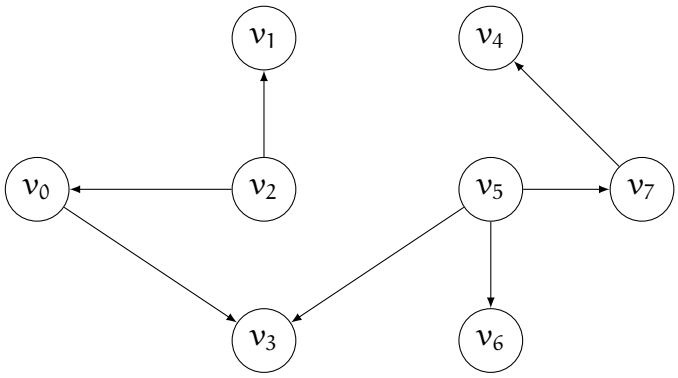


Give a spanning tree of that graph. Recall that given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Phi)$, a subgraph $(\mathcal{V}, \mathcal{E}', \Phi')$ is a spanning tree of $\mathcal{G}$ if:

- $\mathcal{E}' \subseteq \mathcal{E}$,
- $\Phi'(e) = \Phi(e)$ for each $e \in \mathcal{E}'$,
- $(\mathcal{V}, \mathcal{E}', \Phi')$ is a tree.

Solution 06

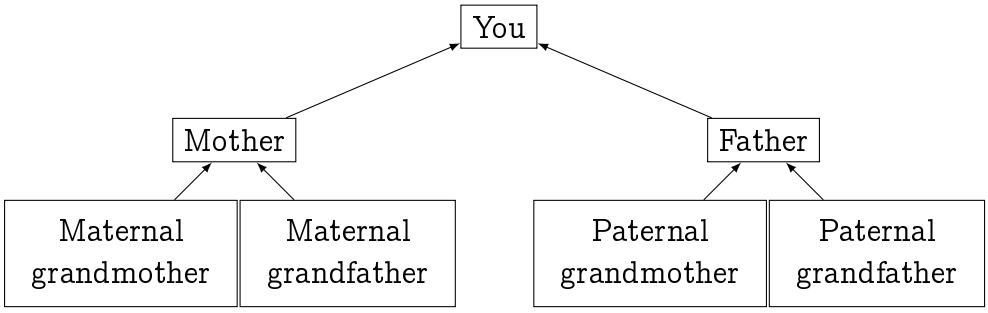
An example of spanning tree is the following:



It is easily verified that it is a subgraph involving all the nodes. Furthermore, it is a connected graph without cycles and thus a tree.

Question 07

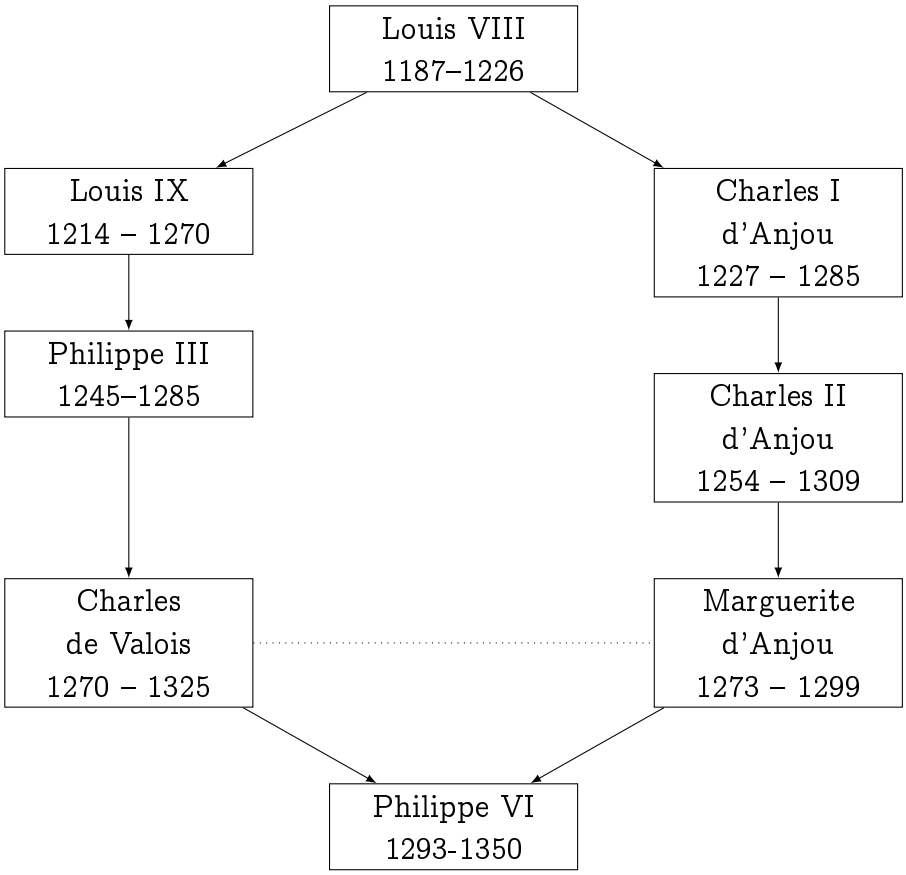
A family tree is a network where each node represents a person, and each arc represents the relation: "is the father/the mother of". Is it indeed a tree? Justify your answer.



Solution 07

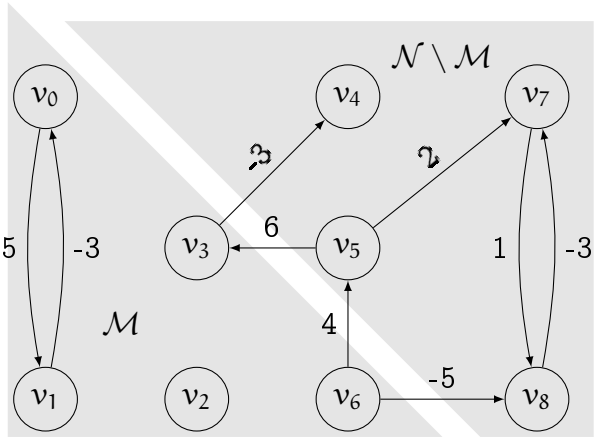
If two persons with a common ancestor have a child, it forms a cycle in the network. Indeed, the child is connected with the common ancestor with two paths: one through her father, and one through her mother. Therefore, the network representing a "family tree" is not a tree.

For example, considering the family tree of the kings of France, there is a cycle through Philippe VI Valois (1293-1350) and Louis VIII (1187–1226), as illustrated below. Louis VIII is the great-grandfather of both parents of Philippe VI Valois.



Question 08

Consider the following network where the flow is reported next to each arc. The gray areas are defining a cut.



1. Write explicitly the cut Γ. What are the sets of forward arcs and backward arcs?

2. Calculate the flow through the cut Γ.

Solution 08

1. The cut is defined as $\Gamma = (\mathcal{M}, \mathcal{N} \setminus \mathcal{M})$ with $\mathcal{M} = \{v_0, v_1, v_2, v_3, v_6\}$ and $\mathcal{N} \setminus \mathcal{M} = \{v_4, v_5, v_7, v_8\}$.

   $\Gamma^{\rightarrow} = \{(v_3, v_4), (v_6, v_5), (v_6, v_8)\}$ is the set of forward arcs.

   $\Gamma^{\leftarrow} = \{(v_5, v_3)\}$ is the set of backward arcs.

2. Recall that the flow through the cut Γ is defined as

$$\mathcal{X}(\Gamma) = \sum_{(i,j) \in \Gamma^{\rightarrow}} x_{ij} - \sum_{(i,j) \in \Gamma^{\leftarrow}} x_{ij}.$$

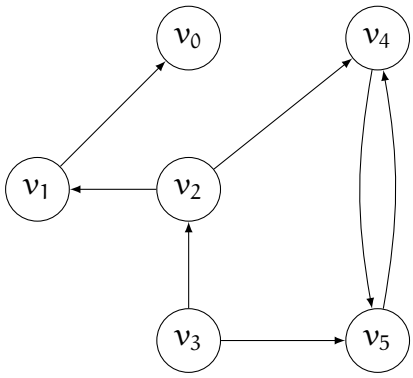
   Then we obtain

$$\mathcal{X}(\Gamma) = (-3 + 4 - 5) - (6) = -10.$$

   The interpretation is that there is a total of 10 units of flow transported from the set $\mathcal{N} \setminus \mathcal{M}$ to the set $\mathcal{M}$.

Question 09

Consider the following network.



Consider all simple paths from v_0 to v_5. For each of them, assume that a quantity f of flow is carried along the path from v_0 to v_5. Identify the corresponding flow on each arc of the network.

Solution 09

There are three simple paths from v_0 to v_5.

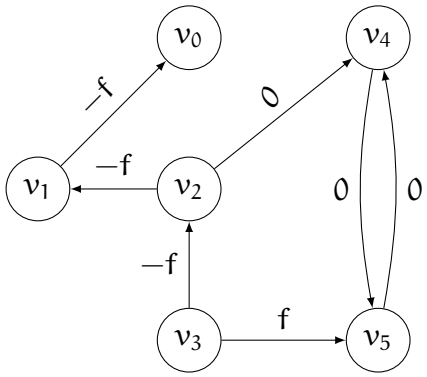
- The first one corresponds to

$$\mathcal{P}_1 = v_0 \leftarrow v_1 \leftarrow v_2 \leftarrow v_3 \rightarrow v_5.$$

  Remember that if a flow f follows a simple path $\mathcal{P}$ from its origin to its destination, then we have

$$x_{ij} = \begin{cases} f & \text{if } (i,j) \in \mathcal{P}^{\rightarrow}, \\ -f & \text{if } (i,j) \in \mathcal{P}^{\leftarrow}, \\ 0 & \text{otherwise.} \end{cases}$$

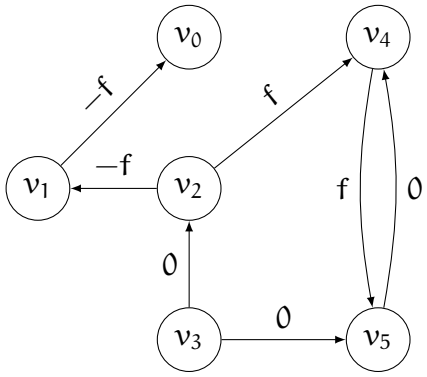
  Here we have $\mathcal{P}_1^{\rightarrow} : \{(v_3, v_5)\}$ and $\mathcal{P}_1^{\leftarrow} = \{(v_1, v_0), (v_2, v_1), (v_3, v_2)\}$. Therefore the simple path flow for $\mathcal{P}_1$ is:



- The second one corresponds to

$$\mathcal{P}_2 : v_0 \leftarrow v_1 \leftarrow v_2 \rightarrow v_4 \rightarrow v_5.$$

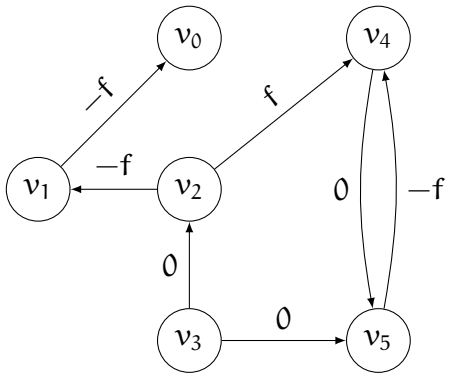
  Here we have $\mathcal{P}_2^{\rightarrow} : \{(v_2, v_4), (v_4, v_5)\}$ and $\mathcal{P}_2^{\leftarrow} = \{(v_1, v_0), (v_2, v_1)\}$. Therefore the simple path flow for $\mathcal{P}_2$ is:



- The third one corresponds to

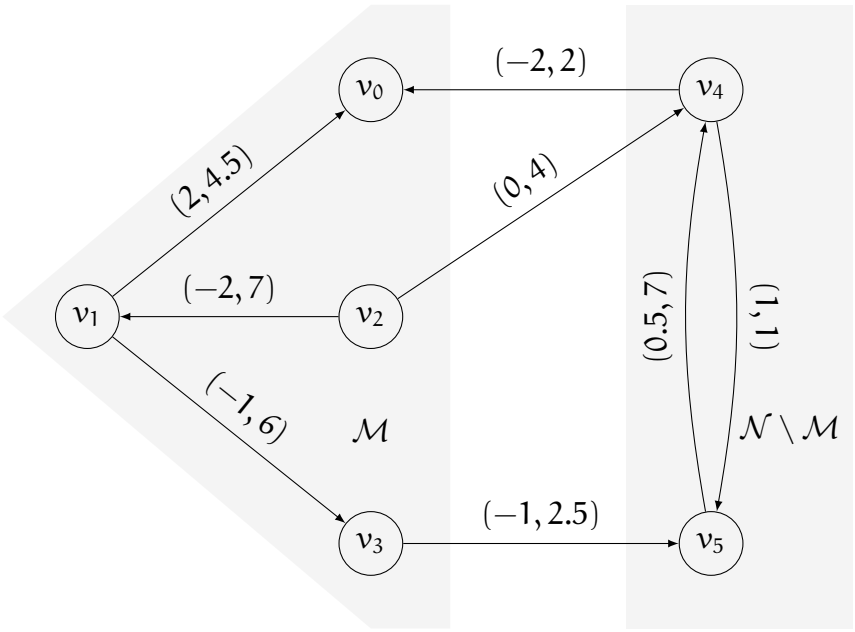
$$\mathcal{P}_3 = v_0 \leftarrow v_1 \leftarrow v_2 \rightarrow v_4 \leftarrow v_5.$$

Here we have $\mathcal{P}_3^{\rightarrow} = \{(v_2, v_4)\}$ and $\mathcal{P}_3^{\leftarrow} = \{(v_1, v_0), (v_2, v_1), (v_5, v_4)\}$. Therefore the simple path flow for $\mathcal{P}_3$ is:



Question 10

Consider the following network. Each arc (i, j) is associated with a pair of numbers (ℓ_{ij}, u_{ij}), representing the lower and upper capacities, respectively. The gray areas are defining a cut.



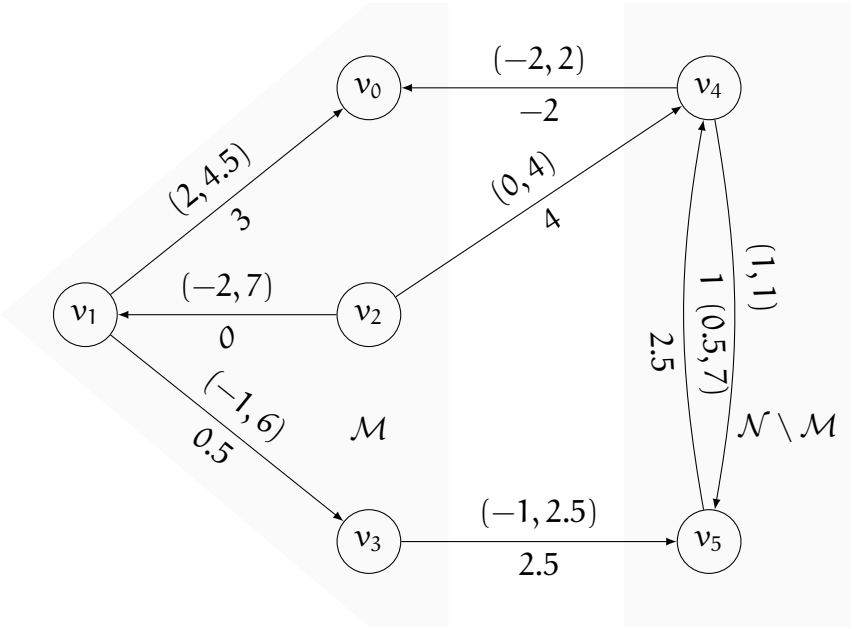
1. Write the cut $\Gamma = \{\mathcal{M}, \mathcal{N} \setminus \mathcal{M}\}$ represented in the figure and the corresponding sets of forward arcs $\Gamma^{\rightarrow}$ and backward arcs $\Gamma^{\leftarrow}$.
2. Calculate the capacity of the cut Γ.
3. Assign a flow on each arc such that the cut is saturated.

Solution 10

1. First, we write the cut and the sets of forward and backward arcs:
   - $\Gamma = (\mathcal{M}, \mathcal{N} \backslash \mathcal{M})$ with $\mathcal{M} = \{v_0, v_1, v_2, v_3\}$ and $\mathcal{N} \backslash \mathcal{M} = \{v_4, v_5\}$.
   - $\Gamma^{\rightarrow} = \{(v_2, v_4), (v_3, v_5)\}$ is the set of forward arcs.
   - $\Gamma^{\leftarrow} = \{(v_4, v_0)\}$ is the set of backward arcs.
2. The capacity of the cut Γ is

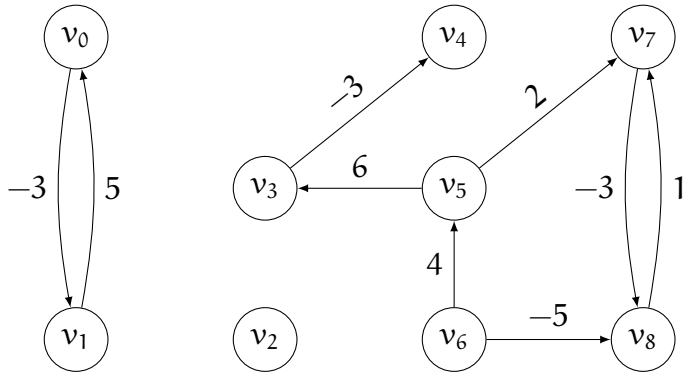
$$\begin{aligned} \mathcal{U}(\Gamma) &= \sum_{(i,j) \in \Gamma^{\rightarrow}} u_{ij} - \sum_{(i,j) \in \Gamma^{\leftarrow}} \ell_{ij} \\ &= (u_{24} + u_{35}) - (\ell_{40}) \\ &= (4 + 2.5) - (-2) \\ &= 8.5. \end{aligned}$$

3. To saturate the cut, the flow on each forward arc must be equal to the upper bound, and the flow on each backward arc must be equal to the lower bound. Therefore, the flow on arc (v_4, v_0) must be -2, the flow on arc (v_2, v_4) must be 4, and the flow on arc (v_3, v_5) must be 2.5. The flows on all the other arcs of the graph are irrelevant for the saturation of the cut. It must be verified that the capacity constraints are verified, though. An example of flow vector is represented in the figure below.


v_0
v_1
v_2
v_3
v_4
v_5
$(-2, 2)$
-2
$(2, 4.5)$
3
$(0, 4)$
4
$(-2, 7)$
0
$(-1, 6)$
0.5
$(-1, 2.5)$
2.5
$(1, 1)$
1
$(0.5, 7)$
2.5
$\mathcal{M}$
$\mathcal{N} \setminus \mathcal{M}$

Question 11

Consider the following network and the corresponding vector of flows:



1. Calculate the divergence of each node. Which nodes are *supply nodes*? Which ones are *demand nodes*? Which ones are *transit nodes*?

2. Verify that the sum of all divergences is zero.

Solution 11

1. Remember that the divergence of a flow vector $x \in \mathbb{R}^n$ at node i is defined as the total quantity of flow that leaves the node minus the total quantity of flow that enters the node:

$$\operatorname{div}(x)_i = \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

   If the divergence of node i is positive, then units of flow are produced at i and it is a *supply node*. If the divergence of node i is negative, then units of flow are consumed at i and it is a *demand node*. If the divergence is zero, it is a *transit node*.

   Here we have:

   - $\operatorname{div}(x)_{v_0} = (-3) - (5) = -8$, therefore v_0 is a demand node.
   - $\operatorname{div}(x)_{v_1} = (5) - (-3) = 8$, therefore v_1 is a supply node.
   - $\operatorname{div}(x)_{v_2} = 0$, therefore v_2 is a transit node.
   - $\operatorname{div}(x)_{v_3} = (-3) - (6) = -9$, therefore v_3 is a demand node.
   - $\operatorname{div}(x)_{v_4} = 0 - (-3) = 3$, therefore v_4 is a supply node.
   - $\operatorname{div}(x)_{v_5} = (6+2) - (4) = 4$, therefore v_5 is a supply node.
   - $\operatorname{div}(x)_{v_6} = (4-5) - 0 = -1$, therefore v_6 is a demand node.
   - $\operatorname{div}(x)_{v_7} = (-3) - (2+1) = -6$, therefore v_7 is a demand node.
   - $\operatorname{div}(x)_{v_8} = 1 - (-3-5) = 9$, therefore v_8 is a supply node.

2. We use the results from the previous step to verify that

$$\sum_{i=0}^{8} \operatorname{div}(x)_{v_i} = -8+8+0-9+3+4-1-6+9 = 0.$$

Question 12

Prove that given a network $(\mathcal{N}, \mathcal{A})$ and its associated vector of divergences, the sum of all divergences is always zero.

Solution 12

For each node $i \in \mathcal{N}$, the divergence of x at node i is defined as

$$\operatorname{div}(x)_i = \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

Therefore, we have

$$\sum_{i\in\mathcal{N}} \operatorname{div}(x)_i = \sum_{i\in\mathcal{N}} \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{i\in\mathcal{N}} \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

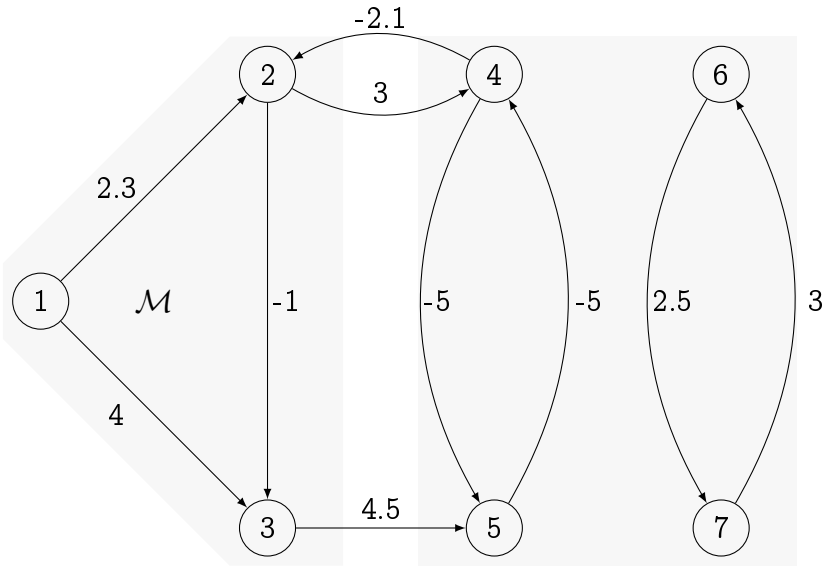
The first term involves a sum over all outgoing arcs of all nodes. It is actually a sum over all the arcs in the network. Similarly, the second involves a sum over all ingoing arcs of all nodes. It is also a sum over all the arcs in the network. Therefore, we have

$$\begin{aligned}\sum_{i\in\mathcal{N}} \operatorname{div}(x)_i &= \sum_{i\in\mathcal{N}} \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{i\in\mathcal{N}} \sum_{k|(k,i)\in\mathcal{A}} x_{ki} \\ &= \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(k,i)\in\mathcal{A}} x_{ki} \\ &= 0.\end{aligned}$$

Intuitively, it means that every unit of flow that is generated somewhere is consumed somewhere else.

Question 13

Consider the following network and the corresponding flow vector:



1. For each node, calculate the divergence.
2. Identify if the node is a supply, demand or transit node.
3. What is the total divergence in the network?
4. Write the cut Γ defined by the gray areas. Calculate the flow through that cut Γ and then calculate the sum of the divergences of the nodes belonging to $\mathcal{M}$. What do you remark?

Solution 13

1. Remember that the divergence of a flow vector $x \in \mathbb{R}^n$ at node i is defined as the total quantity of flow that leaves the node minus the total quantity of flow that enters the node:

$$\operatorname{div}(x)_i = \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

   We obtain:

   - $\operatorname{div}(x)_1 = (2.3 + 4) - 0 = 6.3$,
   - $\operatorname{div}(x)_2 = (-1 + 3) - (-2.1 + 2.3) = 1.8$,
   - $\operatorname{div}(x)_3 = (4.5) - (4 - 1) = 1.5$,
   - $\operatorname{div}(x)_4 = (-5 - 2.1) - (3 - 5) = -5.1$,
   - $\operatorname{div}(x)_5 = (-5) - (-5 + 4.5) = -4.5$,
   - $\operatorname{div}(x)_6 = (2.5) - (3) = -0.5$,
   - $\operatorname{div}(x)_7 = (3) - (2.5) = 0.5$.

2. Nodes 1, 2, 3 and 7 have a positive divergence. They are supply nodes. Nodes 4, 5 and 6 have a negative divergence. They are demand nodes. There is no transit node.

3. The sum of the divergences in the network is *always* zero. It can be checked that

$$6.3 + 1.8 + 1.5 - 5.1 - 4.5 - 0.5 + 0.5 = 0.$$

4. Here, the cut $\Gamma = (\mathcal{M}, \mathcal{N}\backslash\mathcal{M})$ is defined as $\mathcal{M} = \{1, 2, 3\}$ and $\mathcal{N}\backslash\mathcal{M} = \{4, 5, 6, 7\}$.
   The set of forward arcs is

$$\Gamma^{\rightarrow} = \{(2, 4), (3, 5)\},$$

   and the set of backward arcs is

$$\Gamma^{\leftarrow} = \{(4, 2)\}.$$

The flow through the cut Γ is defined as

$$X(\Gamma) = \sum_{(i,j)\in\Gamma^{\rightarrow}} x_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} x_{ij}.$$

Then we obtain

$$X(\Gamma) = (4.5 + 3) - (-2.1) = 9.6.$$

The sum of the divergences of the nodes belonging to $\mathcal{M}$ is

$$\sum_{i\in\mathcal{M}} \mathrm{div}(x)_i = 6.3 + 1.8 + 1.5 = 9.6.$$

Thus the flow through Γ is equal to the sum of the divergences of the nodes belonging to $\mathcal{M}$. This is always the case.

The intuition is the following. Consider an arc (i, j) such that both i and j are in $\mathcal{M}$. When calculating the sum of the divergences in $\mathcal{M}$, the flow of this arc will be involved once with a positive sign for the divergence of j, and once with a negative sign for the divergence of i. Therefore, its contribution to the sum is zero. It means that only arcs (i, j) such that $i \in \mathcal{M}$ and $j \in \mathcal{N} \setminus \mathcal{M}$ will contribute to the sum. But these are exactly the arcs belonging to the cut. Therefore,

$$\sum_{i\in\mathcal{M}} \mathrm{div}(x)_i = \sum_{(i,j)\in\Gamma^{\rightarrow}} x_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} x_{ij} = X(\Gamma).$$

Question 14

You want to travel from Lausanne to Rome and you can choose among the following alternatives.

1. You can take the train from Lausanne to Geneva, which costs 20 CHF and has a travel time of 45 minutes. Then, you fly from Geneva to Rome, which costs 210CHF and has a travel time of 1h30.

2. You can use your car for a total cost of 150 CHF. However, since you have to be concentrated during 9h30 of the journey, it is estimated that each hour spent in the car has an additional cost of 5 CHF.

3. You can travel through Milan with a bus or a train. The bus costs 20CHF and takes 5h, while the train costs 60 CHF and takes 3h30. Once in Milan, you can reach Rome either by bus (40 CHF, 7h) or by train (90 CHF,3h).

We assume that you have a value of time of 30 CHF/hour. It means that each hour of travel is associated with a cost of 30CHF.

Model the resulting problem with a network representation. Mention the cost (in CHF) of each arc. The network must have an injective incidence function, meaning that there must be at most one arc between two nodes.

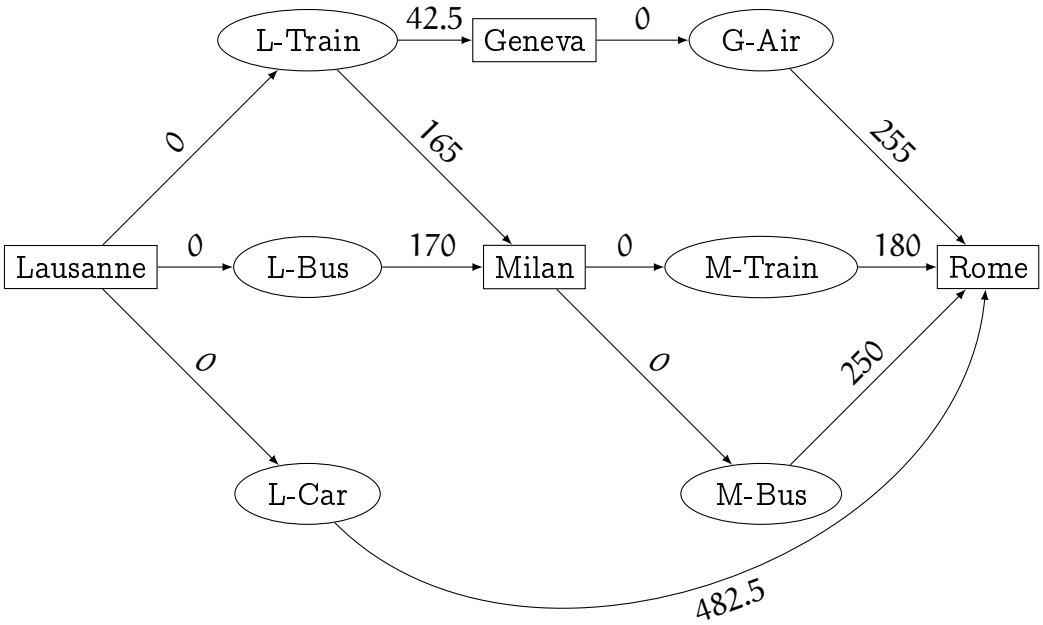
Hint: Consider two sets of nodes, one corresponding to the cities, and one corresponding to all combination of cities and transportation modes.

Solution 14

First, we construct the network. We define two sets of nodes:

- cities, represented with a rectangle shape in the figure below (Lausanne, Geneva, Milan and Rome),
- transportation modes to leave a city, represented with an elliptic shape in the figure below (Lausanne by train, Lausanne by bus, Lausanne by car, Geneva by air, Milan by train and Milan by bus).

An arc between a city and a transportation mode represents the choice of the transportation mode, and is not associated with any cost. An arc between a transportation mode and a city represents the travel, and is associated with a generalized cost that is calculated below.

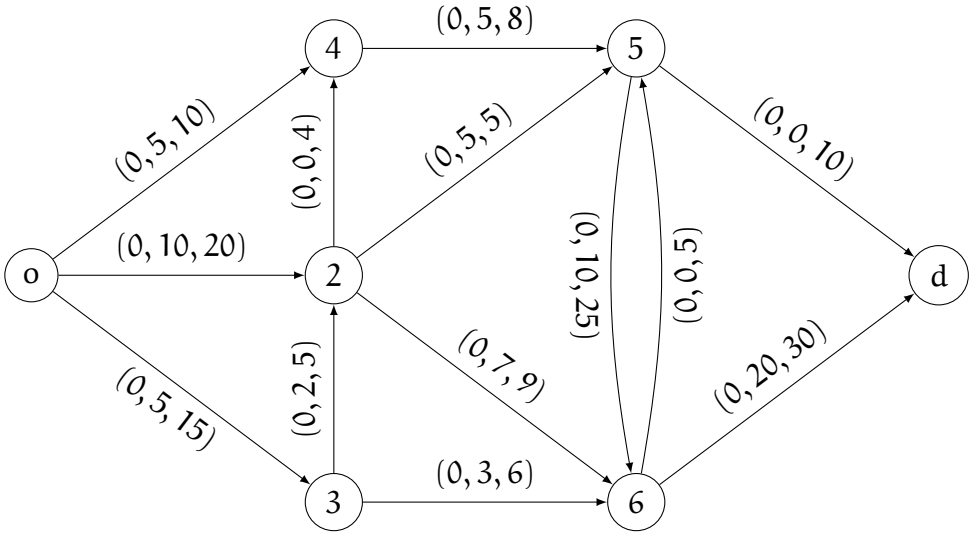


- L-car to Rome represents traveling by car between Lausanne and Rome. The cost in CHF is $150 + 9.5 \cdot 30 + 9.5 \cdot 5 = 482.5$.
- L-train to Geneva represents traveling by train from Lausanne to Geneva. The cost in CHF $20 + 0.75 \cdot 30 = 42.5$.

- L-train to Milan represents traveling by train between Lausanne and Milan. The cost is $60 + 3.5 \cdot 30 = 165$.
- L-bus to Milan represents traveling by bus between Lausanne and Milan. The cost is $20 + 5 \cdot 30 = 170$.
- G-Air to Rome represents traveling by air between Geneva and Rome. The cost in CHF is $210 + 1.5 \cdot 30 = 255$.
- M-train to Rome represents traveling by train from Milan to Rome. The cost is $90 + 3 \cdot 30 = 180$.
- M-bus to Rome represents traveling by bus from Milan to Rome. The cost is $40 + 7 \cdot 30 = 250$.

Question 15

Consider the network represented in the figure below, where each arc (i, j) is associated with its lower bound ℓ_{ij}, its flow x_{ij} and its upper bound u_{ij} in the following way: $(\ell_{ij}, x_{ij}, u_{ij})$. Identify two cuts $\Gamma = (\mathcal{M}, \mathcal{N} \backslash \mathcal{M})$ separating o from d, such that $\Gamma^{\rightarrow}$ contains exactly 4 arcs. For each of them, give the flow through the cut and the capacity of the cut.



Solution 15

Here are two cuts $\Gamma = (\mathcal{M}, \mathcal{N} \backslash \mathcal{N})$ such that $\Gamma^{\rightarrow}$ contains exactly 4 arcs. These are not the only ones.

- For the first one, we consider $\Gamma_1 = (\{o, 3\}, \{2, 4, 5, 6, d\})$.
  The flow through Γ_1 is $\mathcal{X}(\Gamma_1) = 5 + 10 + 5 + 3 = 23$.
  The capacity of that cut is $\mathcal{U}(\Gamma_1) = 10 + 20 + 15 + 6 = 51$.

- For the second one we consider $\Gamma_2 = (\{0, 2, 4, 5\}, \{3, 6, d\})$.
  The flow thought Γ_2 is $\mathcal{X}(\Gamma_2) = 5 - 2 + 7 - 0 + 10 + 0 = 20$.
  The capacity of that cut is $\mathcal{U}(\Gamma_2) = 15 - 0 + 9 - 0 + 25 + 10 = 59$.

Question 16

A software company has sold one of its flagship products to a very important customer. However, the customer was not satisfied with the quality and reported 78 bugs to the company. So the company made an agreement with the client to find a solution and fix all the bugs in a planning period of three months. The resolution of the bugs is planned as follows: solve exactly 28 bugs the first month, 26 the second month and 24 in the last month.
The following table shows the costs to fix a bug and the maximum number of bugs that can be fixed. The costs are different if a computer scientist works during his working hours or if he works overtime. Each bug postponed to the next month is penalized by the customer with 500.- CHF. It is also stipulated that the company must absolutely solve all the bugs in a time period of three months.

Month	Cost to solve one bug in CHF		Maximum number of bugs that can be fixed	
	Regular hours	Supplementary hours	Regular hours	Supplementary hours
1	350	650	22	12
2	350	650	23	10
3	350	650	20	10

Model the situation using a network.

Solution 16

To model the situation, we have to determine the set of nodes and the set of arcs.

1. Set of nodes:
   - The nodes B_i, $i = 1, 2, 3$, represent the bugs which have to be solved during month i. Those nodes are supply nodes with divergence equal to the numbers of bugs (28 bugs for the first month, 26 for the second one and 24 for the last one).
   - The nodes BR_i, $i = 1, 2, 3$, represent the bugs solved during regular hours (thus between 8am and 6pm) throughout month i.
   - The nodes BO_i, $i = 1, 2, 3$, represent the bugs solved during supplementary hours throughout month i.
   - The nodes M_i, $i = 1, 2, 3$, represent the bugs solved during month i.
   - The node O represents the number of bugs which have to be solved at the end of the three months: it is a demand node for which the divergence is equal to the total of bugs that have to be fixed: $\text{div}(x)_O = -78$.
2. Set of arcs:
   - The arcs from the nodes B_i, $i = 1, 2, 3$, to the nodes BR_i, $i = 1, 2, 3$, and to the nodes $B0_i$ $i = 1, 2, 3$. This is no cost for traversing the arc. The lower bound is 0, in order to impose to traverse the arc in the right direction, and the upper bound is infinity. The quantity of flow on each arc corresponds to the number of bugs solved during the regular or supplementary hours of month i.
   - The arcs (B_1, B_2) and (B_2, B_3) represents the bugs which are postponed to the next month. The cost is 500.- CHF per bug.

- The arcs from the nodes for problem solving (BR_i, BO_i, $i = 1, 2, 3$), to the nodes of "delivery" (M_i, $i = 1, 2, 3$). The costs and the upper bounds are given by the data of the problem. Note that the lower bounds must be zero for each arc to impose to follow the direction of the arc.
- The arcs from the "delivery" nodes ($M_i, i = 1, 2, 3$), to the node O to ensure that the total number of bugs solved after 3 months is equal to 78. The lower bound is 0, in order to impose to traverse the arc in the right direction, and the upper bound is infinity.

The resulting graph is shown in the Figure 10 on the following page, where each arc (i, j) is labeled as $c_{ij}[\ell_{ij}, u_{ij}]$, where c_{ij} is the cost, ℓ_{ij} the lower capacity, and u_{ij} the upper capacity. If the divergence of a node is not zero, its value is explicitly provided next to the node.

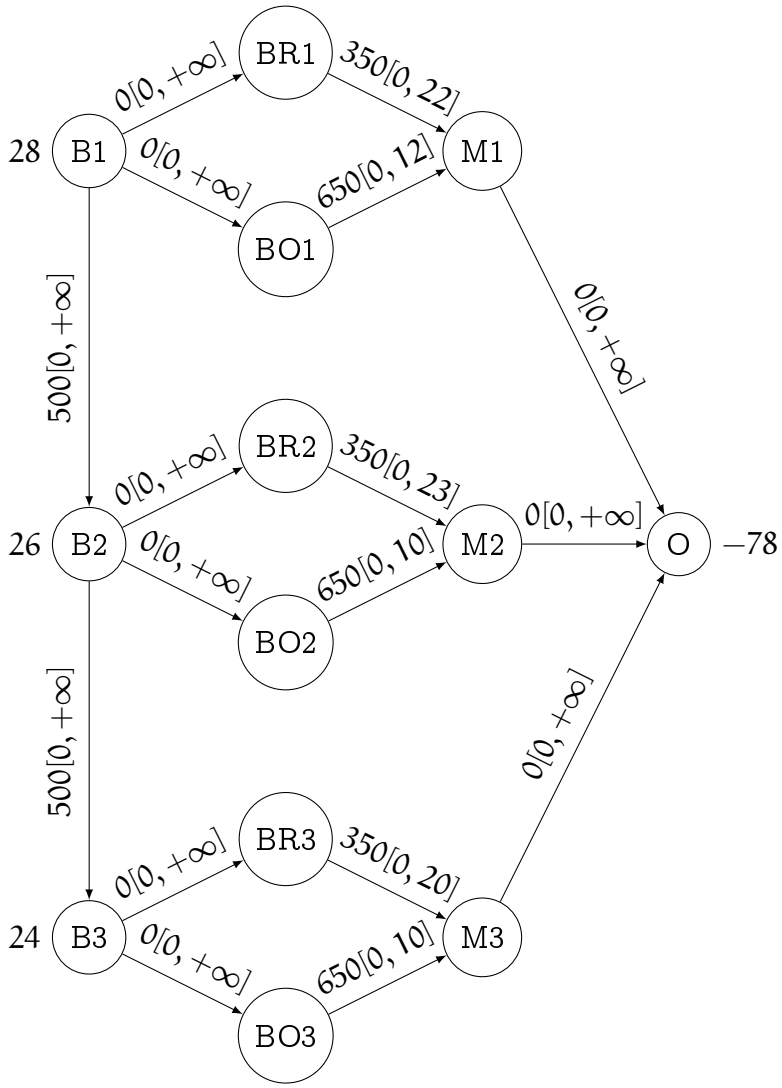
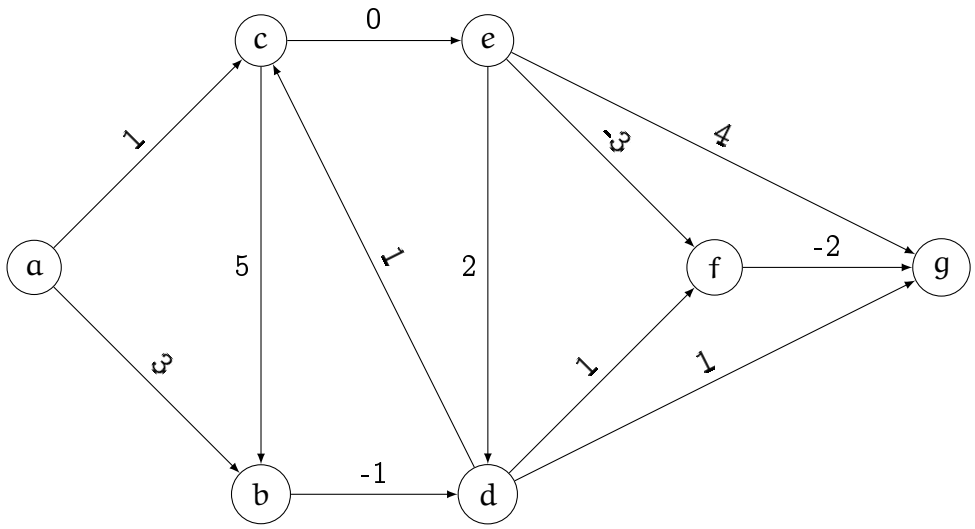


Figure 10: Network representation for the bug fixing problem

Question 17

Consider the following network where the number associated with each arc represents the amount of flow traversing it.



1. What is the indegree, the outdegree and the degree of each node?
2. Give the adjacency matrix of the network.
3. Represent the network using an adjacency list that also store the flows.
4. Is the network connected?
5. Is the network strongly connected?
6. Enumerate all simple forward paths from node a to node g.
7. Give the divergence of the flow vector of each node. What are the supply nodes? What are the demand nodes?
8. Consider the cut $\Gamma = (\mathcal{M}, \mathcal{N} \backslash \mathcal{M})$, defined by the set $\mathcal{M} = \{a, b, c\}$.
   (a) What are the forward arcs of the cut?
   (b) What are the backward arcs of the cut?

(c) What is the flow through the cut? Check that the formula

$$X(\Gamma) = \sum_{i \in \mathcal{M}} \mathrm{div}(x)_i$$

is satisfied.

(d) Assume that the capacities on each arc are -3 for the lower bound and 5 for the upper bound. What is the capacity of the cut?

Solution 17

1. The degree d_i of the node v_i is the number of incident arcs. The outdegree d_i^+ of node i represents the number of incident arcs of the form (i,j) (we say that i is an *upstream node*), and the indegree d_i^- of node i is the number of incident arcs of the form (j,i) (we say that i is a *downstream node*).

$$\begin{array}{lllllll} d_a = 2 & d_b = 3 & d_c = 4 & d_d = 5 & d_e = 4 & d_f = 3 & d_g = 3 \\ d_a^- = 0 & d_b^- = 2 & d_c^- = 2 & d_d^- = 2 & d_e^- = 1 & d_f^- = 2 & d_g^- = 3 \\ d_a^+ = 2 & d_b^+ = 1 & d_c^+ = 2 & d_d^+ = 3 & d_e^+ = 3 & d_f^+ = 1 & d_g^+ = 0 \end{array}$$

2. Considering two nodes $i,j \in \mathcal{N}$ and the set of arcs $\mathcal{A}$, the adjacency matrix $A \in \mathbb{R}^{7\times 7}$ is defined as

$$A(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

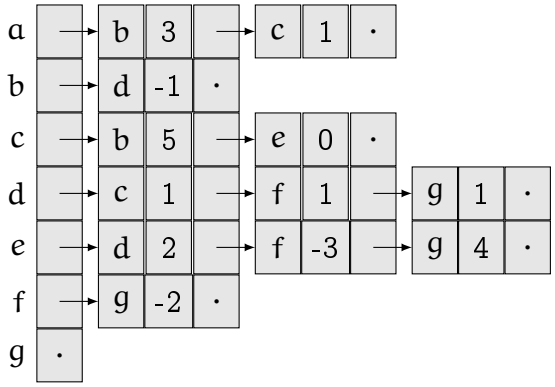
   Thus the adjacency matrix of the network is the following:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

   where row and column 1 correspond to node a, row and column 2 to node b, and so on.

3. The adjacency list corresponding to node $i \in \mathcal{N}$ is a list of length equal to its outdegree d_i^+ (recall that the outdegree of node i is the number of arcs of the form $(i,j) \in \mathcal{A}$). Each element of the list corresponds to an arc going out of i.

   Here we obtain:



4. Since there exists a path between each pair of nodes, the network is connected.

5. Since there is no forward path from g to any other node, then the network is not strongly connected.

6. We are going to enumerate all simple forward path from node a to node g:

   - a → c → e → g,
   - a → c → e → f → g,
   - a → c → e → d → g,
   - a → c → e → d → f → g,
   - a → b → d → g,
   - a → b → d → f → g,
   - a → b → d → c → e → g,
   - a → b → d → c → e → f → g,
   - a → c → b → d → g,
   - a → c → b → d → f → g.

7. Recall that divergence of node i is defined as:

$$\operatorname{div}(x)_i = \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

If this later is negative then node i is a demand node.
If this later is positive then node i is a supply node.
If this later is null then node i is a transit node.

Lets calculate the divergence of each node and then determine its nature.

- $\operatorname{div}(x)_a = 1 + 3 = 4$, therefore a is a supply node.
- $\operatorname{div}(x)_b = -1 - (3 + 5) = -9$, therefore b is a demand node.
- $\operatorname{div}(x)_c = 0 + 5 - (1 + 1) = 3$, therefore c is a supply node.
- $\operatorname{div}(x)_d = 1 + 1 + 1 - (-1 + 2) = 2$, therefore d is a supply node.
- $\operatorname{div}(x)_e = 4 - 3 + 2 - 0 = 3$, therefore e is a supply node.
- $\operatorname{div}(x)_f = -2 - (-3 + 1) = 0$, therefore f is a transit node.
- $\operatorname{div}(x)_g = -(4 - 2 + 1) = -3$, therefore g is a demand node.

8. (a) The set of forward arcs is

$$\Gamma^{\rightarrow} = \{((b, d), (c, e)\}.$$

(b) The set of backward arcs is

$$\Gamma^{\leftarrow} = \{(d, c)\}.$$

(c) The flow through the cut is

$$\begin{aligned} X(\Gamma) &= \sum_{(i,j)\in\Gamma^{\rightarrow}} x_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} x_{ij} \\ &= 0 + (-1) - 1 \\ &= -2. \end{aligned}$$

Lets verify whether the flow through the cut is equal to the sum of divergences of the nodes belonging to that cut:

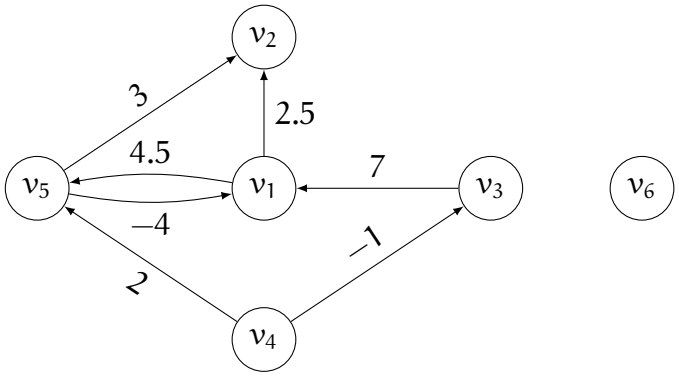
$$\sum_{i\in\mathcal{M}} \operatorname{div}(x)_i = 4 + 3 - 9 = -2 = X(\Gamma).$$

(d) The capacity of the cut is

$$\begin{aligned} U(\Gamma) &= \sum_{(i,j)\in\Gamma^{\rightarrow}} u_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} \ell_{ij} \\ &= 5 + 5 - (-3) \\ &= 13. \end{aligned}$$

Question 18

Consider the following network:



1. What is the adjacency matrix of this network?
2. How would you represent the flows on the arcs?
3. What are the adjacency lists for the nodes of this network?
4. The adjacency list introduced in the lecture allows to easily access the successors of each node. Propose an adjacency list, based on the same concept, that would allow an easy access to the predecessors of each node.

Solution 18

1. The cardinality of the set of nodes $\mathcal{N}$ is $|\mathcal{N}| = 6$. Then, considering two nodes $i, j \in \mathcal{N}$ and the set of arcs $\mathcal{A}$, the adjacency matrix $A \in \mathbb{R}^{6\times 6}$ is defined as

$$A(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

If we consider the natural numbering order from v_1 to v_6, we obtain:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. If we want to store the values of the flows, we can do several things. For example, we can use the same structure:

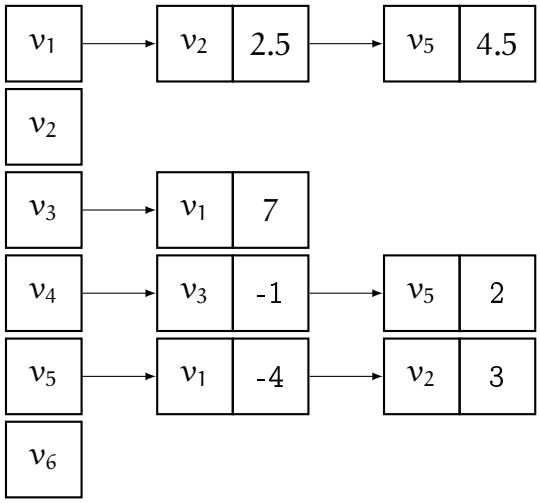
$$F = \begin{pmatrix} 0 & 2.5 & 0 & 0 & 4.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ -4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The disadvantage is that we store a lot of zero values. As suggested in the video, we can also adopt a row-by-row numbering for the arcs, and store the flows into a vector

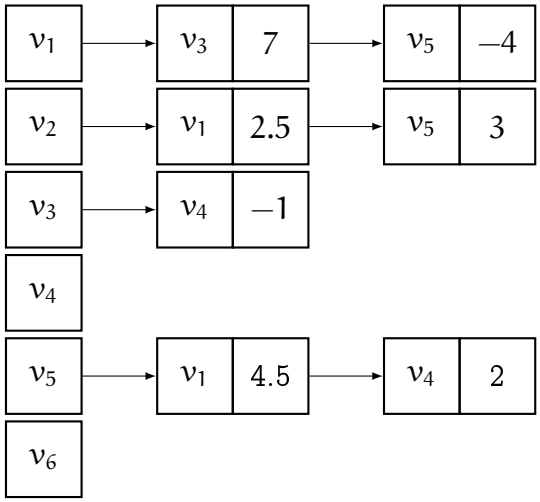
$$F = (2.5, 4.5, 7, -1, 2, -4, 3).$$

3. The adjacency list corresponding to node $i \in \mathcal{N}$ is a list of length equal to its outdegree d_i^+ (recall that the outdegree of node i is the number of arcs of the form $(i,j) \in \mathcal{A}$). Each element of the list corresponds to an arc going out of i.

For each arc, we also store the value of the flow. We obtain:



4. The adjacency lists for predecessor would be as follows:

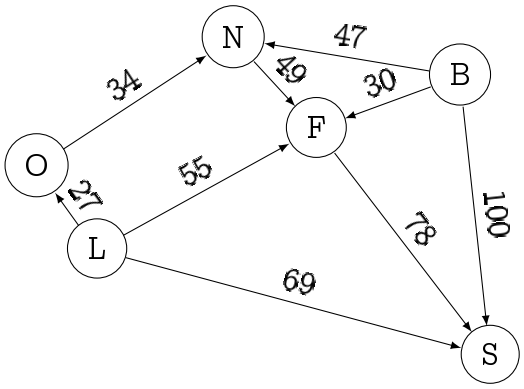


For instance, node v_1 has two predecessors: v_3 and v_5. The arc (v_3, v_1) carries a flow of 7, and the arc (v_5, v_1) carries a flow of -4. Therefore, the adjacency list of node v_1 has two elements: node v_3 associated with the flow on arc (v_3, v_1), that is 7, and node v_5 associated with the flow on arc (v_5, v_1), that is -4. Node v_4 has no predecessor. Therefore, its adjacency list is empty.

6 The transhipment problem

Question 01

A watch company in Switzerland has two factories in Lausanne and Bern. They supply 350 units, and 400 units per day respectively. Customers have a demand of 125 units per day in each of the following cities: Orbe (O), Neuchatel (N), Lausanne (L), Fribourg (F), Berne (B), Sion (S). The logistic system is designed in such a way that transportation between two cities is possible only in one direction, as illustrated in the figure below. The costs for transporting one unit are also reported in the following picture.



The transport from one city to another is limited to 200 units per day.

1. Write down a transhipment problem to minimize the total cost.
2. Transform it into a linear problem in standard form, such that the matrix A of the constraints has exactly two non zero entries in each column: 1 and -1.

Solution 01

1. A transhipment problem has the form:

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \quad \forall i \in \mathcal{N},$$
$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \quad \forall (i,j) \in \mathcal{A},$$

where $\mathcal{A}$ is the set of n arcs, $\mathcal{N}$ is the set of m nodes, $s \in \mathbb{R}^m$ represents the supply/demand of flow, $c \in \mathbb{R}^n$ represents the cost of transporting a unit of flow on each arc and $\ell \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ represent the lower and upper capacity of each arc respectively. The decision variables of the problem are denoted x_{ij}, that is representing the flow on arc (i,j).

The objective function is as follows:

$$\min_{x \in \mathbb{R}^9} 34x_{ON}+27x_{LO}+55x_{LF}+69x_{LS}+49x_{NF}+78x_{FS}+47x_{BN}+30x_{BF}+100x_{BS}.$$

For each node, the flow conservation constraints are

$$\begin{aligned}
x_{ON} - x_{LO} &= s_o = -125, && \text{(demand node)} \\
x_{LO} + x_{LF} + x_{LS} &= s_L = 350 - 125 = 225, && \text{(supply node)} \\
x_{NF} - x_{ON} - x_{BN} &= s_N = -125, && \text{(demand node)} \\
x_{FS} - x_{NF} - x_{BF} - x_{LF} &= s_F = -125, && \text{(demand node)} \\
x_{BN} + x_{BS} + x_{BF} &= s_B = 400 - 125 = 275, && \text{(supply node)} \\
-x_{BS} - x_{FS} - x_{LS} &= s_S = -125. && \text{(demand node)}
\end{aligned}$$

Note that, for the two cities where the units are produced, we have deduced the local demand that does not need to be transported:

- Lausanne: 350 - 125 = 225,
- Bern: 400 - 125 = 275.

The capacity constraints on each arc (i, j) are

$$0 \leq x_{ij} \leq 200.$$

Putting everything together, we have

$$\min_{x \in \mathbb{R}^9} 34x_{ON}+27x_{LO}+55x_{LF}+69x_{LS}+49x_{NF}+78x_{FS}+47x_{BN}+30x_{BF}+100x_{BS},$$

subject to

$$\begin{aligned}
x_{ON} - x_{LO} &= -125, \\
x_{LO} + x_{LF} + x_{LS} &= 225 \\
x_{NF} - x_{ON} - x_{BN} &= -125 \\
x_{FS} - x_{NF} - x_{BF} - x_{LF} &= -125 \\
x_{BN} + x_{BS} + x_{BF} &= 275 \\
-x_{BS} - x_{FS} - x_{LS} &= -125
\end{aligned}$$

$$\begin{aligned}
x_{ON} &\geq 0, & x_{ON} &\leq 200, \\
x_{LO} &\geq 0, & x_{LO} &\leq 200, \\
x_{LF} &\geq 0, & x_{LF} &\leq 200, \\
x_{LS} &\geq 0, & x_{LS} &\leq 200, \\
x_{NF} &\geq 0, & x_{NF} &\leq 200, \\
x_{BN} &\geq 0, & x_{BN} &\leq 200, \\
x_{BF} &\geq 0, & x_{BF} &\leq 200, \\
x_{BS} &\geq 0, & x_{BS} &\leq 200, \\
x_{FS} &\geq 0, & x_{FS} &\leq 200.
\end{aligned}$$

Note that the sum of elements of the vector s is equal to 0:

$$-125 + 225 - 125 - 125 + 275 - 125 = 0.$$

2. The standard form is

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$\begin{aligned} Ax &= b, \\ x &\geq 0. \end{aligned}$$

The objective function is already in standard form (i.e. minimization). We need to modify the constraints. We need to include a slack variable for each inequality constraint different from $x_{ij} \geq 0$. We obtain the following problem in standard form:

$$\min_{x \in \mathbb{R}^{18}} 34x_{ON}+27x_{LO}+55x_{LF}+69x_{LS}+49x_{NF}+78x_{FS}+47x_{BN}+30x_{BF}+100x_{BS},$$

subject to

$$\begin{aligned} x_{ON} - x_{LO} &= -125, \\ x_{LO} + x_{LF} + x_{LS} &= 225, \\ x_{NF} - x_{ON} - x_{BN} &= -125, \\ x_{FS} - x_{NF} - x_{BF} - x_{LF} &= -125, \\ x_{BN} + x_{BS} + x_{BF} &= 275, \\ -x_{BS} - x_{FS} - x_{LS} &= -125, \\ x_{ON} + y_{ON} &= 200, \\ x_{LO} + y_{LO} &= 200, \\ x_{LF} + y_{LF} &= 200, \\ x_{LS} + y_{LS} &= 200, \\ x_{NF} + y_{NF} &= 200, \\ x_{BN} + y_{BN} &= 200, \\ x_{BF} + y_{BF} &= 200, \\ x_{BS} + y_{BS} &= 200, \\ x_{FS} + y_{FS} &= 200, \end{aligned}$$

$$
\begin{aligned}
x_{ON} &\geq 0, & y_{ON} &\geq 0,\\
x_{LO} &\geq 0, & y_{LO} &\geq 0,\\
x_{LF} &\geq 0, & y_{LF} &\geq 0,\\
x_{LS} &\geq 0, & y_{LS} &\geq 0,\\
x_{NF} &\geq 0, & y_{NF} &\geq 0,\\
x_{BN} &\geq 0, & y_{BN} &\geq 0,\\
x_{BF} &\geq 0, & y_{BF} &\geq 0,\\
x_{BS} &\geq 0, & y_{BS} &\geq 0,\\
x_{FS} &\geq 0, & y_{FS} &\geq 0.
\end{aligned}
$$

If we number the arcs using the following order: x_{ON}, x_{LO}, x_{LF}, x_{LS}, x_{NF}, x_{FS}, x_{BN}, x_{BF}, x_{BS}, y_{ON}, y_{LO}, y_{LF}, y_{LS}, y_{NF}, y_{FS}, y_{BN}, y_{BF}, y_{BS}, the non zero entries of the matrix A are

$$
A = \left(\begin{array}{cccccccccc|cccccccccc}
1 & -1 & & & & & & & & & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & & & & & & & \\
-1 & & & & 1 & & -1 & & & & & & & & & & & & & \\
 & & -1 & & -1 & 1 & & -1 & & & & & & & & & & & & \\
 & & & & & & 1 & 1 & 1 & & & & & & & & & & & \\
 & & & -1 & & -1 & & & -1 & & & & & & & & & & & \\
\hline
1 & & & & & & & & & & 1 & & & & & & & & & \\
 & 1 & & & & & & & & & & 1 & & & & & & & & \\
 & & 1 & & & & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & & & & & & & 1 & & & & & & \\
 & & & & 1 & & & & & & & & & & 1 & & & & & \\
 & & & & & 1 & & & & & & & & & & 1 & & & & \\
 & & & & & & 1 & & & & & & & & & & 1 & & & \\
 & & & & & & & 1 & & & & & & & & & & 1 & & \\
 & & & & & & & & 1 & & & & & & & & & & 1 & \\
 & & & & & & & & & 1 & & & & & & & & & & 1
\end{array}\right).
$$

As we will see later in this chapter, the structure of the matrix A plays an important role in the properties of the transhipment problem. In particular, we will exploit the property that each column contains exactly two non zero entries: 1 and -1. The matrix presented above does not have this property. We will therefore apply a simple substitution based on the constraints $x_{ij} + y_{ij} = 200$. Each x_{ij} appearing with a

positive sign in the first set of constraints is replaced by $200 - y_{ij}$. We obtain an equivalent standard form formulation, with the following set of constraints:

$$
\begin{aligned}
-y_{ON} - x_{LO} &= -125 - 200 = -325, \\
y_{LO} - y_{LF} - y_{LS} &= -225 - 600 = -375, \\
y_{NF} - x_{ON} - x_{BN} &= -125 - 200 = -325, - \\
y_{FS} - x_{NF} - x_{BF} - x_{LF} &= -125 - 200 = -325, \\
y_{BN} - -y_{BS} - y_{BF} &= 275 - 600 = -325, \\
x_{BS} - x_{FS} - x_{LS} &= - - 125, \\
x_{ON} + y_{ON} &= 200, \\
x_{LO} + y_{LO} &= 200, - \\
x_{LF} + y_{LF} &= 200, \\
x_{LS} + y_{LS} &= 200, \\
x_{NF} + -y_{NF} &= 200, \\
x_{BN} + y_{BN} &= 200, \\
x_{BF} + y_{BF} &= -200, \\
x_{BS} + y_{BS} &= 200, \\
x_{FS} + y_{FS} &= 200,
\end{aligned}
$$

$$
\begin{aligned}
x_{ON} &\geq 0, & y_{ON} &\geq 0, \\
x_{LO} &\geq 0, & y_{LO} &\geq 0, \\
x_{LF} &\geq 0, & y_{LF} &\geq 0, \\
x_{LS} &\geq 0, & y_{LS} &\geq 0, \\
x_{NF} &\geq 0, & y_{NF} &\geq 0, \\
x_{BN} &\geq 0, & y_{BN} &\geq 0, \\
x_{BF} &\geq 0, & y_{BF} &\geq 0, \\
x_{BS} &\geq 0, & y_{BS} &\geq 0, \\
x_{FS} &\geq 0, & y_{FS} &\geq 0.
\end{aligned}
$$

The matrix A has now the property that each column contains exactly two

non zero entries, a 1 and a -1.

$$A = \left(\begin{array}{cccccccc|cccccccc}
 & -1 & & & & & & & -1 & & & & & & & \\
 & & & & & & & & & -1 & -1 & -1 & & & & \\
-1 & & & & & -1 & & & & & & & -1 & & & \\
 & & -1 & & -1 & & & -1 & & & & & & -1 & & \\
 & & & & & & & & & & & & & & -1 & -1 & -1 \\
 & & & -1 & & -1 & & & & & & & & & & \\
\hline
1 & & & & & & & & 1 & & & & & & & \\
 & 1 & & & & & & & & 1 & & & & & & \\
 & & 1 & & & & & & & & 1 & & & & & \\
 & & & 1 & & & & & & & & 1 & & & & \\
 & & & & 1 & & & & & & & & 1 & & & \\
 & & & & & 1 & & & & & & & & 1 & & \\
 & & & & & & 1 & & & & & & & & 1 & \\
 & & & & & & & 1 & & & & & & & & 1
\end{array}\right).$$

The vector b is

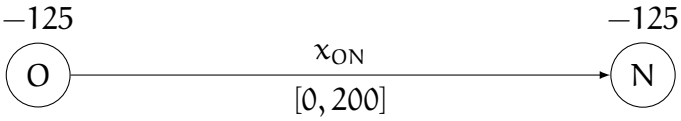
$$b = \begin{pmatrix} -325 \\ -375 \\ -325 \\ -325 \\ -325 \\ -125 \\ 200 \\ 200 \\ 200 \\ 200 \\ 200 \\ 200 \\ 200 \\ 200 \\ 200 \end{pmatrix}.$$

Note that it can be interpreted as a set of supply and demand, as its entries

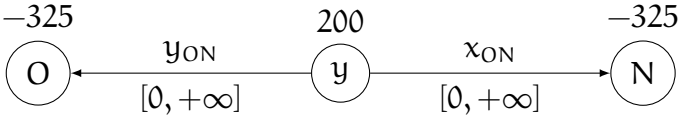
sum up to zero. The vector c is

$$c = \begin{pmatrix} 34 \\ 27 \\ 55 \\ 69 \\ 49 \\ 78 \\ 47 \\ 30 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Interestingly, this formulation can be seen as a standard transhipment problem on a modified network. To see this, consider the arc ON. In the original formulation, the flow on that arc was constrained between 0 and 200. Both O and N are demand nodes, with $s_O = s_N = -125$.



Each slack variable can be associated with a new node and a new arc, in the following configuration.



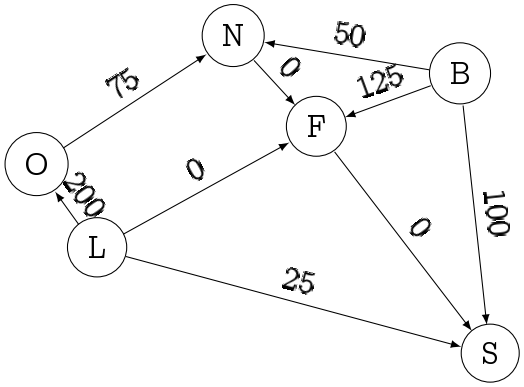
The interpretation is that we inject 200 units of flow in the network at the node corresponding to the slack variable. By construction, the flow x_{ON}

cannot exceed 200, and there is no need to set an upper bound on that arc. The new arc, associated with the slack variable, carries the remaining flow. The constraints $x_{ij} + y_{ij} = 200$ are the flow conservation constraints for the new nodes.

The optimal solution of the problem in standard form is

$$
\begin{aligned}
x_{ON} &= 75, & y_{ON} &= 125,\\
x_{LO} &= 200, & y_{LO} &= 0,\\
x_{LF} &= 0, & y_{LF} &= 200,\\
x_{LS} &= 25, & y_{LS} &= 175,\\
x_{NF} &= 0, & y_{NF} &= 200,\\
x_{FS} &= 0, & y_{FS} &= 200,\\
x_{BN} &= 50, & y_{BN} &= 150,\\
x_{BF} &= 125, & y_{BF} &= 75,\\
x_{BS} &= 100, & y_{BS} &= 100,
\end{aligned}
$$

and is represented on the following picture, where the number associated with each arc is the flow:



Question 02

Consider three factories which produce honey in Boston, New York and Los Angeles, whose supplies are 200 units, 250 units and 300 units per day, respectively. Customers have a demand of 250 units per day in each city. The costs for transporting one unit are the following:

- from Los Angeles to Boston is \$1,
- from Los Angeles to New York is \$0.5,
- from New York to Boston is \$0.8.

A maximum of 40 units per day can be transported from one city to another. Moreover, due to a specific contract with the shipping company, we have to transport at least 20 units from New-York to Boston. We want to find the cheapest way to transport the honey from the factories to the customers.

1. Represent the problem as a network. Notify the supply/demand data s_i next to each node i. Next to each arc (i, j), notify the cost and the capacities using the notation $c_{ij}(\ell_{ij}, u_{ij})$.
2. Write the transhipment problem as a linear optimization problem.
3. Identify the data $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ of the transhipment problem in standard form:

   $$\min_{x\in\mathbb{R}^n} c^T x$$

   subject to

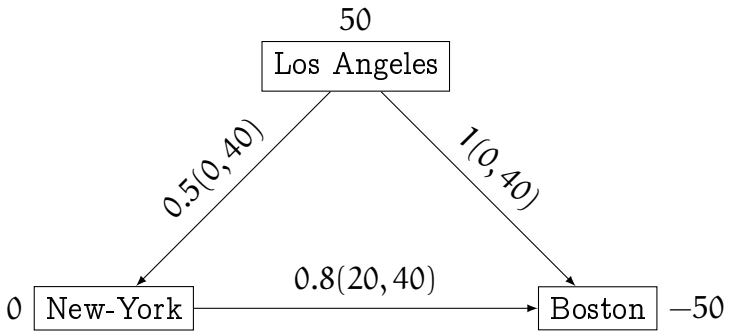
   $$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

   where the matrix A is such that there are only two non zeros entries in each column: 1 and -1.
4. What is the rank of the matrix A?

Solution 02

1. The network corresponding to the problem is represented below. Each city is represented by a node. Each pair of cities is connected with an arc, oriented according to the definition of costs. The supply/demand for each city is

   - Boston: 200 - 250 = -50,
   - New-York: 250 - 250 = 0,
   - Los Angeles: 300 - 250 = 50.



2. The transhipment problem is defined as follows.

$$\min_{x \in \mathbb{R}^3} 0.5x_{LN} + x_{LB} + 0.8x_{NB}$$

   where the decision variables are the flow on each arc, subject to the flow conservation constraints for each node:

$$\begin{aligned} x_{LN} + x_{LB} &= 50, \\ x_{NB} - x_{LN} &= 0, \\ -x_{LB} - x_{NB} &= -50, \end{aligned}$$

   and subject to the capacity constraints for each arc:

$$\begin{aligned} 0 \leq x_{LN} &\leq 40, \\ 0 \leq x_{LB} &\leq 40, \\ 20 \leq x_{NB} &\leq 40. \end{aligned}$$

3. To obtain the standard form, we first set the lower bounds to zero by a change of variables $x'_{ij} = x_{ij} - \ell_{ij}$. We define

$$\begin{aligned} x'_{LN} &= x_{LN}, \\ x'_{LB} &= x_{LB}, \\ x'_{NB} &= x_{NB} - 20. \end{aligned}$$

The problem becomes

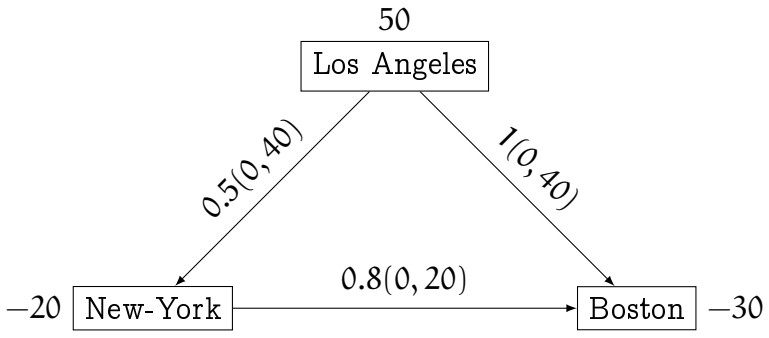
$$\min_{x \in \mathbb{R}^3} 0.5x'_{LN} + x'_{LB} + 0.8x'_{NB} + 16$$

subject to the flow conservation constraints for each node:

$$\begin{aligned} x'_{LN} + x'_{LB} &= 50, \\ x'_{NB} - x'_{LN} &= -20, \\ -x'_{LB} - x'_{NB} &= -30, \end{aligned}$$

$$\begin{aligned} 0 \leq x'_{LN} &\leq 40, \\ 0 \leq x'_{LB} &\leq 40, \\ 0 \leq x'_{NB} &\leq 20. \end{aligned}$$

Note that the constant term in the objective function (16) can be ignored.



Now that all lower bounds are zero, in order to write the problem in standard form, we redesign the network of the problem. We consider

all arcs (i, j) with an upper bound on the flow (all three of them in this example). For each of them, we introduce a new node associated with a supply defined by the upper bound of the arc. Then we replace the original arc by

- an arc from the new node k to the downstream node j of the original arc, with flow x_{kj} and cost $c_{kj} = c_{ij}$ and
- an arc from the new node k to the upstream node i of the original arc, with flow y_{ki} and cost $c_{ki} = 0$.

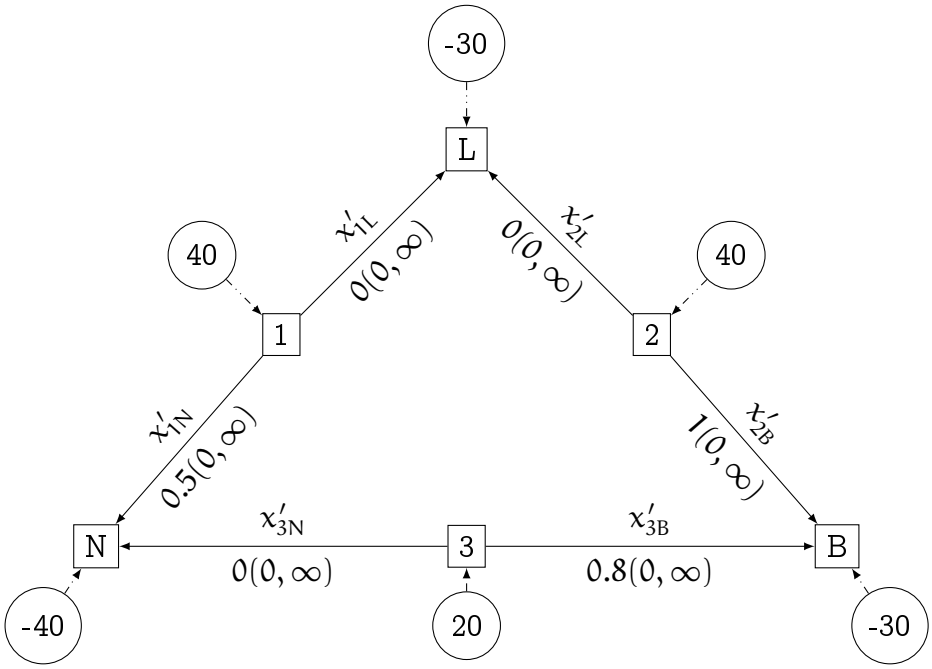
Indeed, the flow y_{ki} is the slack variable, associated with no cost.

We also recalculate the supply/demand of the original upstream nodes using the formula

$$s'_i = s_i - \sum_{(i,j)\in\mathcal{A}} u_{ij}.$$

- $s'_L = s_L - u_{LN} - u_{LB} = 50 - 40 - 40 = -30$,
- $s'_N = s_N - u_{NB} = -20 - 20 = -40$,
- $s'_B = s_B = -30$.

We obtain the following network, where node 1 is associated with arc (L, N), node 2 is associated with arc (L, B) and node 3 is associated with arc (N, B).



The corresponding linear optimization problem is:

$$\min_{x\in\mathbb{R}^6} x'_{2B} + 0.8x'_{3B} + 0.5x'_{1N}$$

subject to the flow conservation constraints

$$\begin{aligned}
-x'_{1L} - x'_{2L} &= -30 && \text{(node L)},\\
x'_{1L} + x'_{1N} &= 40 && \text{(node 1)},\\
-x'_{1N} - x'_{3N} &= -40 && \text{(node N)},\\
x'_{3N} + x'_{3B} &= 20 && \text{(node 3)},\\
-x'_{2B} - x'_{3B} &= -30 && \text{(node B)},\\
x'_{2L} + x'_{2B} &= 40 && \text{(node 2)},
\end{aligned}$$

and the non negativity constraints on each flow

$$x'_{1L}, x'_{1N}, x'_{2L}, x'_{2B}, x'_{3N}, x'_{3B} \geq 0.$$

4. If we order the variables as x'_{1L}, x'_{1N}, x'_{2L}, x'_{2B} ,x'_{3N}, x'_{3B}, the problem is

$$\min_{x'\in\mathbb{R}^n} c^T x'$$

subject to

$$\begin{aligned} Ax' &= b, \\ x' &\geq 0, \end{aligned}$$

where $m = 6$, $n = 6$,

$$A = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} -30 \\ 40 \\ -40 \\ 20 \\ -30 \\ 40 \end{pmatrix}, \; c = \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ 1 \\ 0 \\ 0.8 \end{pmatrix}.$$

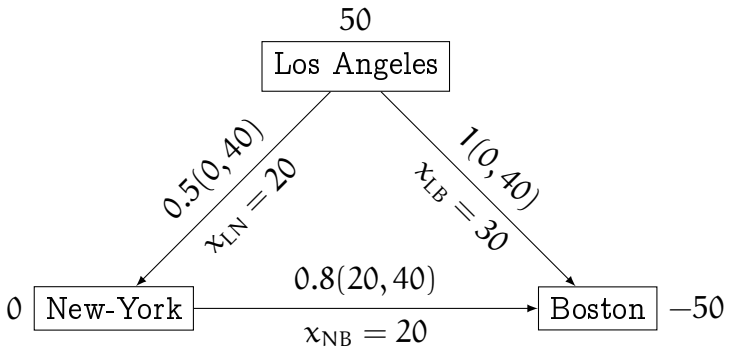
The optimal solution is $x'_{1L} = 20$, $x'_{1N} = 20$, $x'_{2L} = 10$, $x'_{2B} = 30$, $x'_{3N} = 20$, $x'_{3B} = 0$. In the network without slack variables, it means

$$\begin{aligned} x'_{LN} &= 20, \\ x'_{LB} &= 30, \\ x'_{NB} &= 0. \end{aligned}$$

And in the original variables, it means

$$\begin{aligned} x_{LN} &= 20, \\ x_{LB} &= 30, \\ x_{NB} &= 20. \end{aligned}$$

The optimal flows on the original network are represented in the figure below.



5. The matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

It has the property that each column contains exactly two non zero entries: 1 and -1. This property is actually shared with any transhipment problem. It means that if we sum all the rows of the matrix, we obtain the null vector. Therefore, the rows are linearly dependent, and the matrix is not of full rank. Its rank happens to be 5, not 6. There is actually a redundant constraint. Indeed, as the sum of the divergences of all nodes must be zero, one of the flow conservation constraints will be automatically verified if the others are. Consequently, one constraint of a transhipment problem can be removed before solving it.

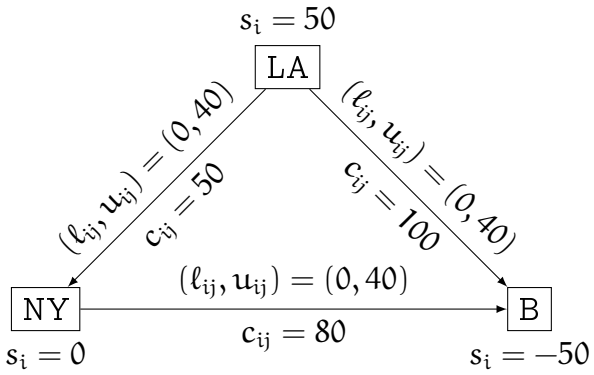
Question 03

Consider three factories producing electric cars in Boston, New-York and Los Angeles, with supplies 200 units, 250 units and 300 units per day respectively. The demand is 250 cars per day and per city. The cost for transporting one car:

- from Los Angeles to Boston is \$100.
- from Los Angeles to New-York is \$50.
- from New-York to Boston is \$80.

No more than 40 cars can be transported between two cities per day.

The corresponding network is:



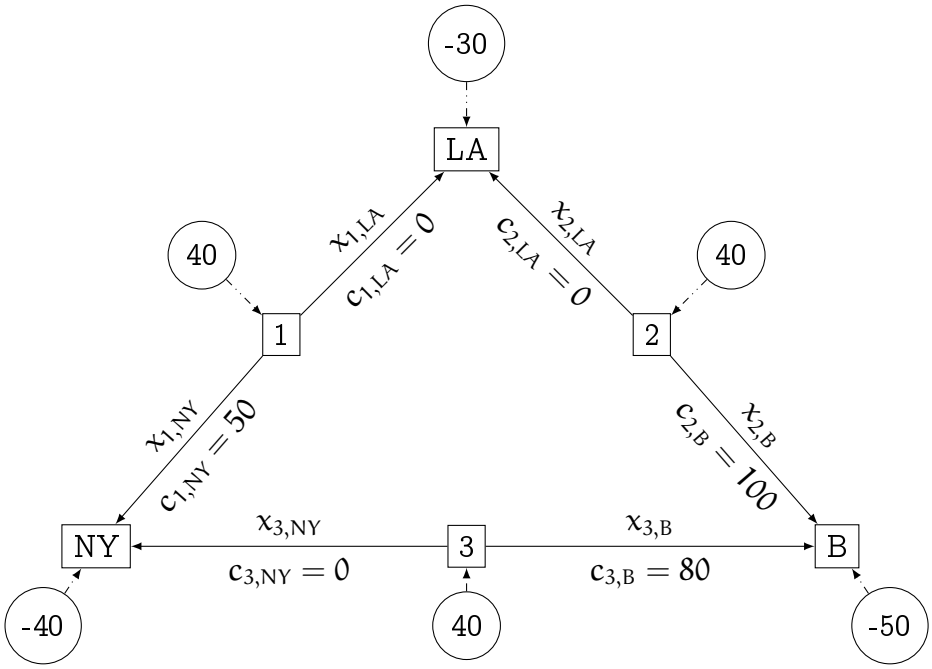
It corresponds to the following transhipment problem:

$$\min_{x \in \mathbb{R}^3} 100x_{LA,B} + 50x_{LA,NY} + 80x_{NY,B}$$

subject to

$$\begin{aligned} x_{LA,NY} + x_{LA,B} &= 50, \\ x_{NY,B} - x_{LA,NY} &= 0, \\ -x_{NY,B} - x_{LA,B} &= -50, \\ 0 \leq x_{LA,NY}, x_{LA,B}, x_{NY,B} &\leq 40. \end{aligned}$$

In order to obtain a formulation in standard form, we redesign the network in the following way:



The corresponding linear optimization problem in standard form is:

$$\min_{x \in \mathbb{R}^6} 100x_{2,B} + 80x_{3,B} + 50x_{1,NY}$$

subject to

$$\begin{aligned}
-x_{1,LA} - x_{2,LA} &= -30, \\
x_{1,LA} + x_{1,NY} &= 40, \\
-x_{1,NY} - x_{3,NY} &= -40, \\
x_{3,NY} + x_{3,B} &= 40, \\
-x_{2,B} - x_{3,B} &= -50, \\
x_{2,LA} + x_{2,B} &= 40, \\
x_{1,LA}, x_{1,NY}, x_{2,LA}, x_{2,B}, x_{3,NY}, x_{3,B} &\geq 0.
\end{aligned}$$

1. Write the dual problem using the Lagrangian method.

2. Solve the dual using optimality conditions. The procedure consists in treating all nodes of the network in a systematic way. The set of nodes that must be treated at a given iteration is denoted by $\mathcal{S}$. The treatment of one node consists in considering all arcs incident to the node, and updating the value of the dual variable of the other incident node using the optimality conditions.

(a) Select one arbitrary node i, set $\lambda_i = 0$ and $\mathcal{S} = \{i\}$.

(b) If $\mathcal{S} \neq \emptyset$, perform the following:

   i. Select a node i in $\mathcal{S}$.

   ii. For each outgoing arc (i, j), if λ_j is not defined yet, or if it violates the optimality conditions, update its value:

$$\text{If } \lambda_j > \lambda_i + c_{ij}, \text{ then } \lambda_j = \lambda_i + c_{ij} \text{ and } \mathcal{S} = \mathcal{S} \cup \{j\}.$$

   iii. For each ingoing arc (j, i), if λ_j is not defined yet, or if it violates the optimality conditions, update its value:

$$\text{If } \lambda_j < \lambda_i - c_{ji}, \text{ then } \lambda_j = \lambda_i - c_{ji} \text{ and } \mathcal{S} = \mathcal{S} \cup \{j\}.$$

   iv. Remove i from $\mathcal{S}$ and go to 2b.

(c) If $\mathcal{S} = \emptyset$, stop.

We suggest to initialize the process with node 1, and then process each node in a counter-clockwise way, that is: 1, NY, 3, B, 2, LA.

Solution 03

1. A dual variable λ_i is associated with each node of the transformed network, and a dual variable μ_{ij} is associated with each arc of the transformed network. The Lagrangian is

$$L(x,\lambda,\mu) = \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + \sum_i \lambda_i\Big(\sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(k,i)\in\mathcal{A}} x_{ki} - s_i\Big) - \sum_{(i,j)\in\mathcal{A}} \mu_{ij}x_{ij},$$

that is

$$\begin{aligned}
L(x,\lambda,\mu) =& 100x_{2,B} + 50x_{1,NY} + 80x_{3,B} \\
&+ \lambda_{LA}(-x_{1,LA} - x_{2,LA} + 30) + \lambda_1(x_{1,NY} + x_{1,LA} - 40) \\
&+ \lambda_{NY}(-x_{3,NY} - x_{1,NY} + 40) + \lambda_2(x_{2,LA} + x_{2,B} - 40) \\
&+ \lambda_B(-x_{2,B} - x_{3,B} + 50) + \lambda_3(x_{3,NY} + x_{3,B} - 40) \\
&- \mu_{1,NY}x_{1,NY} - \mu_{1,LA}x_{1,LA} - \mu_{2,LA}x_{2,LA} \\
&- \mu_{2,B}x_{2,B} - \mu_{3,B}x_{3,B} - \mu_{3,NY}x_{3,NY} \\
=& x_{1,LA}(-\lambda_{LA} + \lambda_1 - \mu_{1,LA}) + x_{1,NY}(50 - \lambda_{NY} + \lambda_1 - \mu_{1,NY}) \\
&+ x_{2,LA}(-\lambda_{LA} + \lambda_2 - \mu_{2,LA}) + x_{2,B}(100 - \lambda_B + \lambda_2 - \mu_{2,B}) \\
&+ x_{3,NY}(-\lambda_{NY} + \lambda_3 - \mu_{3,NY}) + x_{3,B}(80 - \lambda_B + \lambda_3 - \mu_{3,B}) \\
&+ 30\lambda_{LA} + 40\lambda_{NY} + 50\lambda_B - 40\lambda_1 - 40\lambda_2 - 40\lambda_3.
\end{aligned}$$

The coefficient of each flow variable must be zero to get a bounded dual problem:

$$\begin{aligned}
-\lambda_{LA} + \lambda_1 &= \mu_{1,LA}, \\
50 - \lambda_{NY} + \lambda_1 &= \mu_{1,NY}, \\
-\lambda_{LA} + \lambda_2 &= \mu_{2,LA}, \\
100 - \lambda_B + \lambda_2 &= \mu_{2,B}, \\
-\lambda_{NY} + \lambda_3 &= \mu_{3,NY}, \\
80 - \lambda_B + \lambda_3 &= \mu_{3,B}.
\end{aligned}$$

Moreover since $\mu_{ij} \geq 0, \quad \forall i, j$, we can eliminate the μ's, and we obtain the following set of constraints:

$$\begin{aligned} -\lambda_{LA} + \lambda_1 &\geq 0, \\ \lambda_{NY} - \lambda_1 &\leq 50, \\ -\lambda_{LA} + \lambda_2 &\geq 0, \\ \lambda_B - \lambda_2 &\leq 100, \\ -\lambda_{NY} + \lambda_3 &\geq 0, \\ \lambda_B - \lambda_3 &\leq 80. \end{aligned}$$

The objective function is:

$$q(\lambda) = 30\lambda_{LA} + 40\lambda_{NY} + 50\lambda_B - 40\lambda_1 - 40\lambda_2 - 40\lambda_3.$$

The dual problem is therefore:

$$\max 30\lambda_{LA} + 40\lambda_{NY} + 50\lambda_B - 40\lambda_1 - 40\lambda_2 - 40\lambda_3$$

subject to

$$\begin{aligned} -\lambda_{LA} + \lambda_1 &\geq 0, \\ \lambda_{NY} - \lambda_1 &\leq 50, \\ -\lambda_{LA} + \lambda_2 &\geq 0, \\ \lambda_B - \lambda_2 &\leq 100, \\ -\lambda_{NY} + \lambda_3 &\geq 0, \\ \lambda_B - \lambda_3 &\leq 80. \end{aligned}$$

2. The optimality conditions are

   - $c_{ij} + \lambda_i - \lambda_j \geq 0, \quad \forall (i, j) \in \mathcal{A}.$
   - $c_{ij} + \lambda_i - \lambda_j = 0, \quad \forall (i, j) \in \mathcal{A}$ such that $x_{ij} > 0.$

   We are giving tentative values for λ in such a way that they are consistent with the optimality conditions.

   We initialize

   $$\lambda_1 = 0.$$

   and

   $$\mathcal{S} = \{1\}.$$

Treat arcs leaving node 1 Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = ?,\ \lambda_3 = ?,\ \lambda_{LA} = ?,\ \lambda_{NY} = ?,\ \lambda_B = ?.$$

- As λ_{LA} is not defined yet, we define it as $\lambda_{LA} = \lambda_1 + c_{1,LA} = 0 + 0 = 0$.
- As λ_{NY} is not defined yet, we define it as $\lambda_{NY} = \lambda_1 + c_{1,NY} = 0 + 50 = 50$.

We update the set

$$\mathcal{S} = \mathcal{S} \cup \{LA, NY\} \setminus \{1\} = \{LA, NY\}.$$

Treat arcs entering node NY Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = ?,\ \lambda_3 = ?,\ \lambda_{LA} = 0,\ \lambda_{NY} = 50,\ \lambda_B = ?.$$

- As $\lambda_1 \geq \lambda_{NY} - c_{1,NY} = 50 - 50 = 0$, the optimality condition is verified, and nothing needs to be done.
- As λ_3 is not defined yet, we define at as $\lambda_3 = \lambda_{NY} - c_{3,NY} = 50 - 0 = 50$.

We update the set

$$\mathcal{S} = \mathcal{S} \cup \{3\} \setminus \{NY\} = \{LA, 3\}.$$

Treat arcs leaving node 3 Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = ?,\ \lambda_3 = 50,\ \lambda_{LA} = 0,\ \lambda_{NY} = 50,\ \lambda_B = ?.$$

- As $\lambda_{NY} \leq \lambda_3 + c_{3,NY} = 50 + 0 = 50$, the optimality condition is verified, and nothing needs to be done.
- As λ_B is not defined yet, we define it as $\lambda_B = \lambda_3 + c_{3,B} = 50 + 80 = 130$.

We update the set

$$\mathcal{S} = \mathcal{S} \cup \{B\} \setminus \{3\} = \{LA, B\}.$$

Treat arcs entering node B Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = ?,\ \lambda_3 = 50,\ \lambda_{LA} = 0,\ \lambda_{NY} = 50,\ \lambda_B = 130.$$

- As $\lambda_3 \geq \lambda_B - c_{3,B} = 130 - 80 = 50$, the optimality condition is verified, and nothing needs to be done.
- As λ_2 is not defined yet, we define it as $\lambda_2 = \lambda_B - c_{2,B} = 130 - 100 = 30$.

We update the set

$$\mathcal{S} = \mathcal{S} \cup \{2\} \setminus \{B\} = \{LA, 2\}.$$

Treat arcs leaving node 2 Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = 30,\ \lambda_3 = 50,\ \lambda_{LA} = 0,\ \lambda_{NY} = 50,\ \lambda_B = 130.$$

- As $\lambda_B \leq \lambda_2 + c_{2,B} = 30 + 100 = 130$, the optimality condition is verified, and nothing needs to be done.
- As $\lambda_{LA} \leq \lambda_2 + c_{2,LA} = 30 + 0 = 30$, the optimality condition is verified, and nothing needs to be done.

We update the set

$$\mathcal{S} = \mathcal{S} \setminus \{2\} = \{LA\}.$$

Treat arcs entering node LA Current value of the dual variables:

$$\lambda_1 = 0,\ \lambda_2 = 30,\ \lambda_3 = 50,\ \lambda_{LA} = 0,\ \lambda_{NY} = 50,\ \lambda_B = 130.$$

- As $\lambda_2 \geq \lambda_{LA} - c_{2,LA} = 0 - 0 = 0$, the optimality condition is verified, and nothing needs to be done.
- As $\lambda_1 \geq \lambda_{LA} - c_{1,LA} = 0 - 0 = 0$, the optimality condition is verified, and nothing needs to be done.

The optimal values of the dual variables are

$$\begin{aligned}
\lambda_1^* &= 0, \\
\lambda_2^* &= 30, \\
\lambda_3^* &= 50, \\
\lambda_{LA}^* &= 0, \\
\lambda_{NY}^* &= 50, \\
\lambda_B^* &= 130.
\end{aligned}$$

The optimality conditions are summarized in the following table:

Arc	$c_{ij} + \lambda_i - \lambda_j$
$(1, \mathrm{LA})$	$0 + 0 - 0 = 0,$
$(1, \mathrm{NY})$	$50 + 0 - 50 = 0,$
$(2, \mathrm{LA})$	$0 + 30 - 0 = 30,$
$(2, \mathrm{B})$	$100 + 30 - 130 = 0,$
$(3, \mathrm{B})$	$80 + 50 - 130 = 0,$
$(3, \mathrm{NY})$	$0 + 50 - 50 = 0.$

We note that the arc $(2, \mathrm{LA})$ is the only one where the inequality is strict. Therefore, we now that this arc carries no flow:

$$x^*_{2,\mathrm{LA}} = 0.$$

Therefore, in order to verify the flow conservation constraints at node LA, the demand must be served by the arc $(1, \mathrm{LA})$, so that

$$x^*_{1,\mathrm{LA}} = 30.$$

At node 1, the flow conservation constraints are verified if

$$x^*_{1,\mathrm{NY}} = 10.$$

Applying the flow conservation constraints are node NY, we obtain

$$x^*_{3,\mathrm{NY}} = 30.$$

At node 3, we obtain

$$x^*_{3,\mathrm{B}} = 10.$$

At node B, we obtain

$$x^*_{2,\mathrm{B}} = 40.$$

And we can check that the flow conservation constraints are verified at node 2. If we now ignore the slack variables, and come back to the original formulation, this can be interpreted as:

- sending 10 units of flow from LA to NY, for a total cost of 500,
- sending 40 units of flow from LA to B, for a total cost of 4000,
- sending 10 units of flow from NY to B, for a total cost of 800.

The total cost is therefore 5300. Note that the objective function of the dual is

$$\begin{aligned} q(\lambda) &= 30\lambda_{LA} + 40\lambda_{NY} + 50\lambda_{B} - 40\lambda_1 - 40\lambda_2 - 40\lambda_3, \\ &= 0 + 2000 + 6500 + 0 - 1200 - 2000 \\ &= 5300, \end{aligned}$$

consistently with the strong duality theorem.

Question 04

Which of the following matrices are totally unimodular? Explain.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remember that a matrix is totally unimodular if the determinant of each square submatrix is 0, 1 or -1.

Solution 04

A matrix is totally unimodular if the determinant of each square submatrix is 0, 1 or -1. We check whether the matrices A, B, C and D satisfy those conditions.

- Matrix A. There are two entries of the matrix different from 0, 1 or -1. Those are submatrices of size 1 for which the determinant is not 0, 1 or -1. Therefore, the matrix A is not totally unimodular.

- Matrix B. Each element of the matrix, that is each square submatrix of size 1, is 0, 1 or -1. Consider now the square submatrices of size 2:

$$\det\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -1, \quad \det\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix} = 0, \quad \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1.$$

  As there are only two rows, there are no square submatrices of size 3 or above. As all the determinants are 0, 1 or -1, B is totally unimodular.

- Matrix C. Each element of the matrix, that is, each square submatrix of size 1, is 0, 1 or -1. If we consider the submatrix containing the first two columns of C, we have

$$\det\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = 2.$$

  Consequently, C is not totally unimodular.

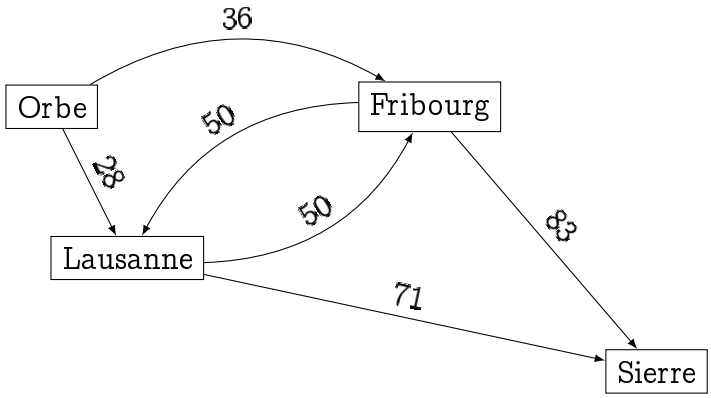
- Matrix D. Each element of the matrix, that is, each square submatrix of size 1, is 0, 1 or -1. We enumerate the submatrices of size two and calculate their determinant:

$$\det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = 1, \quad \det\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = 0, \quad \det\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 0,$$
$$\det\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = 1, \quad \det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = 1, \quad \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1,$$
$$\det\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = 0.$$

They are all 0, 1 or -1. Finally, we consider the unique submatrix of size 3, which is D itself. As $\det(D) = 1$, D is totally unimodular.

Question 05

Consider the four cities represented in the figure below. The travel time (in minutes) by car is shown next to each arc connecting two cities.

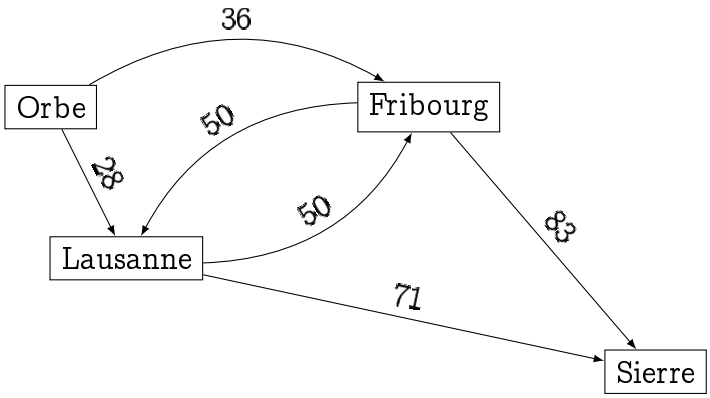


We want to determine the fastest itinerary from Orbe to Sierre.

1. Model the problem as a transhipment problem.
2. Optional: solve it using the simplex algorithm in two phases.

Solution 05

1. We first model the problem as a transhipment problem.



First of all, we denote Orbe (O), Lausanne (L), Fribourg (F) and Sierre (S). A transhipment problem is written

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}.$$

subject to the flow conservation constraints,

$$\mathrm{div}(x)_i = s_i, \ \forall i \in \mathcal{N},$$

and the bounds on the flows:

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \ \forall (i,j) \in \mathcal{A}.$$

In our case,

- $\mathcal{N} = \{O, L, F, S\}$ is the set of nodes,
- $\mathcal{A} = \{(O,L), (O,F), (F,S), (F,L), (L,F), (L,S)\}$ is the set of arcs.
- Costs: $c_{ij} \quad \forall (i,j) \in \mathcal{A}$ is the travel time associated with each arc.
- Supply: as we calculate the shortest path from Orbe to Sierre, we send one unit of flow from Orbe, and consume one unit of flow in Sierre. Therefore,

$$s_O = 1, s_L = 0, s_F = 0, s_S = -1.$$

- Bounds: we impose the flows to be non negative, so that no arc is used in the reverse direction. We do not impose any upper bound here.

The transhipment problem is

$$\min_{x \in \mathbb{R}^6} 28x_{O,L} + 36x_{O,F} + 50x_{L,F} + 50x_{F,L} + 83x_{F,S} + 71x_{L,S}$$

subject to

$$\begin{aligned} x_{O,F} + x_{O,L} &= 1, \\ -x_{F,S} - x_{L,S} &= -1, \\ x_{F,L} + x_{F,S} - x_{O,F} - x_{L,F} &= 0, \\ x_{L,S} + x_{L,F} - x_{F,L} - x_{O,L} &= 0, \\ x_{i,j} &\geq 0, \ \forall (i,j) \in \mathcal{A}. \end{aligned}$$

2. In order to solve it with the simplex algorithm, we first transform it so that the right hand side is non negative:

$$\min_{x \in \mathbb{R}^6} 28x_{O,L} + 36x_{O,F} + 50x_{L,F} + 50x_{F,L} + 83x_{F,S} + 71x_{L,S}$$

subject to

$$\begin{aligned} x_{O,F} + x_{O,L} &= 1, \\ x_{F,S} + x_{L,S} &= 1, \\ x_{F,L} + x_{F,S} - x_{O,F} - x_{L,F} &= 0, \\ x_{L,S} + x_{L,F} - x_{F,L} - x_{O,L} &= 0, \\ x_{i,j} &\geq 0, \ \forall (i,j) \in \mathcal{A}. \end{aligned}$$

Phase I: we introduce 4 auxiliary variables x_1, x_2, x_3, x_4 to obtain the auxiliary problem

$$\min_{x \in \mathbb{R}^{10}} x_1 + x_2 + x_3 + x_4$$

subject to

$$\begin{aligned} x_{O,F} + x_{O,L} + x_1 &= 1, \\ x_{F,S} + x_{L,S} + x_2 &= 1, \\ x_{F,L} + x_{F,S} - x_{O,F} - x_{L,F} + x_3 &= 0, \\ x_{L,S} + x_{L,F} - x_{F,L} - x_{O,L} + x_4 &= 0, \\ x_{i,j} &\geq 0, \ \forall (i,j) \in \mathcal{A}, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

The first tableau is:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	x_1	x_2	x_3	x_4	
1	1	0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	1	0	0	1
-1	0	1	1	-1	0	0	0	1	0	0
0	-1	-1	0	1	1	0	0	0	1	0
0	0	0	-2	0	-2	0	0	0	0	-2

The smallest index corresponding to a negative reduced cost is $p = 4$: $x_{F,S}$ enters the basis. The index corresponding to the smallest distance to the constraint $x_{F,S} \geq 0$ is $q = 3$: x_3 leaves the basis. The pivot is $(3, 4)$.

After pivoting the tableau, we obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	x_1	x_2	x_3	x_4	
1	1	0	0	0	0	1	0	0	0	1
1	0	-1	0	1	1	0	1	-1	0	1
-1	0	1	1	-1	0	0	0	1	0	0
0	-1	-1	0	1	1	0	0	0	1	0
-2	0	2	0	-2	-2	0	0	2	0	-2

The smallest index corresponding to a negative reduced cost is $p = 1$: $x_{O,F}$ enters the basis. The indices corresponding to the smallest distance to the constraint $x_{O,F} \geq 0$ are 1 and 2. We choose $q = 1$: x_1 leaves the basis. The pivot is $(1, 1)$.

After pivoting the tableau, we obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	x_1	x_2	x_3	x_4	
1	1	0	0	0	0	1	0	0	0	1
0	-1	-1	0	1	1	-1	1	-1	0	0
0	1	1	1	-1	0	1	0	1	0	1
0	-1	-1	0	1	1	0	0	0	1	0
0	2	2	0	-2	-2	2	0	2	0	0

The smallest index corresponding to a negative reduced cost is $p = 5$: $x_{L,F}$ enters the basis. The indices corresponding to the smallest distance to the constraint $x_{L,F} \geq 0$ are 2 and 4. We choose $q = 2$: x_2 leaves the basis. The pivot is $(2, 5)$.

After pivoting the tableau, we obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	x_1	x_2	x_3	x_4	
1	1	0	0	0	0	1	0	0	0	1
0	-1	-1	0	1	1	-1	1	-1	0	0
0	0	0	1	0	1	0	1	0	0	1
0	0	0	0	0	0	1	-1	1	1	0
0	0	0	0	0	0	0	2	0	0	0

The last tableau is optimal since all reduced costs are non negative and the optimal value is zero. Therefore, we have identified a feasible solution for the initial problem: $x_{O,F} = 1$, $x_{O,L} = 0$, $x_{F,L} = 0$, $x_{F,S} = 1$, $x_{L,F} = 0$, $x_{L,S} = 0$, which corresponds to the path Orbe, Fribourg, Sierre.

We notice that the auxiliary variable x_4 is basic. But we cannot pivot it out of the basis, since all pivots corresponding to non basic variables from the original problem are zero. Indeed, it corresponds to a redundant constraint.

We remove the corresponding row and column to obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	x_1	x_2	x_3	
1	1	0	0	0	0	1	0	0	1
0	-1	-1	0	1	1	-1	1	-1	0
0	0	0	1	0	1	0	1	0	1
0	0	0	0	0	0	0	2	0	0

There are no auxiliary variables left in the basis anymore, and thus we remove the corresponding columns from the tableau.

Furthermore, to begin Phase II of the algorithm, we recalculate the reduced costs and the optimal value to obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	
1	1	0	0	0	0	1
0	-1	-1	0	1	1	0
0	0	0	1	0	1	1
0	42	100	0	0	-62	-119

The smallest index corresponding to a negative reduced cost is $p = 6$: $x_{L,S}$ enters the basis. The index corresponding to the smallest distance to the constraint $x_{L,S} \geq 0$ is $q = 2$: $x_{L,F}$ leaves the basis. The pivot is $(2, 6)$.

After pivoting the tableau we obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	
1	1	0	0	0	0	1
0	-1	-1	0	1	1	0
0	1	1	1	-1	0	1
0	-20	38	0	62	0	-119

The smallest index corresponding to a negative reduced cost is $p = 2$: $x_{O,L}$ enters the basis. The indices corresponding to the smallest distance to the constraint $x_{O,L} \geq 0$ are 1 and 3. We choose $q = 1$: $x_{O,F}$ leaves the basis. The pivot is $(1, 2)$.

After pivoting the tableau we obtain:

$x_{O,F}$	$x_{O,L}$	$x_{F,L}$	$x_{F,S}$	$x_{L,F}$	$x_{L,S}$	
1	1	0	0	0	0	1
1	0	-1	0	1	1	1
-1	0	1	1	-1	0	0
20	0	38	0	62	0	-99

The last tableau is optimal since all reduced costs are non negative. Thus the optimal solution is:

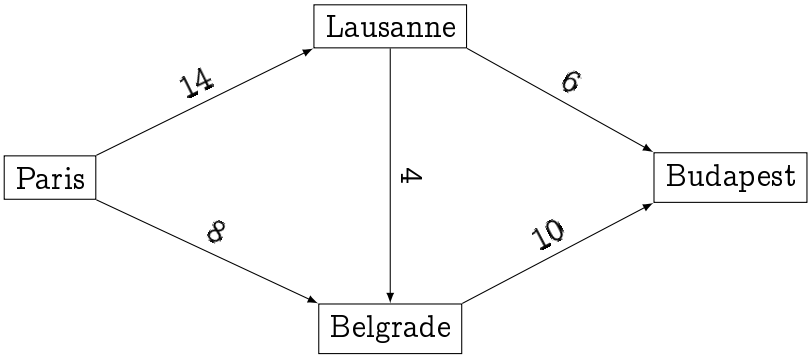
$$\begin{aligned} x_{O,F} &= 0, \\ x_{O,L} &= 1, \\ x_{F,L} &= 0, \\ x_{F,S} &= 0, \\ x_{L,F} &= 0, \\ x_{L,S} &= 1, \end{aligned}$$

and corresponds to the path Orbe, Lausanne, Sierre. Note that the solution is integer, although it has not been explicitly imposed. The optimal travel time is 99.

Question 06

Consider the following network representing a railway network. Each arc is associated with a number representing the maximum number of trains per day that can travel between two cities. What is the maximal number of trains that can be transferred from *Paris* to *Budapest* in one day?

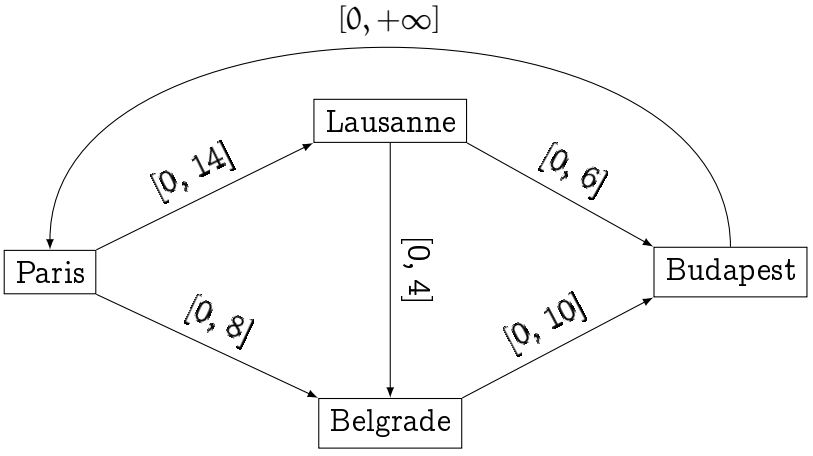
Write the mathematical formulation of the maximum flow problem, as a transhipment problem.



Solution 06

The maximum flow problem has no cost associated with the arcs. To model the maximum flow problem as a transhipment problem, we need to include a dummy arc from Budapest to Paris, associated with a negative cost (-1). We also impose a circulation, which means that we impose that each node is a transit node, with a divergence of zero. Therefore, each unit of flow reaching Budapest is sent back to Paris through the dummy arc. As the real arcs have zero costs, by assigning a cost of -1 to the dummy arc, the opposite of the total cost represents the number of units of flow that are sent back from Budapest to Paris through the dummy arc. Minimizing the total cost is therefore equivalent to maximizing the total flow through the network.

The lower capacity of each real arc is zero, so that the arcs cannot be used in the reverse direction. The upper capacity corresponds to the maximum number of train for one day between the corresponding cities. The artificial arc is also bounded below by zero, but is not bounded above. We use the notation $[\ell_{ij}, u_{ij}]$ to denote the capacity bounds on the network.



A transhipment problem is written

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to the flow conservation constraints,

$$\mathrm{div}(x)_i = s_i \; \forall i \in \mathcal{N},$$

and the bounds on the flows:

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \; \forall (i,j) \in \mathcal{A}.$$

In our case,

- $\mathcal{N}$ is the set of nodes: *Budapest* (Bu), *Paris* (P), *Lausanne* (L) and *Belgrade* (Be).
- $\mathcal{A}^+ = \mathcal{A} \cup \{(\mathrm{Bu}, \mathrm{P})\}$ is the set combining the original arcs and the dummy arc.
- Costs: $c_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$ and $c_{\mathrm{Bu,P}} = -1$.
- Supply: as we impose a circulation, the divergence at each node must be zero: $s_i = 0$, $\forall i \in \mathcal{N}$.
- Bounds: $0 \leq x_{ij} \leq u_{ij}$, $\forall (i,j) \in \mathcal{A}$, and $x_{\mathrm{Bu,P}} \geq 0$.

The transhipment problem is

$$\min_{x \in \mathbb{R}^6} -x_{\mathrm{Bu,P}}.$$

subject to

$$\sum_{j|(i,j)\in\mathcal{A}^+} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}^+} x_{ki} = 0, \quad \forall i \in \mathcal{N},$$
$$x_{ij} \leq u_{ij}, \quad \forall (i,j) \in \mathcal{A},$$
$$x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}^+.$$

In our case, it is

$$\min_{x \in \mathbb{R}^6} -x_{\mathrm{Bu,P}}$$

subject to

$$\begin{aligned}
x_{P,L} + x_{P,Be} - x_{Bu,P} &= 0,\\
-x_{P,L} + x_{L,Be} + x_{L,Bu} &= 0,\\
-x_{P,Be} - x_{L,Be} + x_{Be,Bu} &= 0,\\
-x_{L,Bu} - x_{Be,Bu} + x_{Bu,P} &= 0,\\
x_{P,L} &\leq 14,\\
x_{P,Be} &\leq 8,\\
x_{L,Be} &\leq 4,\\
x_{L,Bu} &\leq 6,\\
x_{Be,Bu} &\leq 10,\\
x_{P,L}, x_{P,Be}, x_{L,Be}, x_{L,Bu}, x_{Be,Bu}, x_{Bu,P} &\geq 0.
\end{aligned}$$

The problem can be solved by the simplex algorithm. We provide here the iterations for reference. We start by writing it in standard form by introducing the slack variables x^s.

$$\min_{x \in \mathbb{R}^{11}} -x_{Bu,P}.$$

subject to

$$\begin{array}{cccccccc}
x_{P,L} & & & & & & +x^s_{P,L} & = 14,\\
& x_{P,Be} & & & & & +x^s_{P,Be} & = 8,\\
& & x_{L,Be} & & & & +x^s_{L,Be} & = 4,\\
& & & x_{L,Bu} & & & +x^s_{L,Bu} & = 6,\\
& & & & x_{Be,Bu} & & +x^s_{Be,Bu} & = 10,\\
x_{P,L} & +x_{P,Be} & & & & -x_{Bu,P} & & = 0,\\
-x_{P,L} & & +x_{L,Be} & +x_{L,Bu} & & & & = 0,\\
& -x_{P,Be} & -x_{L,Be} & & +x_{Be,Bu} & & & = 0,\\
& & & -x_{L,Bu} & -x_{Be,Bu} & +x_{Bu,P} & & = 0,\\
x_{P,L}, & x_{P,Be}, & x_{L,Be}, & x_{L,Bu}, & x_{Be,Bu}, & x_{Bu,P} & & \geq 0,\\
x^s_{P,L}, & x^s_{P,Be}, & x^s_{L,Be}, & x^s_{L,Bu}, & x^s_{Be,Bu} & & & \geq 0.
\end{array}$$

It corresponds to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 14 \\ 8 \\ 4 \\ 6 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector of costs is:

$$c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

An auxiliary variable y_i is associated with each of the equality constraints in order to obtain the auxiliary problem of Phase I. The iterations of Phase I are reported below.

Iteration 0

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	14
0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
1	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
-1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	-1	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
-1	-1	-1	-1	-1	-0	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	-42

Iteration 1

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	-1	0	0	0	1	1	0	0	0	0	1	0	0	0	0	-1	0	0	0	14
0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
1	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	1	1	1	0	-1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	-1	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	1	0	0	0	-42

Iteration 2

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	-1	0	0	0	1	1	0	0	0	0	1	0	0	0	0	-1	0	0	0	14
0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
0	-1	0	-1	0	1	0	0	1	0	0	0	0	1	0	0	-1	-1	0	0	4
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
1	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	1	1	1	0	-1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	1	1	-1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0
0	0	0	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	1	0	0	-1	-2	-1	-1	-1	-1	-1	0	0	0	0	0	2	1	0	0	-42

Iteration 3

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	-1	0	0	0	1	1	0	0	0	0	1	0	0	0	0	-1	0	0	0	14
0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
0	-1	0	-1	0	1	0	0	1	0	0	0	0	1	0	0	-1	-1	0	0	4
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	0	0	-1	0	1	0	0	0	0	1	0	0	0	0	1	-1	-1	-1	0	10
1	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	1	1	1	0	-1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	1	1	-1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	1	0	1	0	-3	-1	-1	-1	-1	-1	0	0	0	0	0	3	2	1	0	-42

Iteration 4

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	0	0	1	0	0	1	0	-1	0	0	1	0	-1	0	0	0	1	0	0	10
0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
0	-1	0	-1	0	1	0	0	1	0	0	0	0	1	0	0	-1	-1	0	0	4
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	1	0	0	0	0	0	0	-1	0	1	0	0	-1	0	1	0	0	-1	0	6
1	0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0	0	-1	0	0	4
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	-1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	-2	0	-2	0	0	-1	-1	2	-1	-1	0	0	3	0	0	0	-1	1	0	-30

Iteration 5

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	0	0	1	0	0	1	0	-1	0	0	1	0	-1	0	0	0	1	0	0	10
0	0	0	0	0	0	0	1	1	0	-1	0	1	1	0	-1	0	0	1	0	2
0	0	0	-1	0	1	0	0	0	0	1	0	0	0	0	1	-1	-1	-1	0	10
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	1	0	0	0	0	0	0	-1	0	1	0	0	-1	0	1	0	0	-1	0	6
1	0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0	0	-1	0	0	4
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	0	0	-2	0	0	-1	-1	0	-1	1	0	0	1	0	2	0	-1	-1	0	-18

Iteration 6

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	0	0	0	0	0	1	0	-1	-1	0	1	0	-1	-1	0	0	1	0	0	4
0	0	0	0	0	0	0	1	1	0	-1	0	1	1	0	-1	0	0	1	0	2
0	0	0	0	0	1	0	0	0	1	1	0	0	0	1	1	-1	-1	-1	0	16
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	1	0	0	0	0	0	0	-1	0	1	0	0	-1	0	1	0	0	-1	0	6
1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	-1	0	0	10
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	0	0	0	0	0	-1	-1	0	1	1	0	0	1	2	2	0	-1	-1	0	-6

Iteration 7

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	0	0	0	0	0	1	0	-1	-1	0	1	0	-1	-1	0	0	1	0	0	4
0	0	0	0	0	0	0	1	1	0	-1	0	1	1	0	-1	0	0	1	0	2
0	0	0	0	0	1	0	0	0	1	1	0	0	0	1	1	-1	-1	-1	0	16
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	1	0	0	0	0	0	0	-1	0	1	0	0	-1	0	1	0	0	-1	0	6
1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	-1	0	0	10
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	0	0	0	0	0	0	-1	-1	0	1	1	0	0	1	2	0	0	-1	0	-2

Iteration 8

$x_{P,L}$	$x_{P,Be}$	$x_{L,Be}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
0	0	0	0	0	0	1	0	-1	-1	0	1	0	-1	-1	0	0	1	0	0	4
0	0	0	0	0	0	0	1	1	0	-1	0	1	1	0	-1	0	0	1	0	2
0	0	0	0	0	1	0	0	0	1	1	0	0	0	1	1	-1	-1	-1	0	16
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	6
0	1	0	0	0	0	0	0	-1	0	1	0	0	-1	0	1	0	0	-1	0	6
1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	-1	0	0	10
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	4
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0

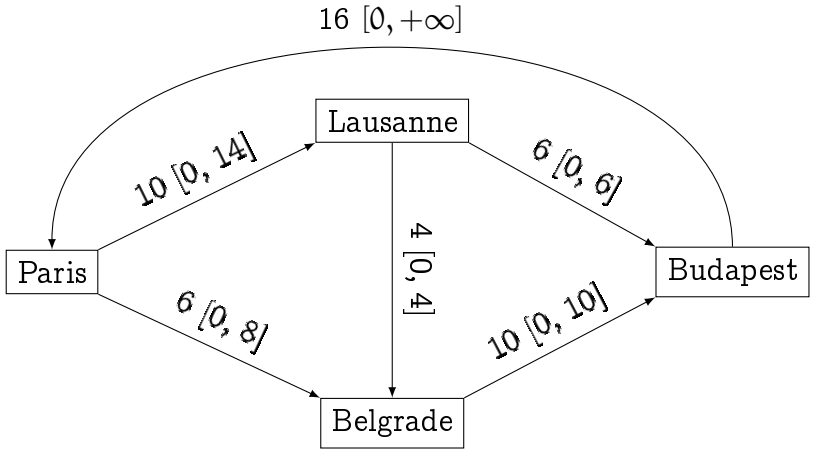
The optimal value is 0. Therefore, we have found a feasible solution of the original problem. But the auxiliary variable y_9 is still in the basis. It is not possible to pivot it out, as all potential pivots are zero. Actually, the transhipment problem as we have formulated it, has always one redundant constraint. As a result, the auxiliary variable stays in the basis. The corresponding constraint can be removed. The columns of the auxiliary variables are then removed, and the reduced costs calculated, to obtain the first tableau for Phase II:

$x_{P,L}$	$x_{P,Be}$	$x_{L,Bu}$	$x_{L,Bu}$	$x_{Be,Bu}$	$x_{Bu,P}$	$x^s_{P,L}$	$x^s_{P,Be}$	$x^s_{L,Be}$	$x^s_{L,Bu}$	$x^s_{Be,Bu}$	
0	0	0	0	0	0	1	0	-1	-1	0	4
0	0	0	0	0	0	0	1	1	0	-1	2
0	0	0	0	0	1	0	0	0	1	1	16
0	0	0	1	0	0	0	0	0	1	0	6
0	1	0	0	0	0	0	0	-1	0	1	6
1	0	0	0	0	0	0	0	1	1	0	10
0	0	1	0	0	0	0	0	1	0	0	4
0	0	0	0	1	0	0	0	0	0	1	10
0	0	0	0	0	0	0	0	0	1	1	16

It is actually optimal, as all reduced costs are non negative. We end up with the optimal solution:

$$\begin{aligned} x_{P,L} &= 10 \\ x_{P,Be} &= 6 \\ x_{L,Bu} &= 6 \\ x_{L,Be} &= 4 \\ x_{Be,Bu} &= 10. \end{aligned}$$

In the following picture, the optimal flow is represented on each arc, next to the bounds.

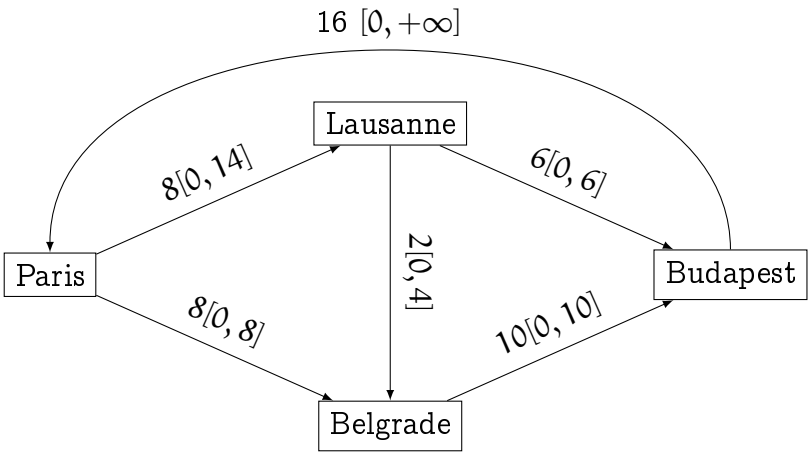


Note that all the values are integers, although not explicitly required. The optimal value is -16, which corresponds to the total flow from Paris to Budapest.

Note also that it's not the only optimal solution. Indeed, in the last tableau, the non basic variable $x^s_{L,Be}$ has a reduced cost equal to 0. Therefore, we can move to another optimal tableau by introducing $x^s_{L,Be}$ in the basis instead of $x^s_{P,Be}$. We obtain:

$$\begin{aligned} x_{P,L} &= 8 \\ x_{P,Be} &= 8 \\ x_{L,Bu} &= 6 \\ x_{L,Be} &= 2 \\ x_{Be,Bu} &= 10. \end{aligned}$$

It is represented by the following picture.



Question 07

There are about 45000 Tesla Model S produced per year, that need batteries. 30000 come from the Gigafactory factory in the Nevada, and 15000 from the Gigafactory in Shanghai. Each battery requires 63kg of lithium. Consequently, the company purchases 2835 tons of lithium per year: 1927 tons from mines in Australia, 605 tons from Chili and 303 tons from China.

The transportation costs from the mining sites to the factories are reported in the following table:

Lithium (transp. cost in $ per ton)

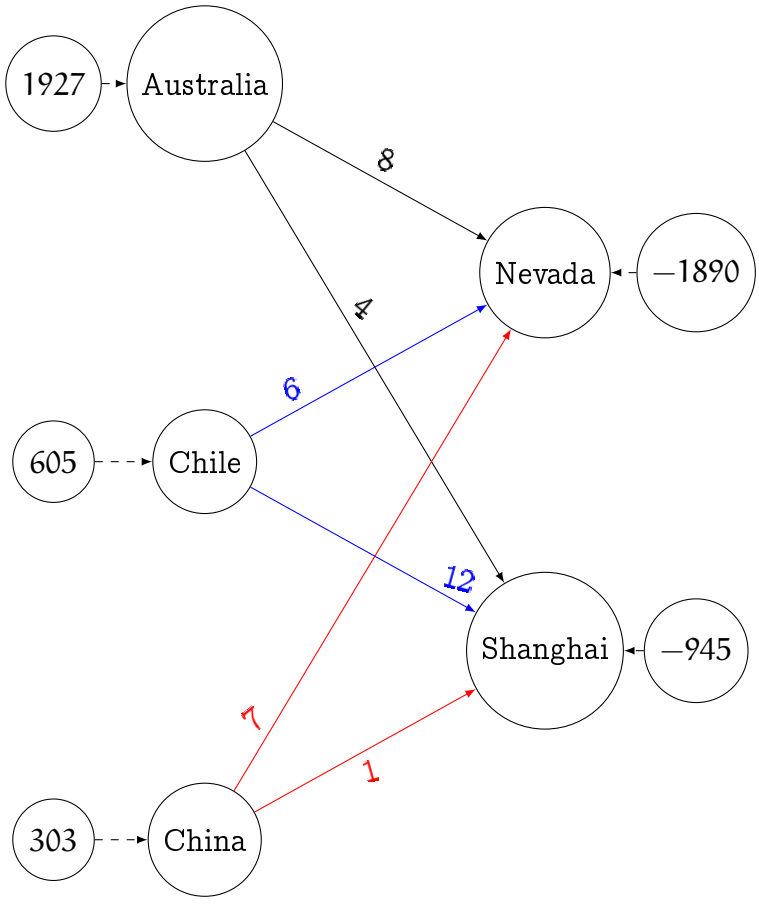
	Mines		
Factories	Australia	Chile	China
Nevada	8	6	7
Shanghai	4	12	1

How should the transportation be organized to supply the factories at minimum cost? Model the problem as a transhipment problem, and draw the corresponding network. It is not asked to solve the optimization problem.

Solution 07

The problem is not associated with a physical network. Still, we define a network in the following way. We define a node for each factory, represented on the right side of the network below, and a node for each mine, represented on the left side.

We introduce an arc between each pair of mine and factory. For example, the arc from Australia to Nevada means that the Australian mine is providing lithium to the Nevada factory. The cost associated with each arc is the transportation price per ton.



A transhipment problem is written

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}.$$

subject to the flow conservation constraints,

$$\mathrm{div}(x)_i = s_i \; \forall i \in \mathcal{N},$$

and the bounds on the flows:

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \; \forall (i,j) \in \mathcal{A}.$$

In our case,

- $\mathcal{N}$ is the set of nodes: Australia (A), Chile (L), China (C), Nevada (N), Shanghai (S).
- $\mathcal{A}$ is the set of possible transactions, that is any pair mine, factory.
- Costs: c_{ij}, $\forall (i,j) \in \mathcal{A}$ corresponding to transportation prices per ton, as reported in the problem description.
- Supply: For each mine, the supply in tons is provided in the problem description: 1927 for Australia, 605 for Chile and 303 for China. The demand in lithium for Nevada is $30000 \cdot 0.063 = 1890$ tons. For Shanghai, it is $15000 \cdot 0.063 = 945$ tons of lithium. For each factory, the supply is the demand with the opposite sign, that is -1890 for Nevada, -945 for Shanghai.
- Bounds: As the transactions can only occur in one direction, we impose $x_{ij} \geq 0$, $\forall (i,j) \in \mathcal{A}$. No upper bound is necessary.

The transhipment problem is

$$\min_{x \in \mathbb{R}^6} 8x_{AN} + 4x_{AS} + 6x_{LN} + 12x_{LS} + 7x_{CN} + x_{CS},$$

subject to

$$\begin{aligned}
x_{AN} + x_{AS} &= 1927, \\
x_{LN} + x_{LS} &= 605, \\
x_{CN} + x_{CS} &= 303, \\
-x_{AN} - x_{LN} - x_{CN} &= -1890, \\
-x_{AS} - x_{LS} - x_{CS} &= -945, \\
x_{ij} &\geq 0, \qquad \forall (i,j) \in \mathcal{A}.
\end{aligned}$$

It was not asked to solve the problem. But, for the sake of illustration, we solve it using the simplex algorithm. First, we need to modify it such that the right-hand side contains only non negative values by multiplying the corresponding constraints by -1. If we do so, and number the variables in the following way:

$$\begin{aligned} x_1 &= x_{AN}, & x_2 &= x_{AS}, \\ x_3 &= x_{LN}, & x_4 &= x_{LS}, \\ x_5 &= x_{CN}, & x_6 &= x_{CS}, \end{aligned}$$

we obtain the following data:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1927 \\ 605 \\ 303 \\ 1890 \\ 945 \end{pmatrix}$$

and

$$c^T = \begin{pmatrix} 8 & 4 & 6 & 12 & 7 & 1 \end{pmatrix}.$$

An auxiliary variable is associated with each of the equality constraints $(x_7, \ldots, x_{11})$ and the Phase I can be started.

Iteration 0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
1	1	0	0	0	0	1	0	0	0	0	1927
0	0	1	1	0	0	0	1	0	0	0	605
0	0	0	0	1	1	0	0	1	0	0	303
1	0	1	0	1	0	0	0	0	1	0	1890
0	1	0	1	0	1	0	0	0	0	1	945
-2	-2	-2	-2	-2	-2	0	0	0	0	0	-5670

Iteration 1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
0	1	-1	0	-1	0	1	0	0	-1	0	37
0	0	1	1	0	0	0	1	0	0	0	605
0	0	0	0	1	1	0	0	1	0	0	303
1	0	1	0	1	0	0	0	0	1	0	1890
0	1	0	1	0	1	0	0	0	0	1	945
0	-2	0	-2	0	-2	0	0	0	2	0	-1890

Iteration 2

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
0	1	-1	0	-1	0	1	0	0	-1	0	37
0	0	1	1	0	0	0	1	0	0	0	605
0	0	0	0	1	1	0	0	1	0	0	303
1	0	1	0	1	0	0	0	0	1	0	1890
0	0	1	1	1	1	-1	0	0	1	1	908
0	0	-2	-2	-2	-2	2	0	0	0	0	-1816

Iteration 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
0	1	0	1	-1	0	1	1	0	-1	0	642
0	0	1	1	0	0	0	1	0	0	0	605
0	0	0	0	1	1	0	0	1	0	0	303
1	0	0	-1	1	0	0	-1	0	1	0	1285
0	0	0	0	1	1	-1	-1	0	1	1	303
0	0	0	0	-2	-2	2	2	0	0	0	-606

Iteration 4

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
0	1	0	1	0	1	1	1	1	-1	0	945
0	0	1	1	0	0	0	1	0	0	0	605
0	0	0	0	1	1	0	0	1	0	0	303
1	0	0	-1	0	-1	0	-1	-1	1	0	982
0	0	0	0	0	0	-1	-1	-1	1	1	0
0	0	0	0	0	0	2	2	2	0	0	0

The optimal value is zero, and we have found a feasible solution of the original problem. The auxiliary variable x_{11} is in the basis, and cannot be pivoted out. It corresponds to the redundant constraint, that can be removed. After cleaning and calculation of the reduced costs, we can apply Phase II:

Phase II: Iteration 0

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	1	0	1	945
0	0	1	1	0	0	605
0	0	0	0	1	1	303
1	0	0	-1	0	-1	982
0	0	0	10	0	-2	-17387

Phase II: Iteration 1

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	1	-1	0	642
0	0	1	1	0	0	605
0	0	0	0	1	1	303
1	0	0	-1	1	0	1285
0	0	0	10	2	0	-16781

The tableau is optimal. Therefore, the total cost is \$16781, and the optimal strategy for the transportation of lithium is summarized in the following table:

Lithium (tons)

	Mines		
Factories	Australia	Chile	China
Nevada	1285	605	0
Shanghai	642	0	303

Question 08

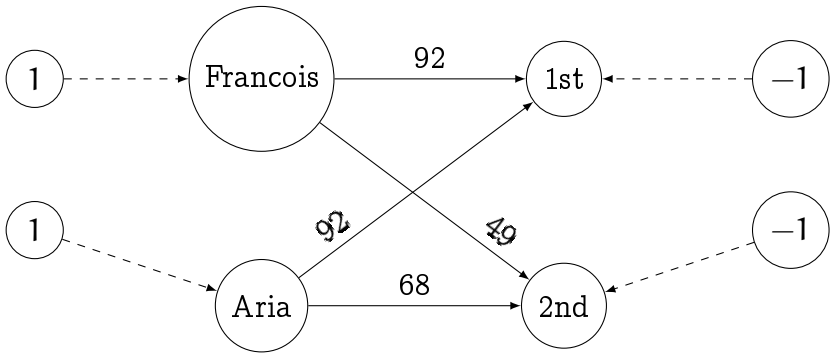
Francois, 24, and Aria, 26, are traveling from Lausanne to Paris. There are only two seats available in the TGV: one first class ticket at €92 and one second class ticket at €68, discounted to €49 if the traveler is 25 or younger. In order to minimize the total costs, it makes sense for Francois to take the second class ticket and for Aria the first class ticket.

1. Model the problem as a transhipment problem, and draw the corresponding network.
2. To which instance of the transhipment problem seen in class does this problem correspond?
3. Prove that the proposed solution is indeed optimal.

Solution 08

1. The problem is not associated with a physical network. Still, we define a network in the following way. We define a node for each ticket, represented on the right of the network below, and a node for each traveler, represented on the left.

   We introduce an arc between each pair of traveler and ticket. For example, the arc from Francois to "1st" means that Francois would buy the first class ticket. The cost associated with each arc is the price of the corresponding transaction.



   A transhipment problem is written

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

   subject to the flow conservation constraints,

$$\mathrm{div}(x)_i = s_i \; \forall i \in \mathcal{N},$$

   and the bounds on the flows:

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \; \forall (i,j) \in \mathcal{A}.$$

   In our case,

   - $\mathcal{N}$ is the set of nodes: Francois (F), Aria (A), 1st (1) and 2nd (2),

- $\mathcal{A}$ is the set of the four possible transactions.
- Costs: c_{ij}, $\forall (i,j) \in \mathcal{A}$ are the transaction costs as reported in the problem description.
- Supply: For each traveler, there is only one transaction. Therefore, the supply is 1. Similarly, each ticket will be be purchased by only one travel, so the demand is -1.
- Bounds: As the transactions can only occur in one direction, we impose $x_{ij} \geq 0$, $\forall (i,j) \in \mathcal{A}$. No upper bound is necessary.

The transhipment problem is

$$\min_{x \in \mathbb{R}^6} 92x_{F1} + 49x_{F2} + 92x_{A1} + 68x_{A2},$$

subject to

$$\begin{aligned} x_{F1} + x_{F2} &= 1, \\ x_{A1} + x_{A2} &= 1, \\ -x_{F1} - x_{A1} &= -1, \\ -x_{F2} - x_{A2} &= -1, \\ x_{ij} &\geq 0, \qquad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

2. This is an assignment problem.
3. Given the small size of the problem, it can trivially be solved by complete enumeration:
   - If Francois takes the second class ticket and Aria the first class ticket, it costs 49+92=€141.
   - If Francois takes the first class ticket and Aria the second class ticket, it costs 92+68=€160.

For the sake illustration, we also solve the problem using the simplex algorithm. If we number the variables in the following way:

$$\begin{aligned} x_1 &= x_{F1}, & x_2 &= x_{F2}, \\ x_3 &= x_{A1}, & x_4 &= x_{A2}, \end{aligned}$$

and multiply some constraints by -1 to obtain a right hand side that is non negative, we obtain the following data:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$c^T = \begin{pmatrix} 92 & 49 & 92 & 68 \end{pmatrix}.$$

An auxiliary variable is associated with each of the equality constraints $(x_5, \ldots, x_8)$ and the Phase I can be started.

Iteration 0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1	1	0	0	1	0	0	0	1
0	0	1	1	0	1	0	0	1
1	0	1	0	0	0	1	0	1
0	1	0	1	0	0	0	1	1
-2	-2	-2	-2	0	0	0	0	-4

Iteration 1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1	1	0	0	1	0	0	0	1
0	0	1	1	0	1	0	0	1
0	-1	1	0	-1	0	1	0	0
0	1	0	1	0	0	0	1	1
0	0	-2	-2	2	0	0	0	-2

Iteration 2

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1	1	0	0	1	0	0	0	1
0	1	0	1	1	1	-1	0	1
0	-1	1	0	-1	0	1	0	0
0	1	0	1	0	0	0	1	1
0	-2	0	-2	0	0	2	0	-2

Iteration 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1	1	0	0	1	0	0	0	1
-1	0	0	1	0	1	-1	0	0
1	0	1	0	0	0	1	0	1
-1	0	0	1	-1	0	0	1	0
2	0	0	-2	2	0	2	0	0

Iteration 4

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1	1	0	0	1	0	0	0	1
-1	0	0	1	0	1	-1	0	0
1	0	1	0	0	0	1	0	1
0	0	0	0	-1	-1	1	1	0
0	0	0	0	2	2	0	0	0

The optimal value is zero, and we have found a feasible solution of the original problem. The auxiliary x_8 is in the basis, and cannot be pivoted out. It corresponds to the redundant constraint, that can be removed. After cleaning and calculation of the reduced costs, we obtain the following tableau to start Phase II:

x_1	x_2	x_3	x_4	
1	1	0	0	1
-1	0	0	1	0
1	0	1	0	1
19	0	0	0	-141

It happens to be optimal and the optimal solution is

$$x_1 = x_{F1} = 0, \qquad x_2 = x_{F2} = 1,$$
$$x_3 = x_{A1} = 1, \qquad x_4 = x_{A2} = 0,$$

for a total cost of 141. This is corresponds to the solution where Francois takes the second class ticket, and Aria the first class. Note that the solution is integer, while this has not been explicitly required.

Question 09

After spending a weekend with friends, Aria has determined the amount of money that they owe each other, as represented in the Table below. Francois requests a simple solution that minimizes the sum of transfers.

Who	How much	To whom
Aria	6	Monia
Aria	16	Gabriel
Monia	0	-
Gabriel	8	Monia
Francois	10	Aria
Francois	16	Monia

1. Model the problem as a mathematical network.
2. Write the problem as a transhipment problem.
3. Write the problem as a transportation problem and solve it using the simplex.

Solution 09

1. First, we model the situation as a mathematical network $(\mathcal{N}, \mathcal{A})$. Each node corresponds to a person. We define an arc between all pairs of persons for which a transfer is possible. The flow on those arcs represents the transfers of money. They have lower bound zero and no upper bound. As the number of transactions must be optimized, we set the cost of each arc to 1, so that each transaction has cost 1.

   We then compute the supply vector. We want that each of the 4 friends receive and give the amount of money needed to set the accounts to zero. Thus we have:

   - Since Aria owes 6.- CHF to Monia and 16.- CHF to Gabriel, and she must receive 10.- CHF from Francois, then the divergence of the corresponding node is $6+16-10 = 12$, and Aria is associated with a supply node.
   - Similarly, the divergence for Monia is $-6-8-16 = -30$ and Monia is associated with a demand node.
   - The divergence for Gabriel is $-16+8 = -8$, and Gabriel is associated with a demand node.
   - The divergence for Francois is $10+16 = 26$, and Francois is associated with a supply node.

   We use the notation $c_{ij}[\ell_{ij}, u_{ij}]$ to specify the costs and capacity bounds. We obtain the following network:

2. We write the problem as a transhipment problem. The decision variables x_{ij} represent the amount of money payed by i to j. Since we want to minimize the sum of transfers, the objective function is

$$\min_{x \in \mathbb{R}^5} \sum_{(i,j) \in \mathcal{A}} x_{ij}.$$

   We then introduce the constraints defined by the supply and demand of each node.

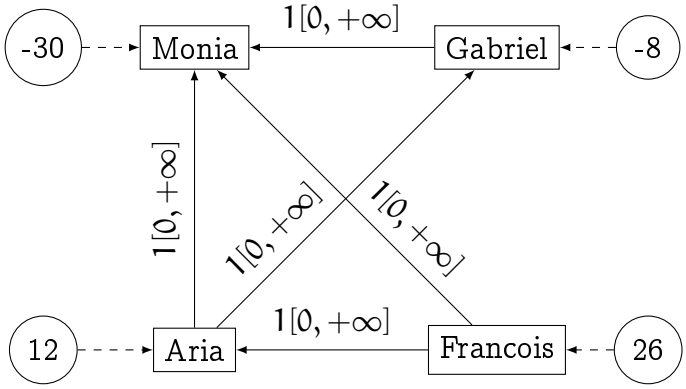


Figure 11: Network for the transhipment problem

$$\begin{aligned} -x_{\text{Aria,Monia}} - x_{\text{Gabriel,Monia}} - x_{\text{François,Monia}} &= -30, \\ x_{\text{François,Monia}} + x_{\text{François,Aria}} &= 26, \\ x_{\text{Aria,Monia}} + x_{\text{Aria,Gabriel}} - x_{\text{François,Aria}} &= 12, \\ -x_{\text{Aria,Gabriel}} + x_{\text{Gabriel,Monia}} &= -8. \end{aligned}$$

The flow on each arc must be greater than 0:

$$x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}.$$

The optimization problem is the following:

$$\min_{x \in \mathbb{R}^5} \sum_{(i,j) \in \mathcal{A}} x_{ij}$$

subject to

$$\begin{aligned} -x_{\text{Aria,Monia}} - x_{\text{Gabriel,Monia}} - x_{\text{François,Monia}} &= -30, \\ x_{\text{François,Monia}} + x_{\text{François,Aria}} &= 26, \\ x_{\text{Aria,Monia}} + x_{\text{Aria,Gabriel}} - x_{\text{François,Aria}} &= 12, \\ -x_{\text{Aria,Gabriel}} + x_{\text{Gabriel,Monia}} &= -8, \\ x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

We have:

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -30 \\ 26 \\ 12 \\ -8 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the sake of illustration, we apply the simplex algorithm to solve the problem. First, we need $b \geq 0$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 26 \\ 12 \\ 8 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We introduce 4 auxiliary variables x_1, x_2, x_3, x_4. The objective function for the auxiliary problem is

$$\min_{x \in \mathbb{R}^9} x_1 + x_2 + x_3 + x_4.$$

The first tableau is

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	x_1	x_2	x_3	x_4	
1	1	1	0	0	1	0	0	0	30
1	0	0	1	0	0	1	0	0	26
0	0	1	-1	1	0	0	1	0	12
0	-1	0	0	1	0	0	0	1	8
-2	0	-2	0	-2	0	0	0	0	-76

The smallest index corresponding to a negative reduced cost is $p = 1$: $x_{F,M}$ enters the basis. The index corresponding to the smallest distance to the constraint $x_{F,M} \geq 0$ is $q = 2$ and x_3 leaves the basis. The pivot is $(2, 1)$. After pivoting the tableau we obtain:

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	x_1	x_2	x_3	x_4	
0	1	1	-1	0	1	-1	0	0	4
1	0	0	1	0	0	1	0	0	26
0	0	1	-1	1	0	0	1	0	12
0	-1	0	0	1	0	0	0	1	8
0	0	-2	2	-2	0	2	0	0	-24

The smallest index corresponding to a negative reduced cost is $p = 3$: $x_{A,M}$ enters the basis. The index corresponding to the smallest distance to the constraint $x_{A,M} \geq 0$ is $q = 1$ and x_1 leaves the basis. The pivot is $(1, 3)$. After pivoting the tableau we obtain:

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	x_1	x_2	x_3	x_4	
0	1	1	-1	0	1	-1	0	0	4
1	0	0	1	0	0	1	0	0	26
0	-1	0	0	1	-1	1	1	0	8
0	-1	0	0	1	0	0	0	1	8
0	2	0	0	-2	2	0	0	0	-16

The smallest index corresponding to a negative reduced cost is $p = 5$: $x_{A,G}$ enters the basis. The smallest index corresponding to the smallest distance to the constraint $x_{A,G} \geq 0$ is $q = 3$ and x_3 leaves the basis. The pivot is $(3, 5)$. After pivoting the tableau we obtain:

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	x_1	x_2	x_3	x_4	
0	1	1	-1	0	1	-1	0	0	4
1	0	0	1	0	0	1	0	0	26
0	-1	0	0	1	-1	1	1	0	8
0	0	0	0	0	1	-1	-1	1	0
0	0	0	0	0	0	2	2	0	0

All reduced costs are non negative and the tableau is optimal. The auxiliary variable x_4 is basic: we want to exchange it with $x_{G,M}$. However, the pivot is null, which means that the constraint is redundant. We remove the corresponding row and the variable x_4 from the tableau to obtain:

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	x_1	x_2	x_3	
0	1	1	-1	0	1	-1	0	4
1	0	0	1	0	0	1	0	26
0	-1	0	0	1	-1	1	1	8
0	0	0	0	0	0	2	2	0

Finally, we remove the column corresponding to the auxiliary variables and calculate the new reduced costs and the opposite of the objective function:

$$\begin{aligned} \bar{c}_{G,M} &= 1 \\ \bar{c}_{F,A} &= 1 \\ -c^T x = -26 - 4 - 8 &= -38. \end{aligned}$$

$x_{F,M}$	$x_{G,M}$	$x_{A,M}$	$x_{F,A}$	$x_{A,G}$	
0	1	1	-1	0	4
1	0	0	1	0	26
0	-1	0	0	1	8
0	1	0	1	0	-38

This tableau is optimal. The solution to the problem is:

- $x_{F,M} = 26$: Francois gives 26.- CHF to Monia,
- $x_{A,M} = 4$: Aria gives 4.- CHF to Monia,
- $x_{A,G} = 8$: Aria gives 8.- CHF to Gabriel.

3. In a transportation problem, there are arcs only between supply nodes and demand nodes. There is no arc linking two demand nodes, or two supply nodes. Here, Aria and Francois correspond to supply nodes, and Monia and Gabriel correspond to demand nodes. Therefore, we can remove the two corresponding arcs. We obtain the following network: The corresponding optimization problem is

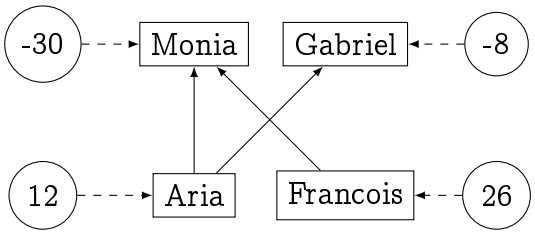


Figure 12: Network for the transportation problem

$$\min_{x \in \mathbb{R}^3} \sum_{(i,j) \in \mathcal{A}} x_{ij}.$$

subject to

$$\begin{aligned} x_{\text{Aria,Monia}} + x_{\text{Aria,Gabriel}} &= 12, \\ x_{\text{François,Monia}} &= 26, \\ x_{\text{Aria,Monia}} + x_{\text{François,Monia}} &= 30, \\ x_{\text{Aria,Gabriel}} &= 8, \\ x_{ij} &\geq 0, \quad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

Therefore, here we have:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 12 \\ 26 \\ 30 \\ 8 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

We use the simplex algorithm in two phases. We introduce 4 slack variables x_1, x_2, x_3, x_4.

The objective function for the auxiliary problem is

$$\min_{x \in \mathbb{R}^7} x_1 + x_2 + x_3 + x_4.$$

The first tableau is

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	x_1	x_2	x_3	x_4	
1	1	0	1	0	0	0	12
0	0	1	0	1	0	0	26
1	0	1	0	0	1	0	30
0	1	0	0	0	0	1	8
-2	-2	-2	0	0	0	0	-78

The pivot is $(1,1)$. After pivoting the tableau we obtain:

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	x_1	x_2	x_3	x_4	
1	1	0	1	0	0	0	12
0	0	1	0	1	0	0	26
0	-1	1	-1	0	1	0	18
0	1	0	0	0	0	1	8
0	0	-2	2	0	0	0	-52

The pivot is $(3,3)$. After pivoting the tableau we obtain:

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	x_1	x_2	x_3	x_4	
1	1	0	1	0	0	0	12
0	1	0	1	1	-1	0	8
0	-1	1	-1	0	1	0	18
0	1	0	0	0	0	1	8
0	-2	0	0	0	2	0	-16

The pivot is $(2,2)$. After pivoting the tableau we obtain:

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	x_1	x_2	x_3	x_4	
1	0	0	0	-1	1	0	4
0	1	0	1	1	-1	0	8
0	0	1	0	1	0	0	26
0	0	0	-1	-1	1	1	0
0	0	0	2	2	0	0	0

The last tableau is optimal since all reduced costs are non negative. We obtain the feasible solution $x_{\text{Aria,Monia}} = 4, \quad x_{\text{Aria,Gabriel}} = 8, \quad x_{\text{François,Monia}} = 26$.

The auxiliary variable x_4 is basic. However, it cannot be exchanged with a variable from the original problem. It means that the corresponding constraint is redundant. We remove the corresponding row and the column x_4 to obtain:

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	x_1	x_2	x_3	
1	0	0	0	-1	1	4
0	1	0	1	1	-1	8
0	0	1	0	1	0	26
0	0	0	2	2	0	0

No auxiliary variable is left in the basis and we can remove the corresponding columns. Furthermore, we have to recalculate the last row. The variables in the basis have reduced cost zero. This is the case for all variables. The objective value is -38.

We obtain the tableau:

$x_{A,M}$	$x_{A,G}$	$x_{F,M}$	
1	0	0	4
0	1	0	8
0	0	1	26
0	0	0	-38

The last tableau is optimal and the optimal solution is

- $x_{F,M} = 26$: Francois gives 26.- CHF to Monia,
- $x_{A,M} = 4$: Aria gives 4.- CHF to Monia,
- $x_{A,G} = 8$: Aria gives 8.- CHF to Gabriel,

which is the same solution as before.

Question 10

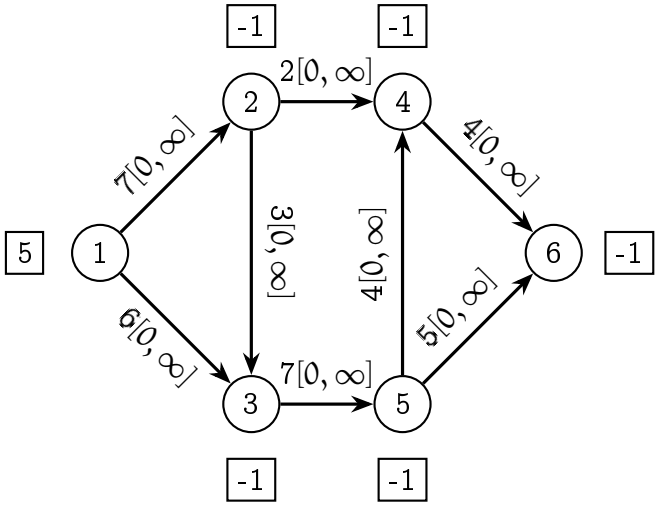
Consider the following transhipment problem. Draw the corresponding graph, mentioning next to each node the supply/demand, and next to each arc its cost, and capacities (lower and upper bounds).

$$\min_{x \in \mathbb{R}^8} 7x_{12} + 6x_{13} + 3x_{23} + 2x_{24} + 7x_{35} + 4x_{54} + 4x_{46} + 5x_{56}$$

$$\begin{aligned}
5 - x_{12} - x_{13} &= 0 \\
x_{12} - x_{23} - x_{24} - 1 &= 0 \\
x_{13} + x_{23} - x_{35} - 1 &= 0 \\
x_{24} + x_{54} - x_{46} - 1 &= 0 \\
x_{35} - x_{54} - x_{56} - 1 &= 0 \\
x_{46} + x_{56} - 1 &= 0 \\
x_{12}, x_{13}, x_{23}, x_{24}, x_{35}, x_{54}, x_{46}, x_{56} &\geq 0
\end{aligned}$$

Solution 10

Each equality constraint corresponds to a node. There are 6 in total. Each variable corresponds to the flow on an arc. For example x_{12} corresponds to the arc linking nodes 1 and 2. There are eight arcs in total. Each equality constraint models the flow conservation constraint at each node. Node 1 is a source and sends 5 units of flow. The other nodes are sinks and absorb one unit of flow each.



Question 11

For security reasons, the city council of Lausanne wants to know how many persons can get from the train station to the bar "Great Escape" during one hour. The possible links which can be used and their corresponding capacity are the following:

- Station to Metro: 1500 persons.
- Station to Place de l'Europe: 2700 persons/hour.
- Metro to the Great Escape: 2500 persons/hour.
- Place de l'Europe to Metro: 1800 persons/hour.
- Place de l'Europe to Great Escape: 2200 persons/hour.

We model this as a maximum flow problem.

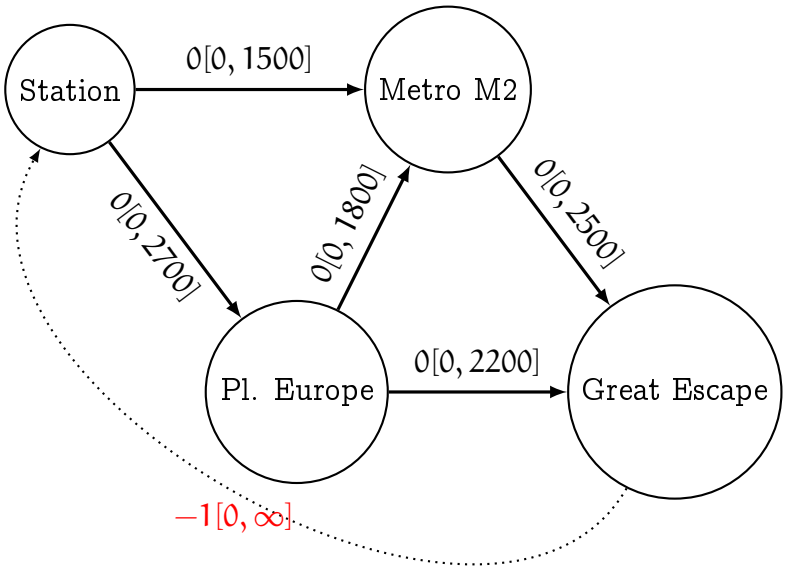
1. Draw the corresponding network. The network must have the upper and lower bounds and the additional dummy arc used for counting the flow.
2. Write down the corresponding optimization problem. Remember that the max flow is a special case of the transhipment problem.
3. The solution to this problem is the following flows:

$$\begin{aligned} x_{S,M2} &= 1500, \\ x_{S,E} &= 2700, \\ x_{E,M2} &= 500, \\ x_{E,GE} &= 2200, \\ x_{M2,GE} &= 2000. \end{aligned}$$

   Identify a saturated cut in the original graph, that does not involve the dummy arc.

Solution 11

1. We introduce one node for each location and one arc for each connection between these locations. We add a fictive arc between the destination (Great Escape) and the origin (Station) with a cost of -1. This arc is used to count the total flow which can be transported through the network by imposing a circulation. In the following figure, each arc (i, j) is associated with the quantities $c_{ij}[\ell_{ij}, u_{ij}]$, where c_{ij} is the cost, ℓ_{ij} the lower bound, and u_{ij} the upper bound.



2. A transhipment problem is written:

$$\min_{x} \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}$$

subject to the conservation constraints,

$$\mathrm{div}(x)_i = s_i \quad \forall i \in \mathcal{N},$$

and the bounds on the flows:

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \ \forall (i,j) \in \mathcal{A},$$

where $\mathcal{A}$ is the set of arcs in the network, $\mathcal{N}$ the set of nodes, ℓ_{ij} and u_{ij} the lower and upper bounds on the flows from node i to node j. For the maximum flow problem, since the costs are 0 on all arcs except the arc returning the flow from the destination to the origin, the objective function is reduced to:

$$\min_x -x_{GE,S}.$$

The bound constraints on the arcs induce the following inequality constraints:

$$\begin{aligned} x_{S,M2} &\leq 1500, \\ x_{S,E} &\leq 2700, \\ x_{E,M2} &\leq 1800, \\ x_{E,GE} &\leq 2200, \\ x_{M2,GE} &\leq 2500, \\ x_i &\geq 0, \quad \forall i. \end{aligned}$$

The flow conservation at each node induces the following equality constraints:

$$\begin{aligned} -x_{GE,S} + x_{S,M2} + x_{S,E} &= 0, && \text{(Station)} \\ -x_{S,E} + x_{E,M2} + x_{E,GE} &= 0, && \text{(Pl. Europe)} \\ -x_{S,M2} - x_{E,M2} + x_{M2,GE} &= 0, && \text{(Metro M2)} \\ -x_{M2,GE} - x_{E,GE} + x_{GE,S} &= 0. && \text{(Great Escape)} \end{aligned}$$

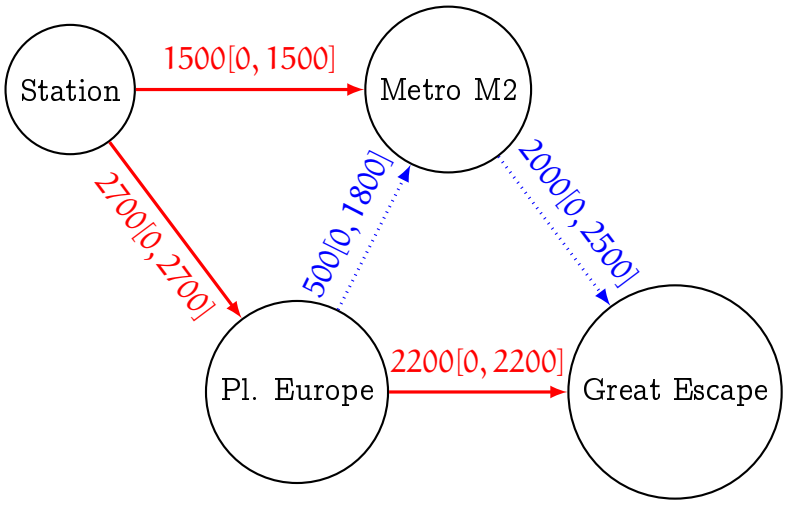
The optimization problem is therefore:

$$\min_{x \in \mathbb{R}^5} -x_{GE,S}$$

subject to

$$\begin{aligned}
x_{S,M2} &\leq 1500, \\
x_{S,E} &\leq 2700, \\
x_{E,M2} &\leq 1800, \\
x_{E,GE} &\leq 2200, \\
x_{M2,GE} &\leq 2500, \\
-x_{GE,S} + x_{S,M2} + x_{S,E} &= 0, \\
-x_{S,E} + x_{E,M2} + x_{E,GE} &= 0, \\
-x_{S,M2} - x_{E,M2} + x_{M2,GE} &= 0, \\
-x_{M2,GE} - x_{E,GE} + x_{GE,S} &= 0, \\
x_{S,M2}, x_{S,E}, x_{E,M2}, x_{E,GE}, x_{M2,GE} &\geq 0.
\end{aligned}$$

3. The solution is represented in the following figure, where each arc (i, j) is associated with the quantities $x_{ij}[\ell_{ij}, u_{ij}]$, where x_{ij} is the flow, ℓ_{ij} the lower bound, and u_{ij} the upper bound. The arcs in red are saturated, which means that the flow along the arc is equal to the capacity of the arc.



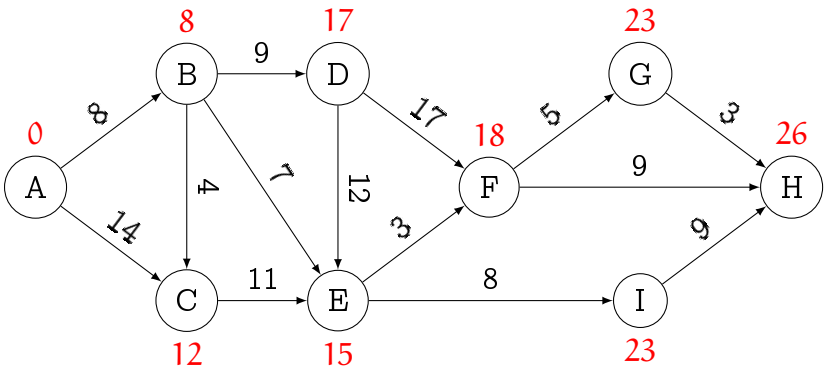
A saturated cut is a cut where all arcs are saturated, hence a saturated cut is a cut where the flow is equal to the capacity of the cut. The cut

$\Gamma = (\mathcal{M}, \mathcal{N} \setminus \mathcal{M})$, where $\mathcal{M} = \{\text{Station}\}$ is a saturated cut. Indeed, the two arcs connecting a node from $\mathcal{M}$ with a node from $\mathcal{N} \setminus \mathcal{M}$ are saturated.

7 Shortest paths

Question 01

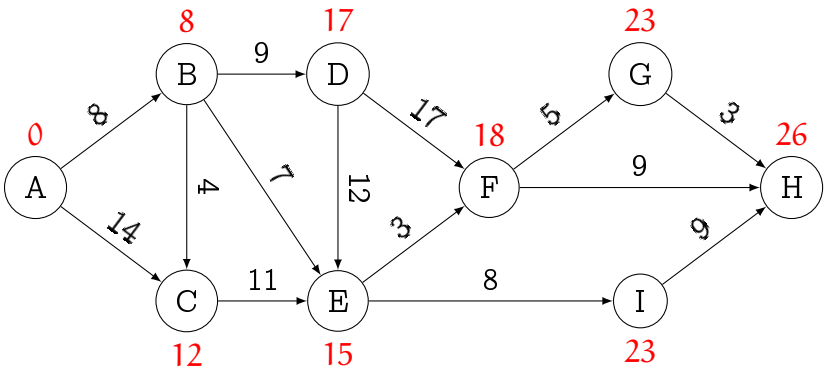
Consider the following network, where each arc (i, j) is associated with its cost c_{ij}, and each node i is associated with a label λ_i (in red).



1. What are the shortest paths between node A and each other node? Provide the shortest path tree.

2. What is the length of the shortest path between node A and node H?

Solution 01



1. The optimality conditions for a shortest path P are

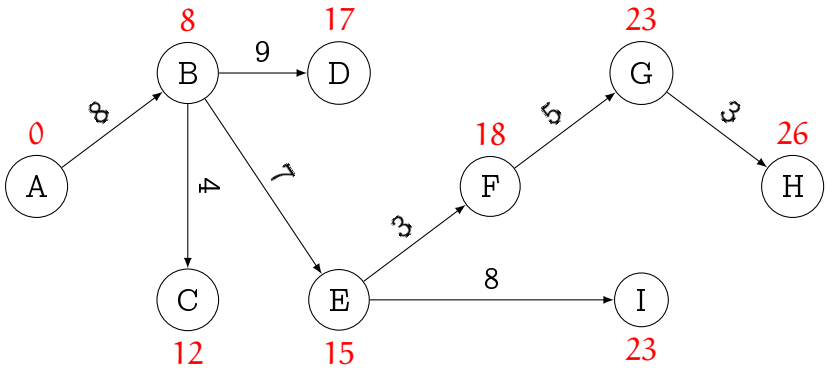
$$\begin{aligned} \lambda_j &\leq \lambda_i + c_{ij}, && \forall (i,j) \in \mathcal{A}, \\ \lambda_j &= \lambda_i + c_{ij}, && \forall (i,j) \in P. \end{aligned}$$

We calculate these quantities for each arc, and compare them.

(i,j)	λ_j		$\lambda_i + c_{ij}$
(A,B)	8	$=$	8,
(A,C)	12	$\leq$	14,
(B,C)	12	$=$	12,
(B,D)	17	$=$	17,
(B,E)	15	$=$	15,
(C,E)	15	$\leq$	23,
(D,E)	15	$\leq$	29,
(D,F)	18	$\leq$	34,
(E,F)	18	$=$	18,
(E,I)	23	$=$	23,
(F,G)	23	$=$	23,
(F,H)	26	$\leq$	27,
(G,H)	26	$=$	26,
(I,H)	26	$\leq$	32.

Note that the condition $\lambda_j \leq \lambda_i + c_{ij}$ is verified for all the arcs in the network.

We consider the subnetwork composed of the arcs that verify the equality condition. We obtain



This is the shortest path tree. For instance, the shortest path between A and D is the only path between A and D in this tree, that is $A \rightarrow B \rightarrow D$. Indeed, the optimality conditions are verified for this path:

- as noticed above, the condition $\lambda_j \leq \lambda_i + c_{ij}$ is verified for all the arcs in the network,
- for the two arcs of path $A \rightarrow B \rightarrow D$, the condition $\lambda_j = \lambda_i + c_{ij}$ is verified, by construction of the tree.

The shortest paths between node A and each other node are

- $A \rightarrow B$,
- $A \rightarrow B \rightarrow C$,
- $A \rightarrow B \rightarrow D$,
- $A \rightarrow B \rightarrow E$,
- $A \rightarrow B \rightarrow E \rightarrow F$,
- $A \rightarrow B \rightarrow E \rightarrow F \rightarrow G$,
- $A \rightarrow B \rightarrow E \rightarrow F \rightarrow G \rightarrow H$,
- $A \rightarrow B \rightarrow E \rightarrow I$.

2. When the set of labels verify the optimality conditions, the length of a shortest path between A and any node i is

$$\lambda_i - \lambda_A = \lambda_i,$$

as $\lambda_A = 0$. Therefore, the length of the shortest path between A and H is $\lambda_H = 26$. It can be verified by considering the path itself, obtained from the shortest path tree:

$$A \to B \to E \to F \to G \to H.$$

Its length is

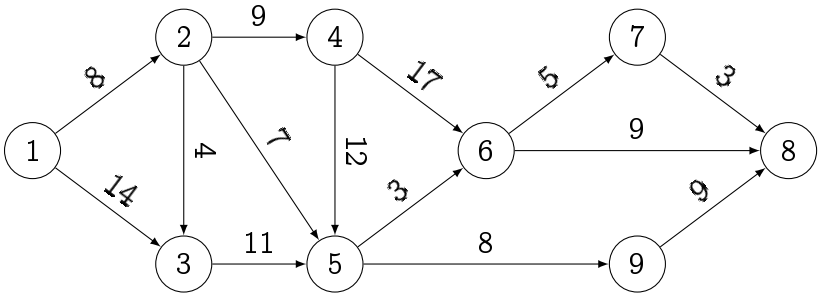
$$8 + 7 + 3 + 5 + 3 = 26.$$

Question 02

The shortest path algorithm, to calculate the shortest paths from node o to all the other nodes in the network, is described as follows.

1. Initialization: $\mathcal{S} \leftarrow \{o\}$, $\lambda_o = 0$, $\lambda_j = +\infty$.
2. At each iteration:
   - Select i in $\mathcal{S}$ and remove it from $\mathcal{S}$.
   - For each (i, j):
     - If $\lambda_j > \lambda_i + c_{ij}$,
       - $\lambda_j \leftarrow \lambda_i + c_{ij}$.
       - If $\lambda_j < 0$ and $\lambda_j < (m-1)\min_{(i,j)\in\mathcal{A}} c_{ij}$: STOP.
       - $\mathcal{S} \leftarrow \mathcal{S} \cup \{j\}$.
   - If $\mathcal{S} = \emptyset$: STOP.
   - Otherwise, start again.

Apply the algorithm to identify the shortest paths from node 1 to all other nodes in the following network, and identify the shortest path tree.



Solution 02

Before we apply the algorithm, we note that all costs in the network are non negative. Therefore, there is no cycle with negative costs, and the problem is bounded.

Each iteration of the algorithm is summarized in Table 1, where

- the first column reports the iteration number,
- the second column reports the content of set $\mathcal{S}$,
- the third column reports the node that is selected to be treated in the next iteration,
- all the other columns contain the labels of all the nodes in the network.

We comment below the first iterations.

Iteration 0 We initialize the set $\mathcal{S}$ with the starting node 1. All labels are set to ∞, except $\lambda_1 = 0$. In column i, we select the node that will be treated by the next iteration. Here, there is only one node: $i = 1$.

Iteration 1 We treat node 1.

- We remove node 1 from $\mathcal{S}$.
- We consider all arcs leaving node 1:
  - Arc (1, 2): the candidate label for node 2 is $\lambda_i + c_{ij} = 0 + 8 = 8$. As it is inferior to the current label (∞), we update $\lambda_2 = 8$ and include node 2 in set $\mathcal{S}$.
  - Arc (1, 3): the candidate label for node 3 is $\lambda_i + c_{ij} = 0 + 14 = 14$. As it is inferior to the current label (∞), we update $\lambda_3 = 14$ and include node 3 in set $\mathcal{S}$.

Iteration 2 We select the node that will be treated by this iteration. We have the choice between 2 and 3. We arbitrarily choose to treat node 3.

- We remove node 3 from $\mathcal{S}$.
- We consider all arcs leaving node 3:

- Arc (3, 5): the candidate label for node 5 is $\lambda_i + c_{ij} = 14 + 11 = 25$. As it is inferior to the current label (∞), we update $\lambda_5 = 25$ and include node 5 in set $\mathcal{S}$.

Iteration 3 We select the node that will be treated by this iteration. We have the choice between 2 and 5. We arbitrarily choose to treat node 2.

- We remove node 2 from $\mathcal{S}$.
- We consider all arcs leaving node 2:
  - Arc (2, 3): the candidate label for node 3 is $\lambda_i + c_{ij} = 8 + 4 = 12$. As it is inferior to the current label (14), we update $\lambda_3 = 12$ and include node 3 in set $\mathcal{S}$.
  - Arc (2, 4): the candidate label for node 4 is $\lambda_i + c_{ij} = 8 + 9 = 17$. As it is inferior to the current label (∞), we update $\lambda_4 = 17$ and include node 4 in set $\mathcal{S}$.
  - Arc (2, 5): the candidate label for node 5 is $\lambda_i + c_{ij} = 8 + 7 = 15$. As it is inferior to the current label (25), we update $\lambda_5 = 15$. We do not need to include node 5 in set $\mathcal{S}$, as it already belongs to it.

Iteration 4 We select the node that will be treated by this iteration. We have the choice between 3, 4 and 5. We arbitrarily choose to treat node 3.

- We remove node 3 from $\mathcal{S}$.
- We consider all arcs leaving node 3:
  - Arc (3,5): the candidate label for node 5 is $\lambda_i + c_{ij} = 12 + 11 = 23$. As it is not inferior to the current label (15), we do not do anything.

The other iterations proceed in the same way, until set $\mathcal{S}$ is empty.

To identify the shortest paths tree, we calculate $\lambda_i + c_{ij}$ and λ_j for each

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
0	$\{1\}$	1	0	∞	∞	∞	∞	∞	∞	∞	∞
1	$\{2,3\}$	3	0	8	14	∞	∞	∞	∞	∞	∞
2	$\{2,5\}$	2	0	8	14	∞	25	∞	∞	∞	∞
3	$\{3,4,5\}$	3	0	8	12	17	15	∞	∞	∞	∞
4	$\{4,5\}$	4	0	8	12	17	15	∞	∞	∞	∞
5	$\{5,6\}$	5	0	8	12	17	15	34	∞	∞	∞
6	$\{6,9\}$	6	0	8	12	17	15	18	∞	∞	23
7	$\{7,8,9\}$	7	0	8	12	17	15	18	23	27	23
8	$\{8,9\}$	8	0	8	12	17	15	18	23	26	23
9	$\{9\}$	9	0	8	12	17	15	18	23	26	23
10	$\{\}$	-	0	8	12	17	15	18	23	26	23

Table 1: Iterations of the shortest path algorithm

arc, where the labels λ are those obtained at the end of the algorithm.

$$
\begin{array}{cccl}
(i,j) & \lambda_j & & \lambda_i + c_{ij} \\
\hline
(1,2) & 8 & = & 0+8, \\
(1,3) & 12 & < & 0+14, \\
(2,3) & 12 & = & 8+4, \\
(2,4) & 17 & = & 8+9, \\
(2,5) & 15 & = & 8+7, \\
(3,5) & 15 & < & 12+11, \\
(4,5) & 15 & < & 17+12, \\
(4,6) & 18 & < & 17+17, \\
(5,6) & 18 & = & 15+3, \\
(5,9) & 23 & = & 15+8, \\
(6,7) & 23 & = & 18+5, \\
(6,8) & 26 & < & 18+9, \\
(7,8) & 26 & = & 23+3, \\
(9,8) & 26 & < & 23+9.
\end{array}
$$

The shortest path tree is composed of the arcs for which the two quantities are equal, as represented in Figure 13.

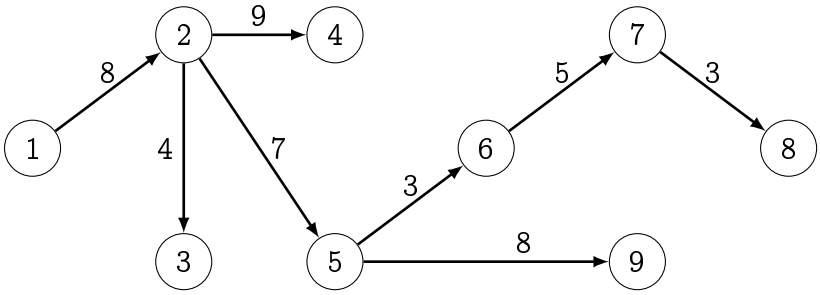


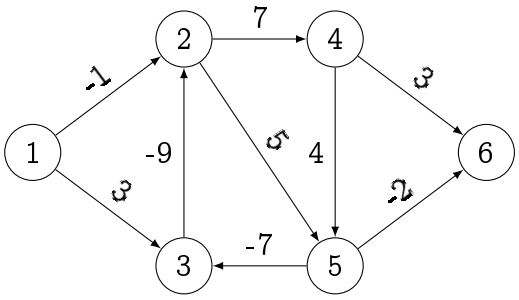
Figure 13: Shortest path tree

Question 03

The shortest path algorithm, to calculate the shortest paths from node o to all the other nodes in the network, is described as follows.

1. Initialization: $\mathcal{S} \leftarrow \{o\}$, $\lambda_o = 0$, $\lambda_j = +\infty$.
2. At each iteration:
   - Select i in $\mathcal{S}$.
   - For each (i, j):
     - If $\lambda_j > \lambda_i + c_{ij}$,
       - $\lambda_j \leftarrow \lambda_i + c_{ij}$.
       - If $\lambda_j < 0$ and $\lambda_j < (m-1)\min_{(i,j)\in\mathcal{A}} c_{ij}$: STOP.
       - $\mathcal{S} \leftarrow \mathcal{S} \cup \{j\}$.
   - $\mathcal{S} \leftarrow \mathcal{S} \setminus \{i\}$.
   - If $\mathcal{S} = \emptyset$: STOP.
   - Otherwise, start again.

Apply the algorithm to identify the shortest paths from node 1 to all other nodes in the following network, and identify the shortest path tree.



Solution 03

As some arcs in the network have negative cost, it is important to verify that the problem is bounded during the course of the algorithm. Consider a network with n arcs and m nodes where $c \geq 0$. Consider an arbitrary pair of different nodes, and P the shortest path between the two. Then,

$$C(P) \geq (m-1) \min_{(i,j)} c_{ij},$$

because a simple path cannot contain more than $m-1$ arcs. In this network, $m = 6$ and $\min_{(i,j)} c_{ij} = -9$. Therefore, no shortest path has a cost lower than -45.

Each iteration of the algorithm is summarized in Table 2, where

- the first column reports the iteration number,
- the second column reports the content of set $\mathcal{S}$,
- the third column reports the node that is selected to be treated in the next iteration,
- all the other columns contain the labels of all the nodes in the network.

The algorithm is interrupted at iteration 15, as the label of node 2 is -50. As it is below the lower bound, it means that the problem is non bounded. Indeed, the cycle $2 \to 5 \to 3 \to 2$ has length $5 - 7 - 9 = -11$. And the cycle $2 \to 4 \to 5 \to 3 \to 2$ has length $7 + 4 - 7 - 9 = -5$. The algorithm cannot identify a shortest path.

Here we have $\lambda_2 < 0$ and

$$\lambda_2 < (m-1) \min_{(i,j) \in \mathcal{A}} c_{ij} = 5 \cdot (-9) = -45.$$

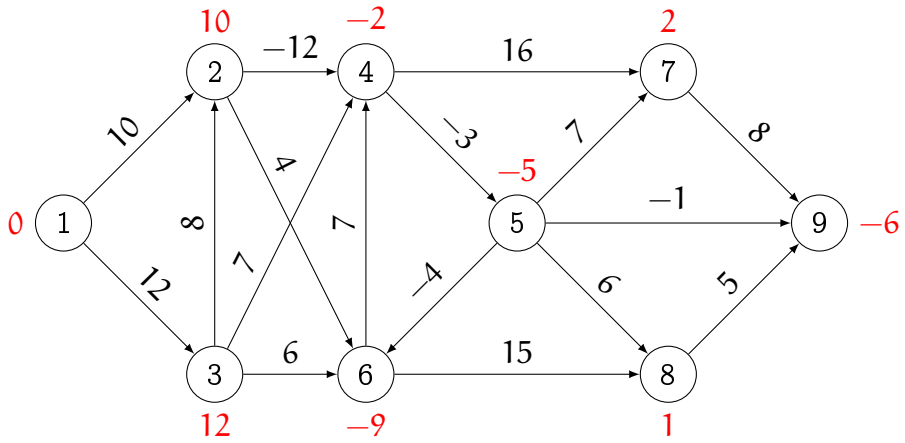
Thus a negative cycle is detected and the algorithm fails as the problem is unbounded. Non negative cycle is indeed an assumption to allow the algorithm to finish after a finite number of iterations.

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	$\{1\}$	1	0	∞	∞	∞	∞	∞
1	$\{2,3\}$	2	0	-1	3	∞	∞	∞
2	$\{3,4,5\}$	3	0	-1	3	6	4	∞
3	$\{2,4,5\}$	2	0	-6	3	6	4	∞
4	$\{4,5\}$	5	0	-6	3	1	-1	∞
5	$\{3,4,6\}$	3	0	-6	-8	1	-1	-3
6	$\{2,4,6\}$	2	0	-17	-8	1	-1	-3
7	$\{4,5,6\}$	5	0	-17	-8	-10	-12	-3
8	$\{3,4,6\}$	3	0	-17	-19	-10	-12	-14
9	$\{2,4,6\}$	2	0	-28	-19	-10	-12	-14
10	$\{4,5,6\}$	5	0	-28	-19	-21	-23	-14
11	$\{3,4,6\}$	3	0	-28	-30	-21	-23	-25
12	$\{2,4,6\}$	2	0	-39	-30	-21	-23	-25
13	$\{4,5,6\}$	5	0	-39	-30	-32	-34	-25
14	$\{3,4,6\}$	3	0	-39	-41	-32	-34	-36
15	$\{2,4,6\}$	-	0	-50	-41	-32	-34	-36

Table 2: Iterations of the shortest path algorithm

Question 04

Consider the following network, where each arc is associated with its cost, and each node is associated with a label (in red) calculated by the shortest path algorithm.



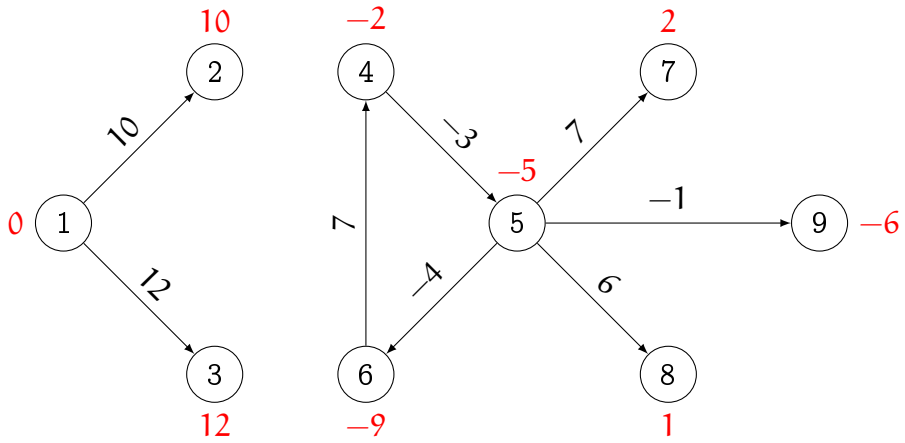
Identify all Bellman's subnetworks in this network. For each of them, mention if it is a shortest path spanning tree or not.

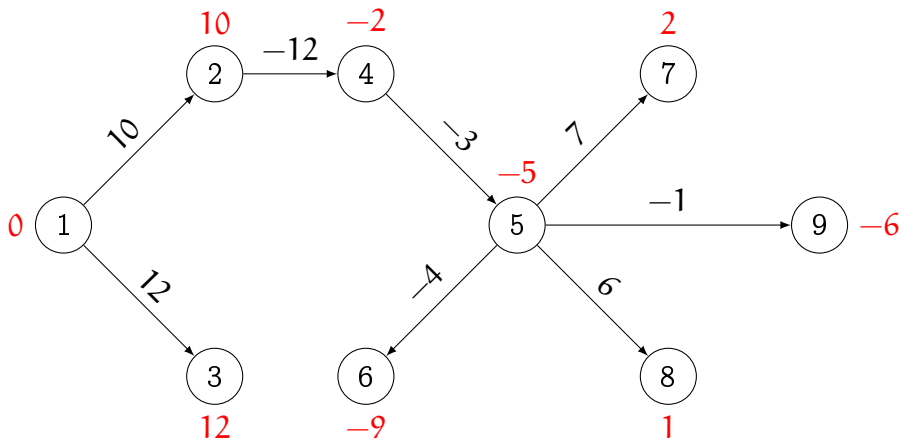
Solution 04

To construct Bellman's subnetwork, we need to consider each node separately, and identify one or several arcs that verify the condition with equality:

- Node 2: arc (1,2) verifies $\lambda_2 = \lambda_1 + c_{12} = 10$.
- Node 3: arc (1,3) verifies $\lambda_3 = \lambda_1 + c_{13} = 12$.
- Node 4: two arcs verify the condition:
  - arc (2,4) verifies $\lambda_4 = \lambda_2 + c_{24} = -2$.
  - arc (6,4) verifies $\lambda_4 = \lambda_6 + c_{64} = -2$.
- Node 5: arc (4,5) verifies $\lambda_5 = \lambda_4 + c_{45} = -5$.
- Node 6: arc (5,6) verifies $\lambda_6 = \lambda_5 + c_{56} = -9$.
- Node 7: arc (5,7) verifies $\lambda_7 = \lambda_5 + c_{57} = 2$.
- Node 8: arc (5,8) verifies $\lambda_8 = \lambda_5 + c_{58} = 1$.
- Node 9: arc (5,9) verifies $\lambda_9 = \lambda_5 + c_{59} = -6$.

As node 4 has two arcs verifying Bellman's condition, we can build two Bellman's subnetworks.





We observe that the first Bellman's subnetwork is not a tree: there is a cycle $4 \to 5 \to 6 \to 4$. Note that it is not inconsistent with the result presented in the lecture, that was based on the assumption that "*each cycle has positive length*". This assumption is not verified here, as the cycle mentioned above has length 0. Therefore, there is no guarantee that Bellman's subnetwork is a tree.

The second subnetwork is indeed a tree. It is a shortest path spanning tree. For instance, the shortest path between node 1 and node 9 is the only path between 1 and 9 in the shortest path spanning tree, that is $1 \to 2 \to 4 \to 5 \to 9$. Its length is

$$10 - 12 - 3 - 1 = -6,$$

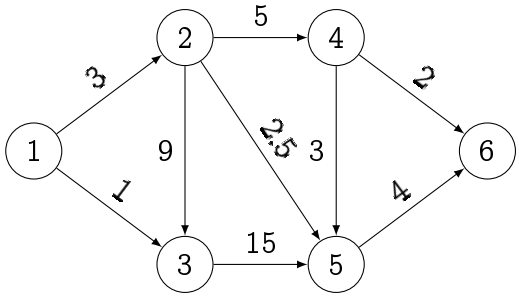
which happens to be the optimal label of node 9.

Question 05

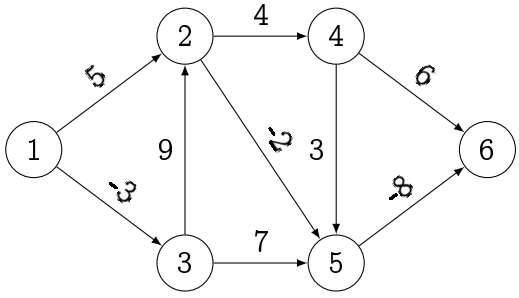
For each of the following networks, we consider the problem of finding the shortest path from node 1 to all other nodes.

- Is Dijkstra's algorithm applicable?
- If so, apply it to find the optimal labels and the shortest path spanning tree. Discuss the properties of the algorithm during iteration 4.
- If not, apply the shortest path algorithm, where Dijkstra's rule for node selection is applied. It means that the node with the smallest label is selected to be processed at each iteration.

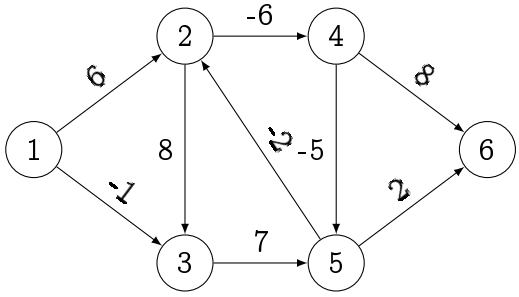
1. Network 1:



2. Network 2:

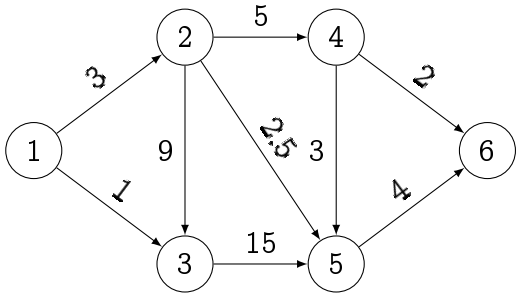


3. Network 3:


1
2
3
4
5
6
6
-1
8
-6
2
-5
8
2
7

Solution 05

1. Network 1:



We first note that all arcs in the network have non negative cost, so that Dijkstra's algorithm can be applied. It is the same algorithm as the shortest path algorithm, with two differences:

- the node that is treated at each iteration is the node in $\mathcal{S}$ with the smallest label,
- there is no need to detect a cycle with negative cost, as all arcs have non negative cost.

The iterations of Dijkstra's algorithm are reported in Table 3 on the next page, where

- the first column reports the number of the iteration,
- the second column reports the set $\mathcal{S}$ of nodes to be treated at the beginning of the iteration,
- the third column reports the node that is treated during the iteration,
- the remaining columns report the values of the labels at the beginning of the iteration.

In order to check how the algorithm works, we focus on the third iteration. At the beginning of iteration 3, both nodes 4 and 5 are in set $\mathcal{S}$. As $\lambda_4 = 8$ and $\lambda_5 = 5.5$, it is decided to treat node 5, as it has the smallest label.

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	$\{1\}$	1	0	∞	∞	∞	∞	∞
1	$\{2,3\}$	3	0	3	1	∞	∞	∞
2	$\{2,5\}$	2	0	3	1	∞	16	∞
3	$\{4,5\}$	5	0	3	1	8	5.5	∞
4	$\{4,6\}$	4	0	3	1	8	5.5	9.5
5	$\{6\}$	6	0	3	1	8	5.5	9.5
6	$\{\}$	-	0	3	1	8	5.5	9.5

Table 3: Iterations of Dijkstra's algorithm for Network 1

The shortest path spanning tree is reported in Figure 14 on the following page. Finally, we discuss the properties of the algorithm for iteration 4. At the beginning of the iteration, the set of permanent labels is

$$\mathcal{T} = \{1, 2, 3, 5\},$$

and the set of nodes to be treated is

$$\mathcal{S} = \{4, 6\}.$$

We verify the properties of the algorithm:

- During iteration 4, only the label of node 6 is updated. And no node in $\mathcal{T}$ is updated.
- No node in $\mathcal{T}$ is included in $\mathcal{S}$ during the iteration. Indeed, after the iteration, $\mathcal{S} = \{6\}$.
- The labels of any node i in $\mathcal{T}$ corresponds to the length of a shortest path from the origin to i. For instance, $\lambda_5 = 5.5$ corresponds to the length of path $1 \to 2 \to 5$, that is the shortest path between 1 and 5.
- The labels of nodes 4 and 6 (nodes not in $\mathcal{T}$) are larger than the labels of any node in $\mathcal{T}$.

2. Network 2: A condition to apply Dijkstra algorithm is that all costs are non negative. Here it's not the case. Therefore, we apply the shortest path algorithm using the same rule as Dijkstra's algorithm for the selection of the node to process at each iteration.

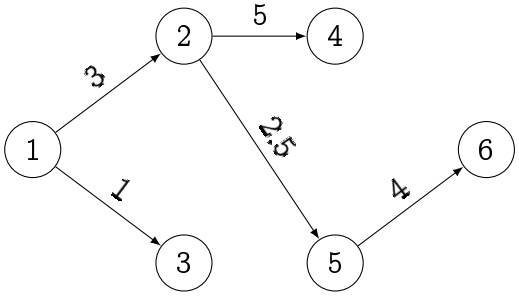
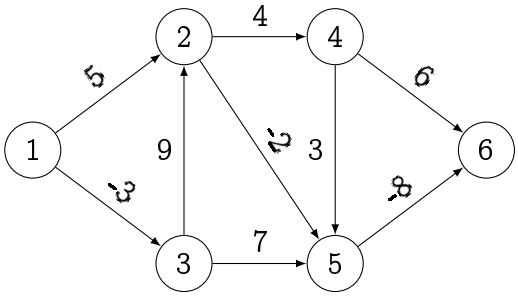


Figure 14: Shortest path spanning tree for Network 1



The iterations of the shortest path algorithm are reported in Table 4 on the next page, and the shortest path spanning tree in Figure 15 on the following page. Note that the lower bound is

$$(m-1)\min_{(i,j)} c_{ij} = 5 \cdot -8 = -40.$$

The algorithm converges without reaching the lower bound. There is no negative costs cycle. It is important to note that, despite the use of Dijkstra's rule for the selection of the node to process, the algorithm does not have the property that each node is processed at most once. In particular, nodes 5 and 6 have been processed twice during the algorithm.

3. Network 3: A condition to apply Dijkstra algorithm is that all costs are non negative. Here it's not the case. Therefore, we apply the shortest path algorithm using the same rule as Dijkstra's algorithm for the selection of the node to process at each iteration.

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	{1}	1	0	∞	∞	∞	∞	∞
1	{2, 3}	3	0	5	-3	∞	∞	∞
2	{2, 5}	5	0	5	-3	∞	4	∞
3	{2, 6}	6	0	5	-3	∞	4	-4
4	{2}	2	0	5	-3	∞	4	-4
5	{4, 5}	5	0	5	-3	9	3	-4
6	{4, 6}	6	0	5	-3	9	3	-5
7	{4}	4	0	5	-3	9	3	-5
8	{}	–	0	5	-3	9	3	-5

Table 4: Iterations of the shortest path algorithm for Network 2

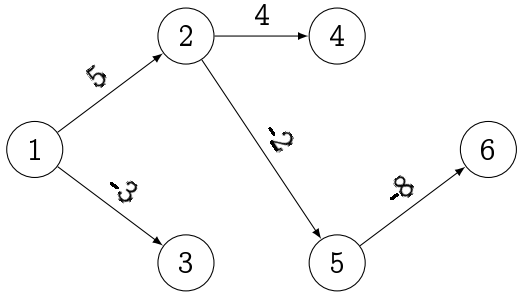
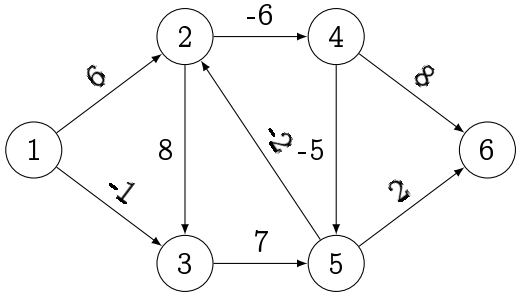


Figure 15: Shortest path spanning tree for Network 2



The iterations of the shortest path algorithm are reported in Table 5 on the next page. Note that the lower bound is

$$(m-1)\min_{(i,j)} c_{ij} = 5 \cdot -6 = -30.$$

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	$\{1\}$	1	0	∞	∞	∞	∞	∞
1	$\{2,3\}$	3	0	6	-1	∞	∞	∞
2	$\{2,5\}$	2	0	6	-1	∞	6	∞
3	$\{4,5\}$	4	0	6	-1	0	6	∞
4	$\{5,6\}$	5	0	6	-1	0	-5	8
5	$\{2,6\}$	2	0	-7	-1	0	-5	-3
6	$\{4,6\}$	4	0	-7	-1	-13	-5	-3
7	$\{5, 6\}$	5	0	-7	-1	-13	-18	-5
8	$\{2, 6\}$	2	0	-20	-1	-13	-18	-16
9	$\{3, 4, 6\}$	4	0	-20	-12	-26	-18	-16
10	$\{3, 5, 6\}$	5	0	-20	-12	-26	-31	-18

Table 5: Iterations of the shortest path algorithm for Network 3

We observe that the algorithm does not converge. There is a negative cost cycle $2 \rightarrow 4 \rightarrow 5 \rightarrow 2$. Its cost is -13. The labels of these nodes decrease over and over. The algorithm stops when the lower bound has been reached. Note that applying Dijkstra's algorithm on this network would generate an infinite number of iterations. Indeed, Dijkstra's algorithm does not check for negative cost cycle, and stops when the set $\mathcal{S}$ is empty, which is never the case here.

Question 06

A troop of young comedians is faced with the problem of planning their next show according to various tasks listed in the table below. The troop wishes to organize the tasks as the preparation of the show lasts as short as possible. Tasks are linked together by precedence constraints, which means that some tasks cannot begin until other tasks are completed. For example, in the table below, we see that B can only begin if A is complete (A precedes B). In the same way, we see that C can only begin if A and B are terminated (A and B precede C), and so on.

Task	Description	Duration [days]	Precedence
A	Writing sketches	30	-
B	Sketches distribution	12	A
C	Choice of the place of the show	8	A and B
D	Technical cuts	4	A and C
E	Preparation of the stage set	7	C and D
F	Rehearsal of the outer sketches	10	A, B, C and D
G	Rehearsal of the inner sketches	12	D, E and F
H	Synchronization	3	F and G
I	Visual accompaniment	14	H
J	Sound accompaniment	7	I and H
K	Mixing	6	I and J
L	General rehearsal	1	K

1. Model the problem using a PERT network.
2. Identify the critical tasks that can not suffer from any delay without delaying the show, and give the minimal duration of the show preparation.
3. Calculate when each task should start at the earliest and the latest to achieve the minimal duration.

Solution 06

1. The PERT network is defined in the following way.

   - A node is associated with each task,
   - an artificial node o is associated with the beginning of the project,
   - an artificial node d is associated with the end of the project.

   The set of arcs is defined as:

   - for each task j, we add an arc (i, j) for each predecessor i,
   - for each task j without predecessor, we define an arc (o, j),
   - for each task i without successor, we define an arc (i, d).

   A cost is associated with each arc (i, j). It is the duration of task i, irrespectively of j. The cost of each arc of the form (o, j) is zero. We obtain the network in Figure 16 on the following page.

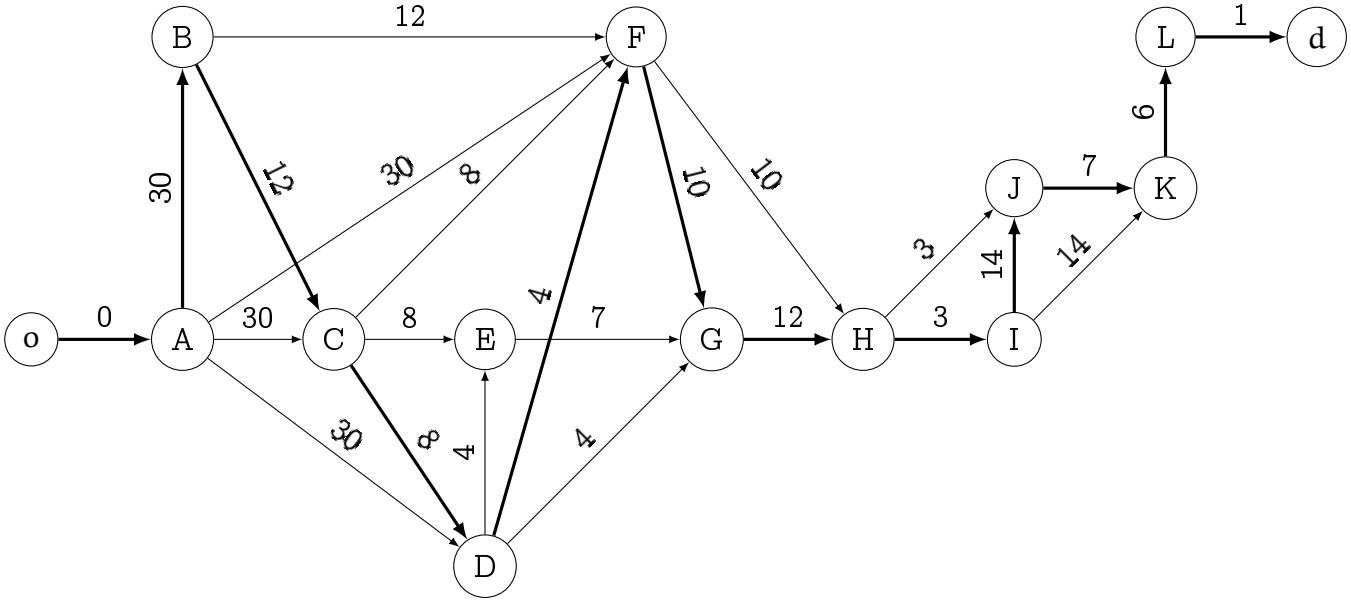


Figure 16: PERT network

2. The critical tasks are found using the longest path from o to d in the network. As the network has no cycle, we apply the shortest path algorithm on the network where all costs have the opposite sign. Indeed, even if all arcs have a negative cost, there is no cycle with negative cost.

   The iterations of the algorithm are reported in Table 6 on the next page. The values of π, capturing the preceding nodes in the shortest paths, are reported in Table 7 on page 443

It.	S	i	λ_o	λ_A	λ_B	λ_C	λ_D	λ_E	λ_F	λ_G	λ_H	λ_I	λ_J	λ_K	λ_L	λ_d
0	{o}	o	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	{A}	A	0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	{B, C, D}	B	0	0	-30	-30	-30	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	{C, D, F}	C	0	0	-30	-42	-30	∞	-42	∞	∞	∞	∞	∞	∞	∞
4	{D, E, F}	D	0	0	-30	-42	-50	-50	-50	∞	∞	∞	∞	∞	∞	∞
5	{E, F, G}	F	0	0	-30	-42	-50	-54	-54	-54	∞	∞	∞	∞	∞	∞
6	{E, G, H}	G	0	0	-30	-42	-50	-54	-54	-64	-64	∞	∞	∞	∞	∞
7	{E, H}	H	0	0	-30	-42	-50	-54	-54	-64	-76	∞	∞	∞	∞	∞
8	{E, I, J}	I	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-79	∞	∞	∞
9	{E, J, K}	J	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-93	∞	∞
10	{E, K}	K	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-100	∞	∞
11	{E, L}	L	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-100	-106	∞
12	{E, d}	d	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-100	-106	-107
13	{E}	E	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-100	-106	-107
13	{}	-	0	0	-30	-42	-50	-54	-54	-64	-76	-79	-93	-100	-106	-107

Table 6: Iterations of the shortest path algorithm in order to find the longest path from o to d.

It.	π_o	π_A	π_B	π_C	π_D	π_E	π_F	π_G	π_H	π_I	π_J	π_K	π_L	π_d
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	o	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	o	A	A	A	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	-1	o	A	B	A	-1	B	-1	-1	-1	-1	-1	-1	-1
4	-1	o	A	B	C	C	C	-1	-1	-1	-1	-1	-1	-1
5	-1	o	A	B	C	D	D	D	-1	-1	-1	-1	-1	-1
6	-1	o	A	B	C	D	D	E	-1	-1	-1	-1	-1	-1
7	-1	o	A	B	C	D	D	E	G	-1	-1	-1	-1	-1
8	-1	o	A	B	C	D	D	E	G	H	H	-1	-1	-1
9	-1	o	A	B	C	D	D	E	G	H	I	I	-1	-1
10	-1	o	A	B	C	D	D	E	G	H	I	J	K	-1
11	-1	o	A	B	C	D	D	E	G	H	I	J	K	L

Table 7: Values of π at each iteration of the shortest path algorithm

The critical path is found by starting with node d, and proceeding backward. Its predecessor is $\pi_d = L$. The predecessor of L is $\pi_L = K$, and so on until node o is reached.

The longest path from o to d is represented in bold in Figure 16 on page 440. Each node along that path corresponds to a critical tasks. The optimal labels of the algorithm (with the opposite sign) correspond to the longest path from the origin to the node. It corresponds to the earliest starting time of each tasks. For instance, $\lambda_D = -50$ means that task D will not start before day 50. Indeed, tasks A (30 days), B (12 days) and C (8 days) must be completed before task D can start. And it is not possible for the project to finish before $-\lambda_d = 107$ days.

3. In order to obtain the latest start for each task, we need to look at the problem in the other direction. Indeed, the "latest start" is the time the closest from the end of the project. Therefore, we reverse the directions of each arc, to obtain the network in Figure 17 on page 445, and we apply the shortest path algorithm while the costs of the arcs stay unchanged. The critical path is the same, but the labels on each node tells us how much time before the end should the task start at

the latest. The optimal labels are

$$\begin{aligned}
\lambda_A &= 107, & \lambda_B &= 77,\\
\lambda_C &= 65, & \lambda_D &= 57,\\
\lambda_E &= 50, & \lambda_F &= 53,\\
\lambda_G &= 43, & \lambda_H &= 31,\\
\lambda_I &= 28, & \lambda_J &= 14,\\
\lambda_K &= 7, & \lambda_L &= 1,\\
\lambda_d &= 0. &&
\end{aligned}$$

The latest starting time of task E is 50 days before the end of the project, that is 107-50=57 days after the beginning of the project.

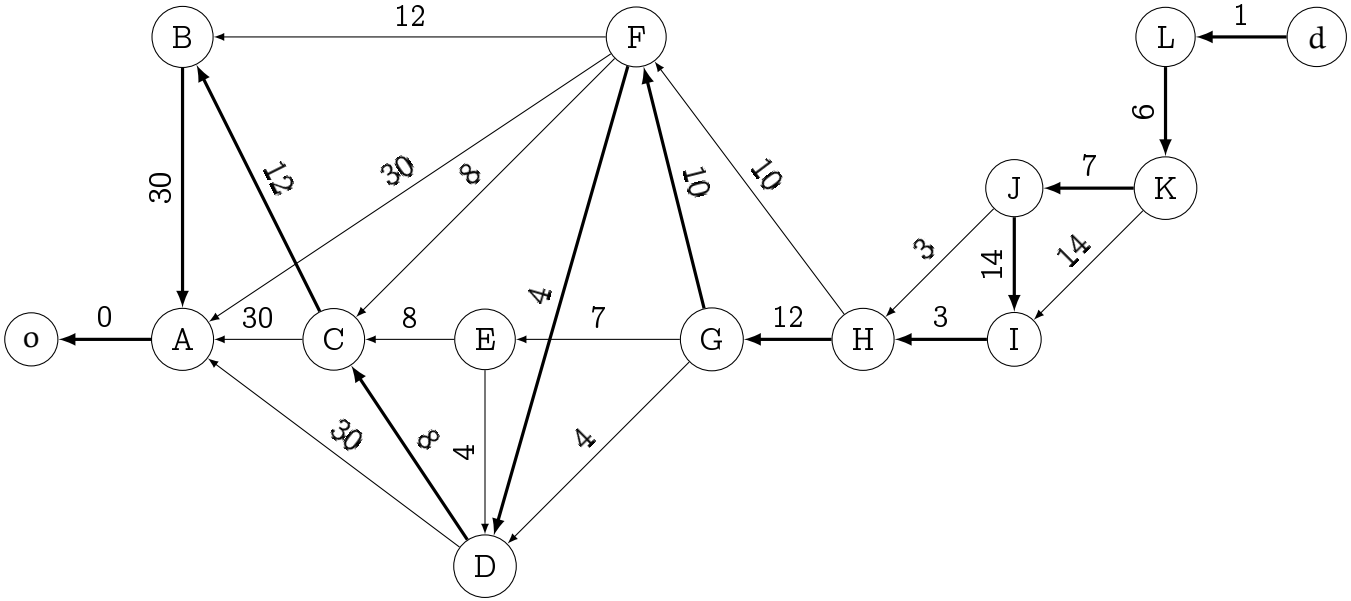


Figure 17: Reverse PERT network

The results are summarized in the table below. The project cannot last less than 107 days. The critical tasks are such that the earliest and the latest time are the same. For the others, there is some flexibility. If the task starts any time between the two dates, the project will not be delayed. In this example, the only non critical task is Task E. It cannot start before day 54, and must not start after day 57.

Ranking	Task	Duration	Precedence	Successor	Earliest start δ_i	Latest start φ_i
1	o	0	-	A	0	0
2	A	30	o	B, C, D, F	0	0
3	B	12	A	C, F	30	30
4	C	8	A, B	D, E, F	42	42
5	D	4	A, C	E, F, G	50	50
6	E	7	C, D	G	54	57
7	F	10	A, B, C, D	G, H	54	54
8	G	12	D, E, F	H	64	64
9	H	3	F, G	I, J	76	76
10	I	14	H	J, K	79	79
11	J	7	H, I	K	93	93
12	K	6	I, J	L	100	100
13	L	1	K	d	106	106
14	d	0	K	-	107	107

Question 07

The renovation of an apartment's living room breaks down into several tasks listed in the table below, that also contains the precedence to be respected during the planning of the works, as well as the duration of each task.

Task	Description	Precedence	Duration [days]
A	Door removal		1/2
B	Sanding and painting doors	A	3
C	Hanging doors	B, J	1/2
D	Peeling off wallpapers		1
E	Pulling electrical wires	D	1
F	Laying electrical outlet	E, H, I	1/2
G	Smoothing walls	E, A	2
H	Sanding of frames	G	2
I	Ceiling painting	G	3
J	Painting frames	H, I	1
K	Ripping off the carpet	H, I, J	1/2
L	Sanding parquet	K	1
M	Impregnation and drying of parquet	L, F	4
N	Balcony painting		2
O	Change of solar protections	N	1

1. What is the mathematical network representation associated with this problem?

2. Identify critical tasks that can not suffer from any delay without delaying the project, and give the minimal timing of the duration of the work.

3. Calculate the earliest and the latest starting dates of each task.

Solution 07

1. We want to model the problem as a mathematical network $(\mathcal{N}, \mathcal{A})$.

   First we define the set of nodes $\mathcal{N}$:

   - A node is associated with each task.
   - We create an artificial node o representing the beginning of living room renovation,
   - and an artificial node d representing its ending.

   The set of arcs is defined as:

   - For each task j, we add an arc (i, j) for each predecessor i.
   - For each task j without predecessor, we define an arc (o, j).
   - For each task i without successor, we define an arc (i, d).

   A cost is associated with each arc (i, j) which is the duration of task i. The cost of each arc of the form (o, j) is zero. We obtain the network represented in Figure 18 on the next page.

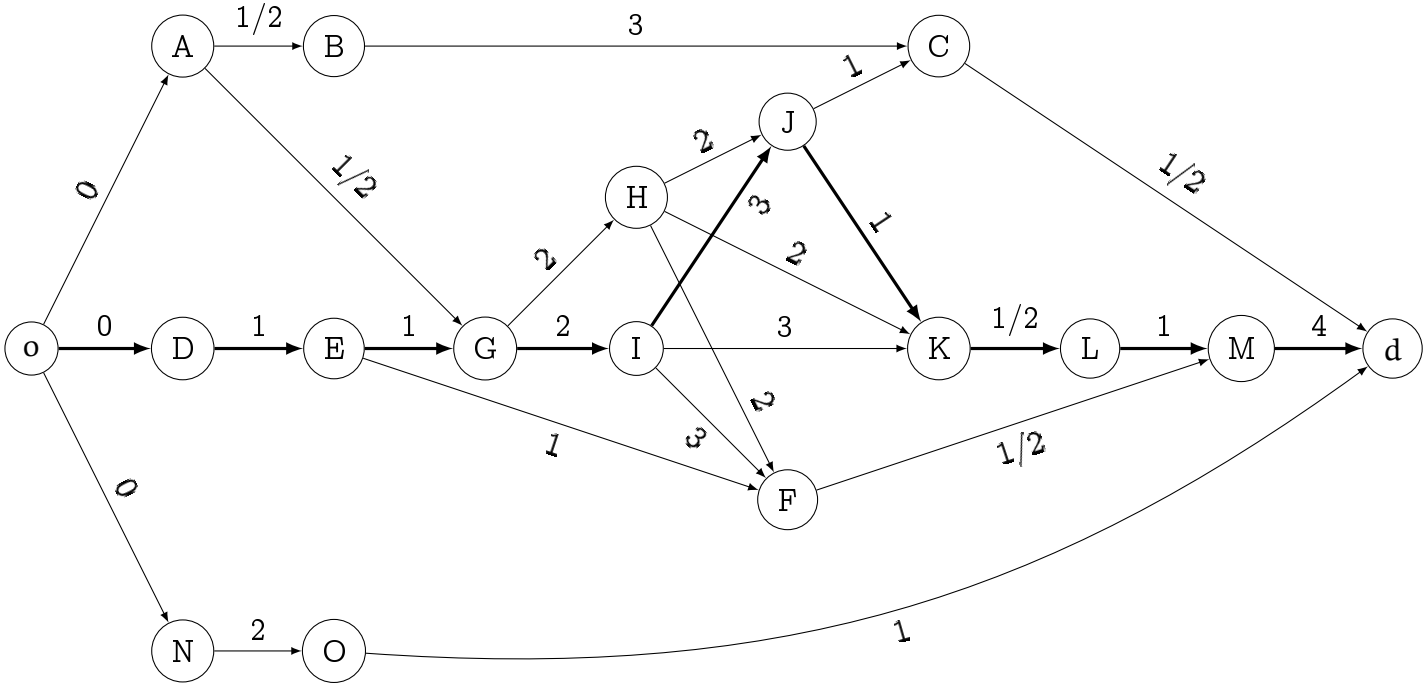


Figure 18: Network representation of the PERT problem

2. To find the longest path from o to d, we apply the shortest path algorithm over the network with the opposite sign of the costs. Each iteration of the algorithm is presented in Table 8 on the following page.

It.	S	i	λ_o	λ_A	λ_B	λ_C	λ_D	λ_E	λ_F	λ_G	λ_H	λ_I	λ_J	λ_K	λ_L	λ_M	λ_N	λ_O	λ_d
0	{o}	o	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	{A, D, N}	D	0	0	∞	∞	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	∞	∞
2	{A, E, N}	E	0	0	∞	∞	0	-1	∞	∞	∞	∞	∞	∞	∞	∞	0	∞	∞
3	{A, F, G, N}	G	0	0	∞	∞	0	-1	-2	-2	∞	∞	∞	∞	∞	∞	0	∞	∞
4	{A, F, H, I, N}	I	0	0	∞	∞	0	-1	-2	-2	-4	-4	∞	∞	∞	∞	0	∞	∞
5	{A, F, H, J, K, N}	J	0	0	∞	∞	0	-1	-7	-2	-4	-4	-7	-7	∞	∞	0	∞	∞
6	{A, C, F, H, K, N}	K	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	∞	∞	0	∞	∞
7	{A, C, F, H, L, N}	L	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	∞	0	∞	∞
8	{A, C, F, H, M, N}	M	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	∞
9	{A, C, F, H, N}	C	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
10	{A, F, H, N}	F	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
11	{A, H, N}	H	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
12	{A, N}	A	0	0	∞	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
13	{B, N}	B	0	0	-1/2	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
14	{N}	N	0	0	-1/2	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	∞	-13.5
15	{O}	O	0	0	-1/2	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	-2	-13.5
16	{}	-	0	0	-1/2	-8	0	-1	-7	-2	-4	-4	-7	-8	-8.5	-9.5	0	-2	-13.5

Table 8: Iterations of the shortest path algorithm to find the longest path from o to d.

According to each of the steps of shortest path algorithm we have the vector π

$$\pi_o = -1, \quad \pi_A = o, \quad \pi_B = A, \quad \pi_C = J, \quad \pi_D = o, \quad \pi_E = D, \quad \pi_F = I,$$

$$\pi_G = E, \quad \pi_H = G, \quad \pi_I = G, \quad \pi_J = I, \quad \pi_K = J, \quad \pi_L = K,$$

$$\pi_M = L, \quad \pi_N = o, \quad \pi_O = N \text{ and } \pi_d = M.$$

The longest path from o to d is represented bold in the above network. Each node among that path corresponds to a critical task. The optimal labels of the algorithm are the earliest start for each task (column δ_i in the Table below with opposite signs).

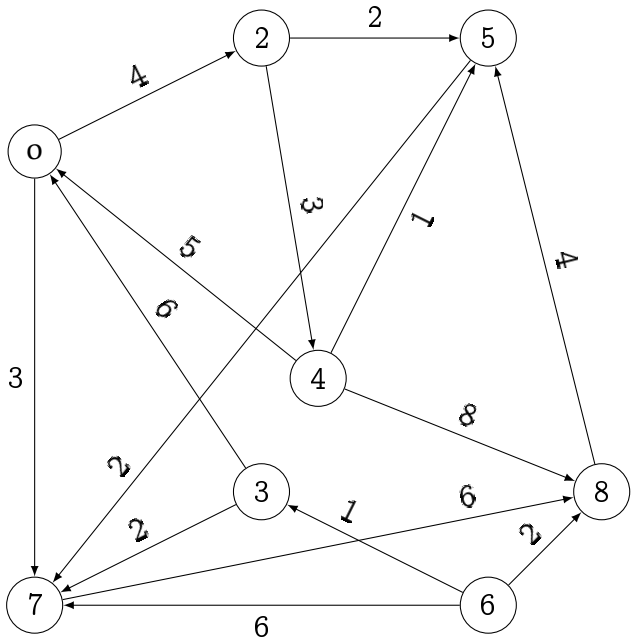
3. In order to obtain the latest start for each task, we have to invert the direction for each arc, and then calculate the longest path from d to o. This path is the same, with a cost of 13.5. For each node, we remove the optimal label 13.5 to obtain the latest start of the corresponding task (column φ_i in the Table below). Consider task O for instance. It does not precede any task, and lasts 1 day. Therefore, it can start at the latest at day $\phi_O = 13.5 - 1 = 12.5$, without delaying the project.

Ranking	Task	Duration	Precedence	Successor	Earliest start δ_i	Latest start φ_i
1	α	0	-	A, D, N	0	0
2	A	0.5	α	B, G	0	1.5
3	D	1	α	E	0	0
4	N	2	α	O	0	10.5
5	B	3	A	C	0.5	10
6	E	1	D	G, F	1	1
7	O	1	N	ω	2	12.5
8	G	2	A, E	H, I	2	2
9	H	2	G	F, J, K	4	5
10	I	3	G	F, J, K	4	4
11	J	1	H, I	C, K	7	7
12	F	0.5	E, H, I	M	7	9
13	C	0.5	B, J	ω	8	13
14	K	0.5	H, I, J	L	8	8
15	L	1	K	M	8.5	8.5
16	M	4	L, F	ω	9.5	9.5
17	ω	0	C,M,O	-	13.5	13.5

The minimum duration of the work is 13.5 days. The critical tasks are D, E, G, I, J, K, L and M (there are the only critical path).

Question 08

Consider the network represented in the figure below, where the value associated with each arc represents its length. Determine, when they exist, the shortest paths from node o to all other nodes in the network.



Solution 08

As all costs are non negative, we apply Dijkstra's algorithm in order to find the shortest paths from node o to all other nodes, if they exist. The iterations are detailed in Table 9.

Iteration	S	i	λ_o	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
0	$\{o\}$	o	0	∞	∞	∞	∞	∞	∞	∞
1	$\{2,7\}$	7	0	4	∞	∞	∞	∞	3	∞
2	$\{2,8\}$	2	0	4	∞	∞	∞	∞	3	9
3	$\{4,5,8\}$	5	0	4	∞	7	6	∞	3	9
4	$\{4,8\}$	4	0	4	∞	7	6	∞	3	9
5	$\{8\}$	8	0	4	∞	7	6	∞	3	9
6	$\{\}$	-	0	4	∞	7	6	∞	3	9

Table 9: Iterations of Dijkstra's algorithm.

We observe that there is no path between node o and nodes 3 and 6, as the labels of these nodes is ∞ at the end of the algorithm. Table 10 reports the values of π at each iteration of the shortest path algorithm, that is the preceding node in the shortest path (or -1 if the node has not been processed). It will allow us to build the tree of all shortest paths.

Iteration	π_0	π_2	π_3	π_4	π_5	π_6	π_7	π_8
0	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	o	-1	-1	-1	-1	o	-1
2	-1	o	-1	-1	-1	-1	o	-1
3	-1	o	-1	2	2	-1	o	7
4	-1	o	-1	2	2	-1	o	7
5	-1	o	-1	2	2	-1	o	7
6	-1	o	-1	2	2	-1	o	7

Table 10: Values of π at each iteration of the shortest path algorithm

Bellman's subnetwork is represented in Figure 19 on the next page, where the plain arcs are belong to the subnetwork, and dotted arcs are arcs from

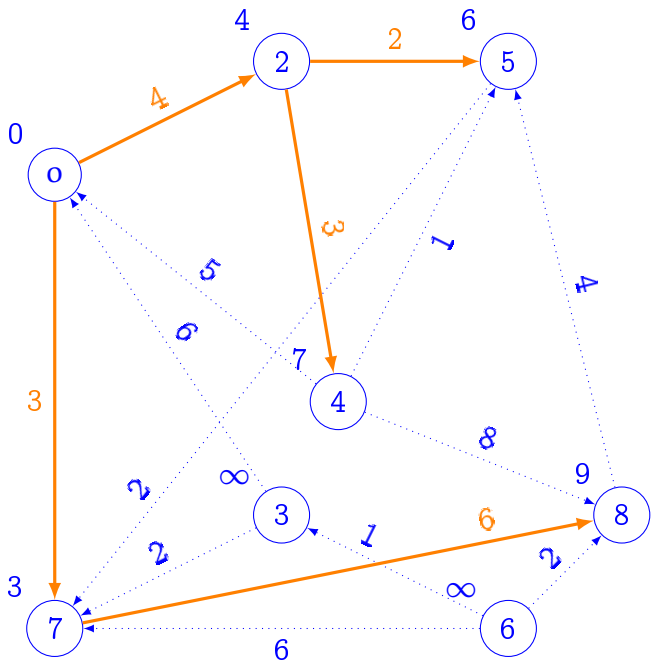


Figure 19: Bellman's subnetwork

the original network that do not belong to the subnetwork. It can be verified that

$$\lambda_j = \lambda_i + c_{ij},$$

for each arc (i, j) in the subnetwork. Bellman's subnetwork is the shortest path spanning tree.

Question 09

A museum hires attendants during opening hours, from 9:00 to 17:00. It has the possibility of hiring several people, each of them being available for several hours during the day (the museum can hire an attendant for only a portion of his available hours), at a given cost per hour, as reported in the table below.

Identify who the museum should hire so that there is at least one attendant present in the museum at any time, in order to minimize the total cost.

Hint: Model the problem as a shortest path problem. To do so, create an origin node from which one have to find the shortest path to all slot times.

Name	from	to	cost
Durin	9:00	13:00	30
Isumbras	9:00	11:00	18
Hamfast	12:00	15:00	14
Isengrim	12:00	17:00	38
Arathorn	14:00	17:00	16
Bilbo	13:00	16:00	22
Gelmir	16:00	17:00	9

Solution 09

We first model the problem as a mathematical network $(\mathcal{N}, \mathcal{A})$. We define the set of nodes $\mathcal{N}$ as the following:

- There is a node for each attendant.
- There is a node for each time slot.
- We define an artificial node o which is the origin.

The set of arcs $\mathcal{A}$ is defined as follow:

- There is an arc from the origin o to each attendant with cost zero.
- There is an arc between all pairs of attendant and time slot for which the attendant can work, with associated cost given in the table.

The network is represented in Figure 20 on the following page.

The objective is to minimize the total cost. Thus, we have to find the shortest paths from o to all other nodes. The details of the iterations are reported in Tables 11 and 12 on page 460.

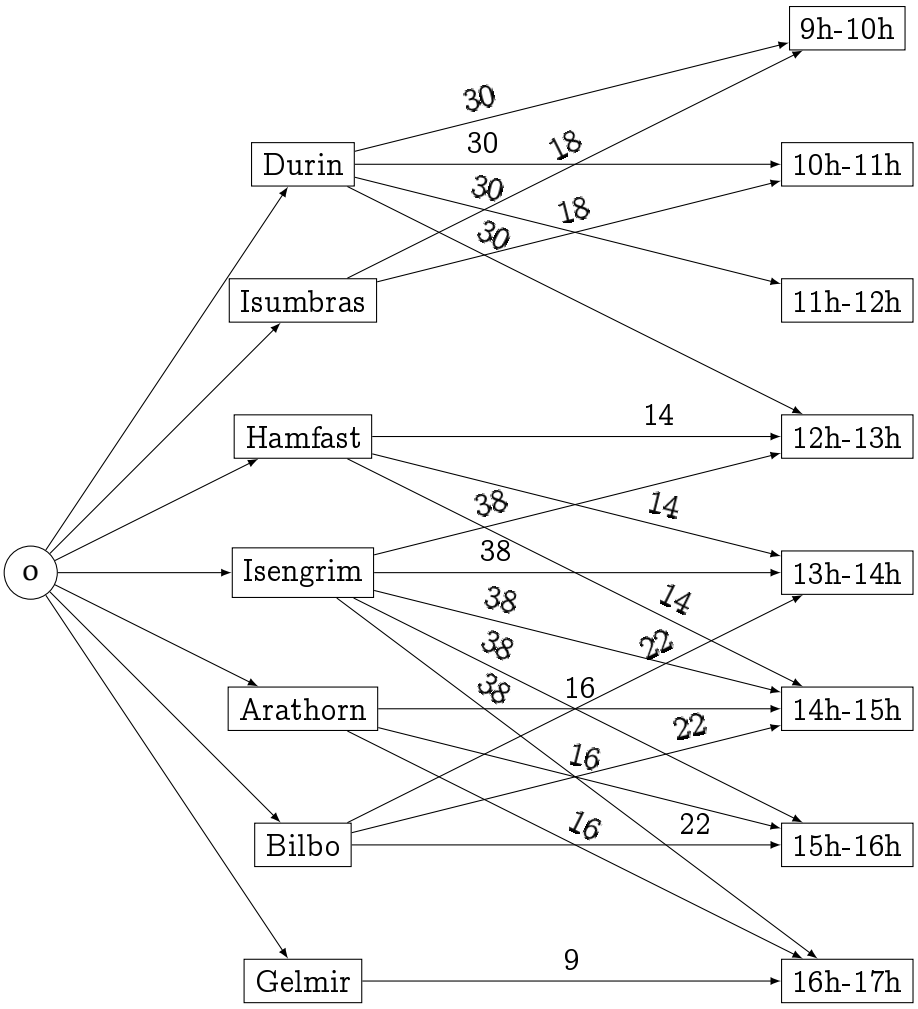


Figure 20: Network representation

It.	S	i	λ_o	λ_D	λ_{Isu}	λ_{Ha}	λ_5Ise	λ_A	λ_B	λ_G	λ_{9h-10h}	$\lambda_{10h-11h}$	$\lambda_{11h-12h}$	$\lambda_{12h-13h}$	$\lambda_{13h-14h}$	$\lambda_{14h-15h}$	$\lambda_{15h-16h}$	$\lambda_{16h-17h}$
0	{o}	o	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	{D, Isu, Ha, Ise, A, B, G}	D	0	0	0	0	0	0	0	0	∞	∞	∞	∞	∞	∞	∞	∞
2	{Isu, Ha, Ise, A, B, G}	Isu	0	0	0	0	0	0	0	0	30	30	30	30	∞	∞	∞	∞
3	{Ha, Ise, A, B, G}	Ha	0	0	0	0	0	0	0	0	18	18	30	30	∞	∞	∞	∞
4	{Ise, A, B, G}	Ise	0	0	0	0	0	0	0	0	18	18	30	14	14	14	∞	∞
5	{A, B, G}	A	0	0	0	0	0	0	0	0	18	18	30	14	14	14	38	38
6	{B, G}	B	0	0	0	0	0	0	0	0	18	18	30	14	14	14	16	16
7	{G}	G	0	0	0	0	0	0	0	0	18	18	30	14	14	14	16	16
6	{}	-	0	0	0	0	0	0	0	0	18	18	30	14	14	14	16	9

Table 11: Iteration of the shortest algorithm to find the shortest path from o to all nodes.

It.	π_o	π_D	π_{Isu}	π_{Ha}	π_5Ise	π_A	π_B	π_G	π_{9h-10h}	$\pi_{10h-11h}$	$\pi_{11h-12h}$	$\pi_{12h-13h}$	$\pi_{13h-14h}$	$\pi_{14h-15h}$	$\pi_{15h-16h}$	$\pi_{16h-17h}$
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	o	o	o	o	o	o	o	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	o	o	o	o	o	o	o	D	D	D	D	-1	-1	-1	-1
3	-1	o	o	o	o	o	o	o	Isu	Isu	D	D	-1	-1	-1	-1
4	-1	o	o	o	o	o	o	o	Isu	Isu	D	Ha	Ha	Ha	-1	-1
5	-1	o	o	o	o	o	o	o	Isu	Isu	D	Ha	Ha	Ha	Ise	Ise
6	-1	o	o	o	o	o	o	o	Isu	Isu	D	Ha	Ha	Ha	A	A
7	-1	o	o	o	o	o	o	o	Isu	Isu	D	Ha	Ha	Ha	A	A
6	-1	o	o	o	o	o	o	o	Isu	Isu	D	Ha	Ha	Ha	A	G

Table 12: Predecessor of each node for each iteration of the shortest path algorithm.

The optimal solution is represented in Figure 21 on the following page.

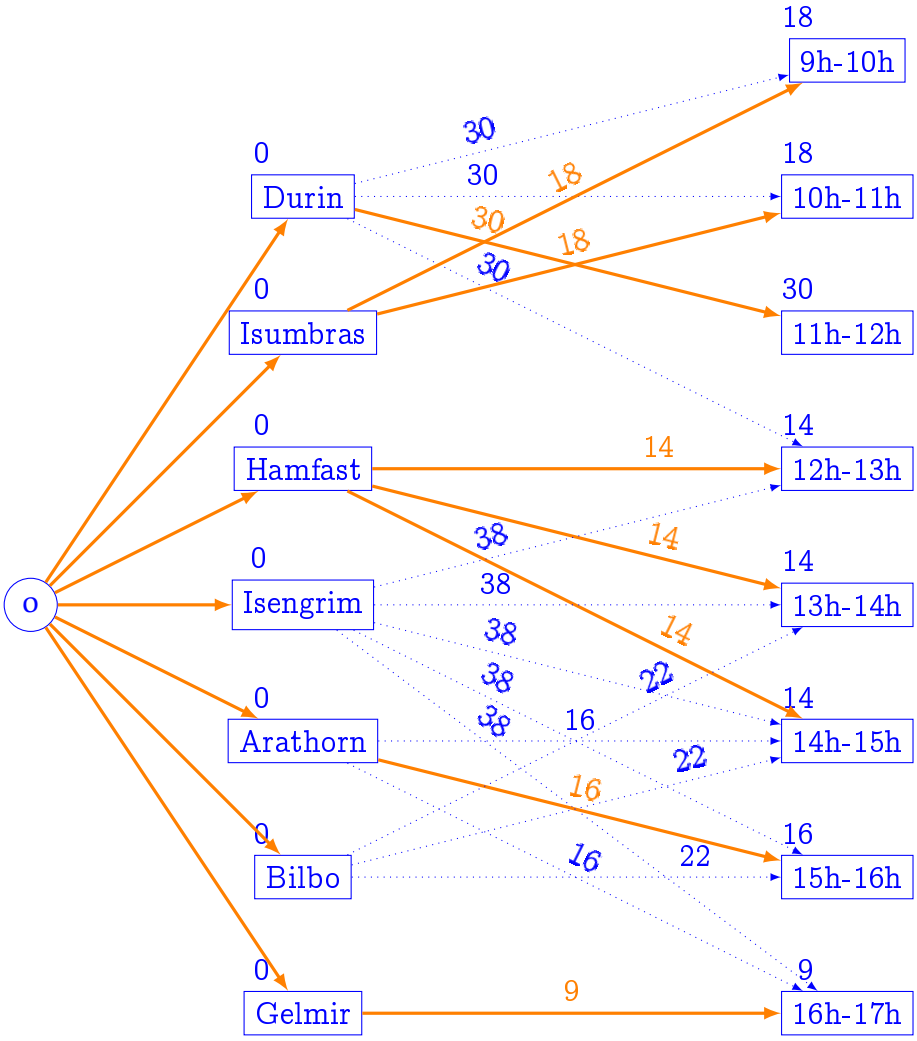
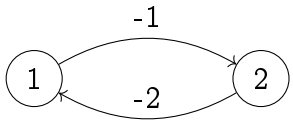


Figure 21: Representation of the optimal solution

Question 10

Consider the following network, with two nodes and two arcs. The cost of each arc is written next to it.



The shortest simple path between node 1 and node 2 is the arc (1,2). Check the optimality conditions (Theorem 23.1). What do you conclude?

Solution 10

Let λ_1 be the label of node 1. As arc (1,2) is in the shortest path, (23.4) tells us that

$$\lambda_2 = \lambda_1 + c_{12} = \lambda_1 - 1.$$

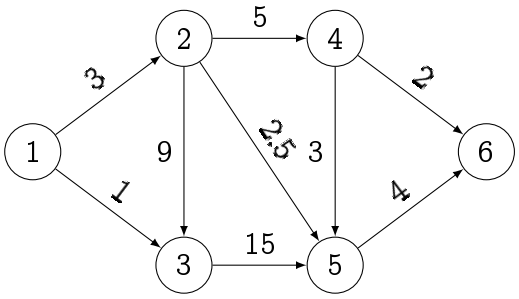
Consider now arc (2,1). The condition (23.3) says that

$$\lambda_1 \leq \lambda_2 + c_{21} = \lambda_1 - 1 - 2,$$

which is not possible. Indeed, the network contains a cycle with a negative cost (-3). In that case, the shortest **simple** path between 1 and 2 is indeed the arc (1,2). But the shortest path does not exist. Indeed, each time a path goes around the cycle, its cost decreases by 3 units, and the problem is unbounded. Finding the shortest **simple** path is a difficult problem, as there is no characterization with optimality conditions. Thanks to the optimality conditions of Theorem 23.1 (related to the transhipment problem), finding the shortest path is a simple problem. But it has no solution if the network contains a cycle with a negative cost (Theorem 23.2).

Question 11

Consider the following network, where the cost of each arc is reported next to it.



1. Apply Dijkstra's algorithm in order to find the shortest path from node 1 to all other nodes.

2. Select one iteration, and verify the invariants of Theorem 23.14.

Solution 11

1. We first note that all arcs in the network have non negative cost, so that Dijkstra's algorithm applies. It is the same algorithm as the shortest path algorithm, with two differences:

   - the node that is treated at each iteration is the node in $\mathcal{S}$ with the smallest label,
   - there is no need to identify a cycle with negative cost, as all arcs have non negative cost.

   The iterations of Dijkstra's algorithm are reported in the following table, where

   - the first column reports the number of the iteration,
   - the second column reports the set $\mathcal{S}$ of nodes to be treated at the beginning of the iteration,
   - the third column reports the node that is treated during the iteration,
   - the remaining columns report the values of the labels at the beginning of the iteration.

Iteration	S	i	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	$\{1\}$	1	0	∞	∞	∞	∞	∞
1	$\{2,3\}$	3	0	3	1	∞	∞	∞
2	$\{2,5\}$	2	0	3	1	∞	16	∞
3	$\{4,5\}$	5	0	3	1	8	5.5	∞
4	$\{4,6\}$	4	0	3	1	8	5.5	9.5
5	$\{6\}$	6	0	3	1	8	5.5	9.5
6	$\{\}$	-	0	3	1	8	5.5	9.5

   At the beginning of iteration 3, both nodes 4 and 5 can be treated. As $\lambda_4 = 8$ and $\lambda_5 = 5.5$, it is decided to treat node 5, as it has the smallest label.

2. We observe the invariants of the algorithm for iteration 4. Remember that the set of permanent labels is the set of nodes with a label different from $+\infty$, and not in $\mathcal{S}$. At the beginning of the iteration, the set of permanent labels is

$$\mathcal{T} = \{1, 2, 3, 5\},$$

and the set of nodes to be treated is

$$\mathcal{S} = \{4, 6\}.$$

We verify the invariants of the algorithm:

- During iteration 4, only the label of node 6 is updated. And no node in $\mathcal{T}$ is updated.
- No node in $\mathcal{T}$ is included in $\mathcal{S}$ during the iteration. Indeed, after the iteration, $\mathcal{S} = \{6\}$.
- The labels of any node i in $\mathcal{T}$ corresponds to the length of a shortest path from the origin to i. For instance, $\lambda_5 = 5.5$ corresponds to the length of path $1 \to 2 \to 5$, that is the shortest path between 1 and 5.
- The labels of nodes 4 and 6 (nodes not in $\mathcal{T}$) are larger than the labels of any node in $\mathcal{T}$.

Question 12

A traveling sales person is exploring the different routes to visit his customers. Since he can only visit one customer per day, he must always return home after visiting a customer. Therefore, he seeks the shortest path to all of his customers from his home, represented by node A. The network with the travel times is represented in Figure 22. To help the sales person, answer the following questions:

1. Dijkstra's algorithm can be used on this graph. Why?
2. How many iterations of Dijkstra's algorithm are needed to find the shortest path to all nodes from node A ? Justify your answer.
3. By using Dijkstra's algorithm, find the shortest path to all nodes from node A. Report the labels of all the nodes at each iteration.
4. Provide the shortest path spanning tree from node A. Specify the arcs and the labels of each node.
5. Justify for all arcs entering or leaving node E whether they belong or not to the shortest path spanning tree using Bellman's equation.

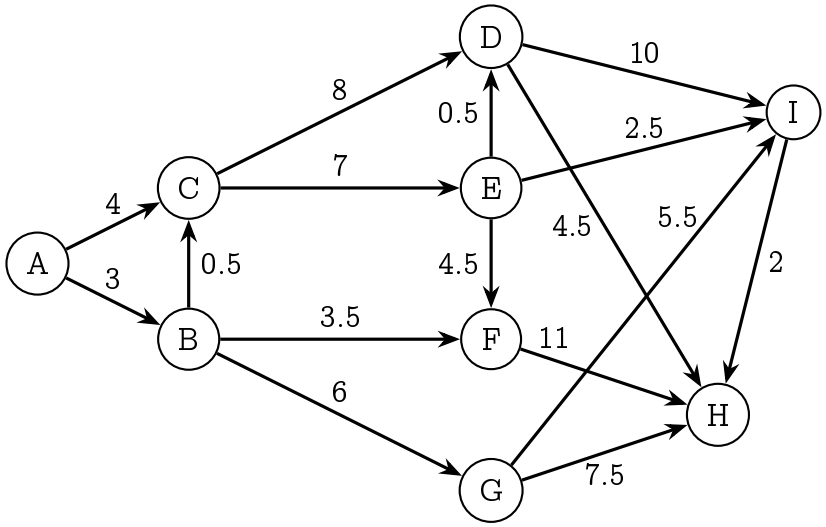


Figure 22: Graph with travel times.

Solution 12

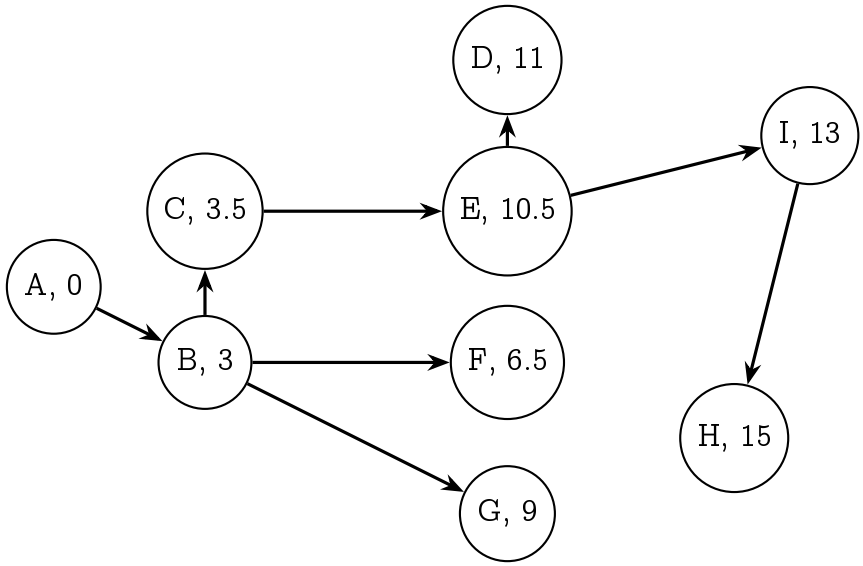
1. Because all costs are non negative.

2. There are 9 iterations. There is one iteration per node since all nodes must be visited to find the shortest path to all nodes.

3. The iterations of the algorithm are reported in the table below. The first column reports the number of the iteration. The second column reports the set of active nodes. The third column report the node that is treated at the corresponding iteration. And the following columns report the label of each node.

it.	S	i	λ_A	λ_B	λ_C	λ_D	λ_E	λ_F	λ_G	λ_H	λ_I
0	{A}	A	0	∞	∞	∞	∞	∞	∞	∞	∞
1	{B,C}	B	0	3	4	∞	∞	∞	∞	∞	∞
2	{C,F,G}	C	0	3	3.5	∞	∞	6.5	9	∞	∞
3	{D,E,F,G}	F	0	3	3.5	11.5	10.5	6.5	9	∞	∞
4	{D,E,G,H}	G	0	3	3.5	11.5	10.5	6.5	9	17.5	∞
5	{D,E,H,I}	E	0	3	3.5	11.5	10.5	6.5	9	16.5	14.5
6	{D,H,I}	D	0	3	3.5	11	10.5	6.5	9	16.5	13
7	{H,I}	I	0	3	3.5	11	10.5	6.5	9	15.5	13
8	{H}	H	0	3	3.5	11	10.5	6.5	9	15	13
9	$\emptyset$		0	3	3.5	11	10.5	6.5	9	15	13

4. The tree is represented below.



5. One arc enters E, and three leave. For the incoming arc, we need to check if the optimality condition is verified with equality or not.

$$\lambda_E \stackrel{?}{=} \lambda_C + c_{C,E} \rightarrow \quad 10.5 = 3.5 + 7 \rightarrow \text{in tree} \tag{37}$$

$$\tag{38}$$

and the outgoing arc:

$$\lambda_D \stackrel{?}{=} \lambda_E + c_{E,D} \rightarrow 11 = 10.5 + 0.5 \rightarrow \text{in tree} \tag{39}$$

$$\lambda_I \stackrel{?}{=} \lambda_E + c_{E,I} \rightarrow 13 = 10.5 + 2.5 \rightarrow \text{in tree} \tag{40}$$

$$\lambda_F \stackrel{?}{=} \lambda_E + c_{E,F} \rightarrow 6.5 < 10.5 + 4.5 \rightarrow \text{not in tree} \tag{41}$$

Therefore only the arc (E, F) is not in the shortest spanning tree. Arcs (C, E), (E, D) and (E, I) are in the tree.

8 Discrete optimization

Question 01

In order to participate in a cooking competition called "Top Chef", Frank must build a team of 10 excellent chefs. He has access to a pool of candidates (see Table 13). For each of them, he knows the age, the restaurant where they come from, and their cooking specialties (some candidates have only one, some others have two). He also knows the number of points that each of them has been able to collect in previous similar competitions. And he wants to build a team with the maximum number of such points. But several constraints apply:

1. The team must contain at least 3 chefs who are skilled in preparing appetizers.
2. The team must contain at least 4 chefs who are skilled in preparing fish.
3. The team must contain at least 4 chefs who are skilled in preparing meat.
4. The team must contain at least 3 chefs who are skilled in preparing dessert.
5. To promote young people, the team must contain at least 2 chefs who are 20 or less.
6. Yoda and Obi-Wan cannot stand each other, and it is not possible to have both of them in the team.
7. Han and Sheev have so much experience in working together that if one of the two is included in the team, the other must be too.
8. For the sake of fairness, not more than 3 chefs from the same restaurant should be hired in the same team.

Write an integer optimization problem to help Frank build his team.

Name	Age	Restaurant	Points	First specialty	Second specialty
Anakin	49	Lausanne	124	Fish	Meat
Chewbacca	39	Noirmont	105	Fish	Meat
Dooku	49	Crissier	118	Meat	Dessert
Gial	33	Mézières	134	Dessert	
Han	35	Crissier	160	Appetizer	
Jabba	47	Crissier	184	Fish	
Jubnuk	28	Neuchatel	101	Fish	
Lando	17	Lucens	116	Dessert	
Leia	18	Sierre	187	Appetizer	Fish
Lorth	41	Baulmes	132	Dessert	
Luke	37	Vufflens	181	Appetizer	
Obi-Wan	22	Vufflens	199	Fish	Meat
Padme	38	Cossonay	192	Appetizer	
Qui-Gon	19	Cossonay	136	Fish	
Sebulba	43	Valeyres	123	Meat	
Sheev	21	Crissier	127	Meat	
Teebo	32	Sierre	132	Meat	
Watto	20	Sierre	131	Meat	
Yoda	39	Noirmont	102	Appetizer	Fish
Zuckuss	42	Vufflens	123	Dessert	

Table 13: Data about each chef

Solution 01

In this problem, Frank wants to maximize the number of points of his team, given some constraints. First, the decision variables are the different candidates:

x_1: Anakin,	x_{11}: Luke,
x_2: Chewbacca,	x_{12}: Obi-Wan,
x_3: Dooku,	x_{13}: Padme,
x_4: Gial,	x_{14}: Qui-Gon,
x_5: Han,	x_{15}: Sebulba,
x_6: Jabba,	x_{16}: Sheev,
x_7: Jubnuk,	x_{17}: Teebo,
x_8: Lando,	x_{18}: Watto,
x_9: Leia,	x_{19}: Yoda,
x_{10}: Lorth,	x_{20}: Zuckuss.

Let c_i denote the number of points of candidate i. Since Frank wants to maximize the number of points in his team, we get the objective function

$$\max_{x \in \{0,1\}^{20}} \sum_{i=1}^{20} c_i x_i$$

Then, we have to model each constraint:

1. The team must contain 10 chefs:
$$\sum_{i=1}^{20} x_i = 10.$$
2. The team must contain at least 3 chefs who are skilled in preparing appetizers:
$$x_5 + x_9 + x_{11} + x_{13} + x_{19} \geq 3.$$
3. The team must contain at least 4 chefs who are skilled in preparing fish:
$$x_1 + x_2 + x_6 + x_7 + x_9 + x_{12} + x_{14} + x_{19} \geq 4.$$
4. The team must contain at least 4 chefs who are skilled in preparing meat:
$$x_1 + x_2 + x_3 + x_{12} + x_{15} + x_{16} + x_{17} + x_{18} \geq 4.$$

5. The team must contain at least 3 chefs who are skilled in preparing dessert.
$$x_3 + x_4 + x_8 + x_{10} + x_{20} \geq 3.$$

6. To promote young people, the team must contain at least 2 chefs who are 20 or less.
$$x_8 + x_9 + x_{14} + x_{18} \geq 2.$$

7. Yoda and Obi-Wan cannot stand each other, and it is not possible to have both of them in the team.
$$x_{12} + x_{19} \leq 1.$$

8. Han and Sheev have so much experience in working together that if one of the two is included in the team, the other must be too.
$$x_5 = x_{16}.$$

9. For the sake of fairness, not more than 3 chefs from the same restaurant should be hired in the same team. We observe that 4 candidates come from Crissier but this is the only restaurant with more than 3 chefs.
$$x_3 + x_5 + x_6 + x_{16} \leq 3.$$

The optimization problem is
$$\max_{x \in \{0,1\}^{20}} \sum_{i=1}^{20} c_i x_i$$

subject to

$$\begin{aligned}
\sum_{i=1}^{20} x_i &= 10,\\
x_5 + x_9 + x_{11} + x_{19} &\geq 3,\\
x_1 + x_2 + x_6 + x_7 + x_9 + x_{12} + x_{14} + x_{19} &\geq 4,\\
x_1 + x_2 + x_3 + x_{12} + x_{15} + x_{16} + x_{17} + x_{18} &\geq 4,\\
x_3 + x_4 + x_8 + x_{10} + x_{20} &\geq 3,\\
x_8 + x_9 + x_{14} + x_{18} &\geq 2,\\
x_{12} + x_{19} &\leq 1,\\
x_5 &= x_{16},\\
x_3 + x_5 + x_6 + x_{16} &\leq 3,\\
x_i \in \{0,1\}, \quad \forall i &= 1, ..., 20.
\end{aligned}$$

The best team verifying all the constraints is composed of

- Anakin,
- Dooku,
- Gial,
- Jabba,
- Leia,
- Lorth,
- Luke,
- Obi-Wan,
- Padme, and
- Watto,

for a total of 1582 points. In terms of specialties, there are

- 3 chefs specialized in appetizers,
- 4 chefs specialized in fish,

- 4 chefs specialized in meat,
- 3 chefs specialized in desserts.

There are two youngs chefs. As Obi-Wan is part of the team, Yoda is not. Han and Sheev have not been included in the team. In terms of restaurant, we have

- 1 chef from Lausanne,
- 2 chefs from Crissier,
- 1 chef from Mézières,
- 2 chefs from Sierre,
- 1 chef from Baulmes,
- 2 chefs from Vufflens,
- 1 chef from Cossonay.

Question 02

You want to hire a group of workers for your job site. You have at your disposal 7 candidates:

- Asmodeus,
- Belphegor,
- Beelzebub,
- Leviathan,
- Lucifer,
- Mammon,
- Satan.

Each decision is modeled by a binary variable x_i which is 1 if the worker i is hired, and 0 otherwise. Model the following constraints:

1. not all candidates can be hired;
2. at least one candidate must be hired;
3. Asmodeus can not be hired if Beelzebub is;
4. Leviathan can be hired only if Belphegor is;
5. Asmodeus and Lucifer are not separable, and have to be hired both, or not at all;
6. at least one of the two following constraints must be verified: (i) at least one of the candidates in the group Asmodeus, Belphegor, Beelzebub is chosen, or (ii) at least two in the group Belphegor, Leviathan, Lucifer, Mammon.

Provide a feasible solution.

Solution 02

The following binary decision variables are equal to 1 if the corresponding candidate is hired, and 0 otherwise:

- x_1: Asmodeus,
- x_2: Belphegor,
- x_3: Beelzebub,
- x_4: Leviathan,
- x_5: Lucifer,
- x_6: Mammon,
- x_7: Satan.

The constraints are modeled as follows.

1. Not all candidates can be hired. It means that not all variables can take the value one. We require the sum not to exceed 6:

$$\sum_{i=1}^{7} x_i \leq 6.$$

2. At least one candidate must be hired. At least one variable must take the value one. We require the sum to be at least equal to one.

$$\sum_{i=1}^{7} x_i \geq 1.$$

3. Asmodeus can not be hired if Beelzebub is. The fact that Asmodeus is not hired is represented by a negation: $1 - x_1$. The implication is then modeled as

$$x_3 \leq 1 - x_1,$$

or, equivalently,

$$x_1 + x_3 \leq 1.$$

4. Leviathan can be hired only if Belphegor is. If $x_4 = 1$, then $x_2 = 1$. It is modeled as

$$x_4 \leq x_2,$$

or equivalently,

$$x_2 - x_4 \geq 0.$$

5. Asmodeus and Lucifer are not separable, and have to be hired both, or not at all. Basically, it means that the decision for one is the same as the decision for the other one. The constraint is

$$x_1 = x_5,$$

or, equivalently,

$$x_1 - x_5 = 0.$$

6. At least one of the two following constraints must be verified: (i) at least one of the candidates in the group Asmodeus, Belphegor, Beelzebub is chosen, or (ii) at least two in the group Belphegor, Leviathan, Lucifer, Mammon. If means, either

$$x_1 + x_2 + x_3 \geq 1, \tag{42}$$

or

$$x_2 + x_4 + x_5 + x_6 \geq 2. \tag{43}$$

Let's first remind the theory on disjunctive constraints. We consider two constraints:

- Constraint 1: $f(x) \geq a$.
- Constraint 2: $g(x) \geq b$.
- One of them must be verified, but not necessarily both.

We assume that f and g are bounded from below.

$$f(x) \geq L_f \text{ and } g(x) \geq L_g \text{ are always true.}$$

We introduce a binary variable y, and define the two following constraints:

$$\begin{aligned} f(x) &\geq L_f + (a - L_f)y \\ g(x) &\geq L_g + (b - L_g)(1 - y). \end{aligned}$$

In our case, we have

$$\begin{aligned} f(x) &= x_1 + x_2 + x_3, \\ a &= 1, \\ L_f &= 0, \\ g(x) &= x_2 + x_4 + x_5 + x_6, \\ b &= 2, \\ L_g &= 0. \end{aligned}$$

Therefore, the two constraints are

$$\begin{aligned} x_1 + x_2 + x_3 &\geq y, \\ x_2 + x_4 + x_5 + x_6 &\geq 2(1 - y). \end{aligned}$$

Note that, if $y = 0$, the first constraint is

$$x_1 + x_2 + x_3 \geq 0,$$

and is always verified, for any x. The second constraint is

$$x_2 + x_4 + x_5 + x_6 \geq 2,$$

which is constraint (43).

If $y = 1$, the first constraint is

$$x_1 + x_2 + x_3 \geq 1,$$

which is constraint (42). The second constraint is

$$x_2 + x_4 + x_5 + x_6 \geq 0,$$

and is always verified. Note also that it is possible for the two constraints to be verified at the same time. This is illustrated in the feasible solution provided below.

One feasible solution is

$$\begin{aligned} x_1 &= 0, & x_2 &= 1, \\ x_3 &= 1, & x_4 &= 1, \\ x_5 &= 0, & x_6 &= 1, \\ x_7 &= 1, & y &= 1. \end{aligned}$$

It consists in hiring Belphegor, Beelzebub, Leviathan, Mammon, and Satan. We verify the constraints:

$$\begin{aligned}
\sum_{i=1}^{7} x_i = 5 &\leq 6, \\
\sum_{i=1}^{7} x_i = 5 &\geq 1, \\
x_1 + x_3 = 1 &\leq 1, \\
x_2 - x_4 = 0 &\geq 0, \\
x_1 - x_5 = 0, & \\
x_1 + x_2 + x_3 - y = 1 &\geq 0, \\
x_2 + x_4 + x_5 + x_6 + 2y - 2 = 3 &\geq 0.
\end{aligned}$$

Question 03

Patricia is about to spend several days on a long hike in the mountain to climb the Bishorn. She is now preparing her knapsack and thinking about which items to take. The alpine guides strongly recommend carrying no more than 7kg. Therefore, Patricia has to decide which items to carry and which items to leave at home. Each of them has a different level of importance. While it is critical to carry water and food it is less critical to carry a laptop. For each item i considered by Patricia, she knows its weight in kilos $w_i \geq 0$ and its value $u_i \in \mathbb{R}$.

Item	Value	Weight
Laptop	1	1
Water	4	3
Food	5	4
Clothes	7	5

1. Write the mathematical model for this integer optimization problem.
2. Determine which items Patricia should take in order to maximize the total utility while satisfying the maximum weight constraints.

Solution 03

We introduce a binary variable x_i for each item. It is one if the item is included in the sack, and 0 otherwise.

The knapsack problem is formulated as follows:

$$\max_{x \in \{0,1\}^4} \sum_{i=1}^{4} u_i x_i,$$

subject to:

$$\sum_{i=1}^{4} w_i x_i \leq W,$$
$$x_i \in \{0,1\},\ i = 1, \ldots, 4,$$

where u_i is the utility of item i, w_i is the weight of item i and W is the capacity of the sack. In our case,

$$\max_{x \in \{0,1\}^4} x_1 + 4x_2 + 5x_3 + 7x_4,$$

subject to

$$x_1 + 3x_2 + 4x_3 + 5x_4 \leq 7,$$
$$x_i \in \{0,1\},\ i = 1, \ldots, 4.$$

To identify the best sack, we enumerate all possible solutions. There are $2^4 = 16$ of them. For each of them, we calculate their total weight, and total utility. Then, we select the combination associated with the highest total utility among those that verify the capacity constraint.

x_1	x_2	x_3	x_4	$u^T x$	$w^T x$	Feasible?
0	0	0	0	0	0	YES
0	0	0	1	7	5	YES
0	0	1	0	5	4	YES
0	0	1	1	12	9	NO
0	1	0	0	4	3	YES
0	1	0	1	11	8	NO
0	**1**	**1**	**0**	**9**	**7**	**YES**
0	1	1	1	16	12	NO
1	0	0	0	1	1	YES
1	0	0	1	8	6	YES
1	0	1	0	6	5	YES
1	0	1	1	13	10	NO
1	1	0	0	5	4	YES
1	1	0	1	12	9	NO
1	1	1	0	10	8	NO
1	1	1	1	17	13	NO

The optimal solution consists in taking water and food, and no laptop and no clothes. The total utility is 9, and the total weight is 7.

Question 04

You need to recruit software developers from the following team of outstanding computer scientists: Alan, Bjarne, Tim, and Donald. Each of them is expert in at least one programming language:

- Alan in C and C++,
- Bjarne in C++ and Java,
- Tim in C++, Ruby and Python,
- Donald in C and Java.

Your job is to hire a team of experts, with the following requirements:

1. each language must be covered, in the sense that, for each language, there should be at least one person in the team mastering it,
2. the team must be as small as possible.

Model that set covering problem as an integer optimization problem. Provide also a feasible solution.

Solution 04

The set $\mathcal{U}$ of elements is the set of programming languages:

$$\mathcal{U} = \{\text{C, C++, Java, Python, Ruby}\}.$$

Each programmer $i = 1, \ldots, 4$ is associated with a subset S_i of $\mathcal{U}$, which corresponds to the known programming languages. Thus, in our case we have $n = 4$ subsets:

- $S_1 = \{\text{C, C++}\}$,
- $S_2 = \{\text{Java, C++}\}$,
- $S_3 = \{\text{Ruby, C++, Python}\}$,
- $S_4 = \{\text{C, Java}\}$.

They are characterized by binary parameters a_{ij} that is 1 if language j is known by person i, and 0 otherwise.

$$\begin{aligned}
a_{1,\text{C}} &= 1, & a_{2,\text{C}} &= 0, & a_{3,\text{C}} &= 0, & a_{4,\text{C}} &= 1,\\
a_{1,\text{C++}} &= 1, & a_{2,\text{C++}} &= 1, & a_{3,\text{C++}} &= 1, & a_{4,\text{C++}} &= 0,\\
a_{1,\text{Java}} &= 0, & a_{2,\text{Java}} &= 1, & a_{3,\text{Java}} &= 0, & a_{4,\text{Java}} &= 1,\\
a_{1,\text{Python}} &= 0, & a_{2,\text{Python}} &= 0, & a_{3,\text{Python}} &= 1, & a_{4,\text{Python}} &= 0,\\
a_{1,\text{Ruby}} &= 0, & a_{2,\text{Ruby}} &= 0, & a_{3,\text{Ruby}} &= 1, & a_{4,\text{Ruby}} &= 0.
\end{aligned}$$

The decision variables are x_1, x_2, x_3, x_4. They are binary variables representing whether person i is involved in the team. For each language, we need a constraint that requires that at least one person masters it. For example, for C we have

$$\begin{aligned}
\sum_{i=1}^{4} a_{i,\text{C}} x_i &\geq 1,\\
x_1 + x_4 &\geq 1.
\end{aligned}$$

The set of covering constraints is therefore:

$$\begin{aligned} x_1 + x_4 &\geq 1, \\ x_1 + x_2 + x_3 &\geq 1, \\ x_2 + x_4 &\geq 1, \\ x_3 &\geq 1, \\ x_3 &\geq 1. \end{aligned}$$

As we want to minimize the number of subsets, the costs c_i of each subset must all have the same value, say 1. The optimization problem is therefore

$$\min_{x \in \{0,1\}^4} x_1 + x_2 + x_3 + x_4,$$

subject to

$$\begin{aligned} x_1 + x_4 &\geq 1, \\ x_1 + x_2 + x_3 &\geq 1, \\ x_2 + x_4 &\geq 1, \\ x_3 &\geq 1, \\ x_3 &\geq 1, \\ x_i &\in \{0,1\},\ i = 1, \ldots, 4. \end{aligned}$$

It is clear that x_3 must be included, as Tim is the only one who can cover Python and Ruby. It is also clear that at least another person must be hired, as Tim does not know C and Java. These two languages can be covered by Donald, so that we cover all languages with two persons. The optimal solution is therefore:

$$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1.$$

The total cost is 2, and the constraints are satisfied:

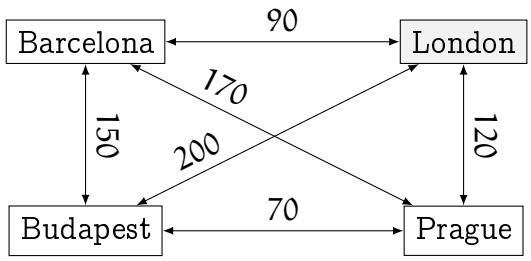
$$\begin{aligned} x_1 + x_4 &= 1 \geq 1, \\ x_1 + x_2 + x_3 &= 1 \geq 1, \\ x_2 + x_4 &= 1 \geq 1, \\ x_3 &= 1 \geq 1, \\ x_3 &= 1 \geq 1. \end{aligned}$$

Question 05

Anna and Tom live in London and want to plan a road trip during their holidays. They want to visit 3 cities: Barcelona, Budapest and Prague. They would like to minimize the cost of their trip by ordering the cities in a smart way. The cost to go

- between Barcelona and Budapest is 150.- CHF,
- between Barcelona and Prague is 170.- CHF,
- between Barcelona and London is 90.- CHF,
- between London and Prague is 120.- CHF,
- between London and Budapest is 200.- CHF,
- between Budapest and Prague is 70.- CHF,

as summarized in the figure below.



1. Model that traveling salesman problem as an integer optimization problem.
2. Consider one possible tour. Provide the corresponding values of the decision variables of the optimization problem, and show that they verify the constraints.

Solution 05

We denote $\mathcal{N}$ the set of n nodes. The decision variables are binary variables x_{ij}, with $i \in \mathcal{N}$, $j \in \mathcal{N}$ that are equal to 1 if Anna and Tom visit city j just after city i, and is equal to 0 otherwise. As Anna and Tom want to minimize their travel costs, the objective function is:

$$\begin{aligned}\min_{x\in\{0,1\}^{12}} \; & 150x_{B,Bu} + 170x_{B,P} + 90x_{B,L} \\ & + 120x_{L,P} + 200x_{L,Bu} + 90x_{L,B} \\ & + 70x_{Bu,P} + 150x_{Bu,B} + 200x_{Bu,L} \\ & + 170x_{P,B} + 120x_{P,L} + 70x_{P,Bu}.\end{aligned}$$

Each city must have exactly one successor, that is

$$\begin{aligned} x_{B,Bu} + x_{B,P} + x_{B,L} &= 1, \\ x_{L,P} + x_{L,Bu} + x_{L,B} &= 1, \\ x_{Bu,P} + x_{Bu,B} + x_{Bu,L} &= 1, \\ x_{P,B} + x_{P,L} + x_{P,Bu} &= 1. \end{aligned}$$

Furthermore, each city must have exactly one predecessor, that is

$$\begin{aligned} x_{Bu,B} + x_{P,B} + x_{L,B} &= 1, \\ x_{P,L} + x_{Bu,L} + x_{B,L} &= 1, \\ x_{P,Bu} + x_{B,Bu} + x_{L,Bu} &= 1, \\ x_{B,P} + x_{L,P} + x_{Bu,P} &= 1. \end{aligned}$$

Finally, to eliminate subtours, we introduce the integer variables y_i, for each city i except London, modeling the position of city i in the tour. The subtour elimination constraint imposes that, if $x_{ij} = 1$, then $y_j \geq y_i + 1$. It can be modeled as

$$x_{ij}(n-1) \leq y_j - y_i + n - 2, \qquad \forall i, j \in \mathcal{N} \setminus \{\text{London}\},$$

where $n = 4$. If $x_{ij} = 1$, that is, if city j is visited just after city i, the constraint becomes

$$1 \cdot 3 \leq y_j - y_i + 4 - 2,$$

or, equivalently,

$$y_j \geq y_i + 1,$$

that is, the position of city j in the tour must be after city j.

If $x_{ij} = 0$, that is if city j is not the city visited after city i, the constraint becomes

$$0 \leq y_j - y_i + 4 - 2,$$

or, equivalently,

$$y_i - y_j \leq 2,$$

which is always verified. Indeed, as London is not numbered, the numbering can be 1, 2, or 3, and the largest difference is 2. In general, any difference $y_j - y_i$ is less or equal to $n - 2$.

The subtour elimination constraints are:

$$\begin{aligned}
3x_{B,Bu} + y_B - y_{Bu} &\leq 2,\\
3x_{Bu,B} + y_{Bu} - y_B &\leq 2,\\
3x_{P,B} + y_P - y_B &\leq 2,\\
3x_{B,P} + y_B - y_P &\leq 2,\\
3x_{Bu,P} + y_{Bu} - y_P &\leq 2,\\
3x_{P,Bu} + y_P - y_{Bu} &\leq 2.
\end{aligned}$$

To sum up, the mathematical model for the given problem is the following:

$$\begin{aligned}\min_{x\in\{0,1\}^{12}}\ & 150x_{B,Bu} + 170x_{B,P} + 90x_{B,L} \\ & + 120x_{L,P} + 200x_{L,Bu} + 90x_{L,B} \\ & + 70x_{Bu,P} + 150x_{Bu,B} + 200x_{Bu,L} \\ & + 170x_{P,B} + 120x_{P,L} + 70x_{P,Bu}\end{aligned}$$

subject to

$$\begin{aligned}
&x_{B,Bu} + x_{B,P} + x_{B,L} = 1,\\
&x_{L,P} + x_{L,Bu} + x_{L,B} = 1,\\
&x_{Bu,P} + x_{Bu,B} + x_{Bu,L} = 1,\\
&x_{P,B} + x_{P,L} + x_{P,Bu} = 1,\\
&x_{Bu,B} + x_{P,B} + x_{L,B} = 1,\\
&x_{P,L} + x_{Bu,L} + x_{B,L} = 1,\\
&x_{P,Bu} + x_{B,Bu} + x_{L,Bu} = 1,\\
&x_{B,P} + x_{L,P} + x_{Bu,P} = 1,\\
&3x_{B,Bu} + y_B - y_{Bu} \le 2,\\
&3x_{Bu,B} + y_{Bu} - y_B \le 2,\\
&3x_{P,B} + y_P - y_B \le 2,\\
&3x_{B,P} + y_B - y_P \le 2,\\
&3x_{Bu,P} + y_{Bu} - y_P \le 2,\\
&3x_{P,Bu} + y_P - y_{Bu} \le 2,\\
&y_i \in \{1,2,3\}, \qquad \forall i \in \mathcal{N}\setminus\{L\},\\
&x_{i,j} \in \{0,1\}, \qquad \forall i,j \in \mathcal{N}.
\end{aligned}$$

Consider the tour London $\to$ Barcelona $\to$ Budapest $\to$ Prague. The corresponding decision variables are

$$\begin{aligned}
x_{L,B} &= 1, & x_{L,Bu} &= 0, & x_{L,P} &= 0,\\
x_{B,L} &= 0, & x_{B,Bu} &= 1, & x_{B,P} &= 0,\\
x_{Bu,L} &= 0, & x_{Bu,B} &= 0, & x_{Bu,P} &= 1,\\
x_{P,L} &= 1, & x_{P,B} &= 0, & x_{P,Bu} &= 0,\\
y_B &= 1, & y_{Bu} &= 2, & y_P &= 3.
\end{aligned}$$

We can verify the constraints:

$$
\begin{aligned}
x_{B,Bu} + x_{B,P} + x_{B,L} &= 1 + 0 + 0 && = 1,\\
x_{L,P} + x_{L,Bu} + x_{L,B} &= 0 + 0 + 1 && = 1,\\
x_{Bu,P} + x_{Bu,B} + x_{Bu,L} &= 1 + 0 + 0 && = 1,\\
x_{P,B} + x_{P,L} + x_{P,Bu} &= 0 + 1 + 0 && = 1,\\
x_{Bu,B} + x_{P,B} + x_{L,B} &= 0 + 0 + 1 && = 1,\\
x_{P,L} + x_{Bu,L} + x_{B,L} &= 1 + 0 + 0 && = 1,\\
x_{P,Bu} + x_{B,Bu} + x_{L,Bu} &= 0 + 1 + 0 && = 1,\\
x_{B,P} + x_{L,P} + x_{Bu,P} &= 0 + 0 + 1 && = 1,\\
3x_{B,Bu} + y_B - y_{Bu} &= 3 + 1 - 2 = 2 && \leq 2,\\
3x_{Bu,B} + y_{Bu} - y_B &= 2 - 1 = 1 && \leq 2,\\
3x_{P,B} + y_P - y_B &= 3 - 1 = 2 && \leq 2,\\
3x_{B,P} + y_B - y_P &= 1 - 3 = -2 && \leq 2,\\
3x_{Bu,P} + y_{Bu} - y_P &= 3 + 2 - 3 = 2 && \leq 2,\\
3x_{P,Bu} + y_P - y_{Bu} &= 3 - 2 = 1 && \leq 2.
\end{aligned}
$$

Question 06

The company Alpiq is investing in hydro-electricity. The engineers have identified four potential location to build new dams. For each location, they have assessed the investment costs as well as the long-term expected revenues:

	Location	Cost (mCHF)	Revenues (mCHF)
1	China	50	160
2	Iran	70	220
3	Brazil	40	120
4	India	30	80

The company has a budget of 140 mCHF. Where should the company invest in new dams?

1. What type of optimization problem is it?
2. Formulate the problem as an integer optimization problem.
3. Solve the relaxation of the problem, and use the solution of the relaxation to advise the company.
4. Solve the problem by full enumeration, and use the solution to advise the company. Compare the two advises.

Solution 06

1. This is a knapsack problem.

2. The decision variables are binary variables: x_i is 1 if a dam is built at location i, and 0 otherwise.

   The objective is to maximize the expected revenues:

   $$\max_{x \in \{0,1\}^4} 16x_1 + 22x_2 + 12x_3 + 8x_4.$$

   Note that we are modeling the problem using 10mCHF as the base unit.

   The budget constraint is given by

   $$5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14.$$

   Therefore, the integer optimization problem is:

   $$\max_{x \in \{0,1\}^4} 16x_1 + 22x_2 + 12x_3 + 8x_4,$$

   subject to

   $$\begin{aligned} 5x_1 + 7x_2 + 4x_3 + 3x_4 &\leq 14, \\ x_1, x_2, x_3, x_4 &\in \{0, 1\}. \end{aligned}$$

   It is actually a knapsack problem.

3. The relaxation of the problem is

   $$\max_{x \in \{0,1\}^4} 16x_1 + 22x_2 + 12x_3 + 8x_4,$$

   subject to

   $$\begin{aligned} 5x_1 + 7x_2 + 4x_3 + 3x_4 &\leq 14, \\ x_1, x_2, x_3, x_4 &\geq 0, \\ x_1, x_2, x_3, x_4 &\leq 1. \end{aligned}$$

   The optimal solution of the relaxation can be calculated using the simplex algorithm. It is

   $$x_1 = 1,\ x_2 = 1,\ x_3 = 0.5,\ x_4 = 0.$$

For the company, this suggests to build dams 1 and 2, and not to build dam 4. Regarding dam 3, building it would not fit the budget. Building dams 1 and 2 will cost 120 mCHF, and the expected benefit is 380 mCHF.

4. We enumerate the feasible solutions. For each of them, we calculate the revenues, the cost, and we check if it is feasible for the budget.

x_1	x_2	x_3	x_4	Revenues	Cost	Feasible?
0	0	0	0	0	0	YES
0	0	0	1	8	3	YES
0	0	1	0	12	4	YES
0	0	1	1	20	7	YES
0	1	0	0	22	7	YES
0	1	0	1	30	10	YES
0	1	1	0	34	11	YES
0	1	1	1	42	14	YES
1	0	0	0	16	5	YES
1	0	0	1	24	8	YES
1	0	1	0	28	9	YES
1	0	1	1	36	12	YES
1	1	0	0	38	12	YES
1	1	0	1	46	15	NO
1	1	1	0	50	16	NO
1	1	1	1	58	19	NO

The optimal solution is to build dams 2, 3 and 4. The budget is completely invested, and the expected revenues are 420 mCHF. Although the first dam has the highest return, it consumes too much of the budget. It is therefore better not to build it, and invest in the three others. It illustrates the fact that using the relaxed problem to make the decision yields to a sub-optimal solution.

Question 07

Consider the following mixed integer optimization problem:

$$\min 10x_1 + x_2$$

subject to

$$\begin{aligned} 5x_1 + 2x_2 &\geq 10, \\ x_1 + 3x_2 &\geq 7, \\ x_1 &\leq 5, \\ x_2 &\leq 5, \\ x_1, x_2 &\geq 0, \\ x_1 &\in \mathbb{N}. \end{aligned}$$

1. Find the optimal solution of the linear relaxation using the graphical method.
2. Find the optimal solution of the mixed integer optimization problem using the graphical method.

Solution 07

1. Figure 23 illustrates the solution of the linear relaxation using the graphical method. The constraint polyhedron is represented in orange. Several level lines of the objective function are represented as dashed red. The arrows attached to them correspond to the opposite direction of the gradient, that is

$$-c = \begin{pmatrix} -10 \\ -1 \end{pmatrix}.$$

   This is the direction to follow, as we are solving a minimization problem. We move the level lines as far as we can in that direction, while keeping a non empty intersection with the polyhedron. The optimal solution is represented as a red circle:

$$x_r^* = \begin{pmatrix} 0.4 \\ 4 \end{pmatrix},$$

   with value 8.

2. Figure 24 illustrates the solution of the linear relaxation using the graphical method. The constraint polyhedron is represented in orange. The integrality constraints are represented by dotted blue lines. One way to solve the problem is to enumerate the feasible values for x_1, and solve the corresponding subproblems for the real variable x_2. The optimal solution for the initial problem will be equal to the best optimal solution of the 6 subproblems. The possible values for x_1 are 1, 2, 3, 4, and 5. Indeed, any solution such that $x_1 = 0$ is infeasible. For a given value of x_1 (represented by a dotted blue line), the optimal value of x_2 corresponds to the lowest possible solution along the line, as the coefficient of x_2 in the objective function is positive. These solutions, represented by circles in the Figure, are

   - $x_1 = 1$, $x_2 = 5/2$,
   - $x_1 = 2$, $x_2 = 5/3$,
   - $x_1 = 3$, $x_2 = 4/3$,

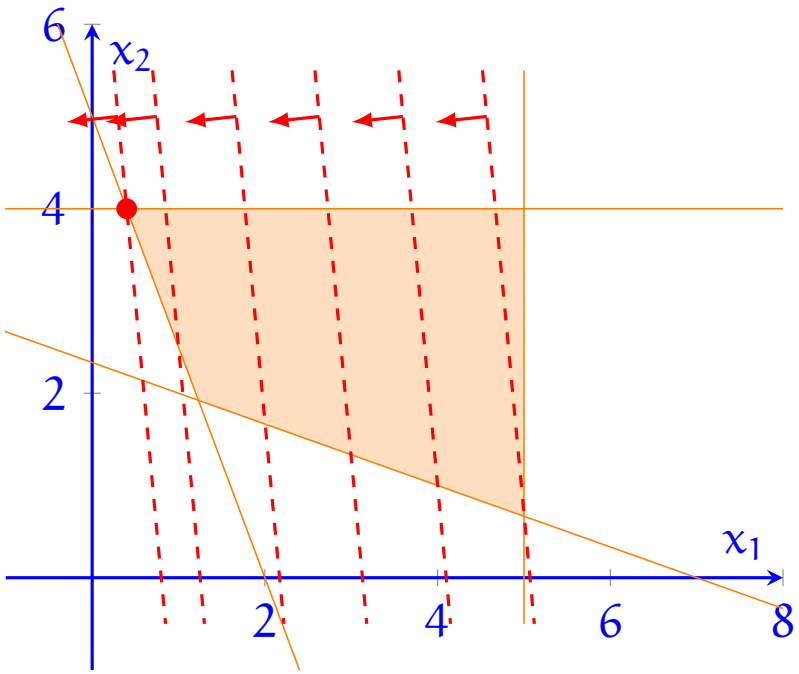


Figure 23: Graphical method for the linear relaxation

- $x_1 = 4$, $x_2 = 1$,
- $x_1 = 5$, $x_2 = 2/3$.

Now, we consider the value of the objective function for each of these solutions, represented by the level lines. The one corresponding to the lowest value is

$$x^* = \begin{pmatrix} 1 \\ 5/2 \end{pmatrix},$$

with value 12.5.

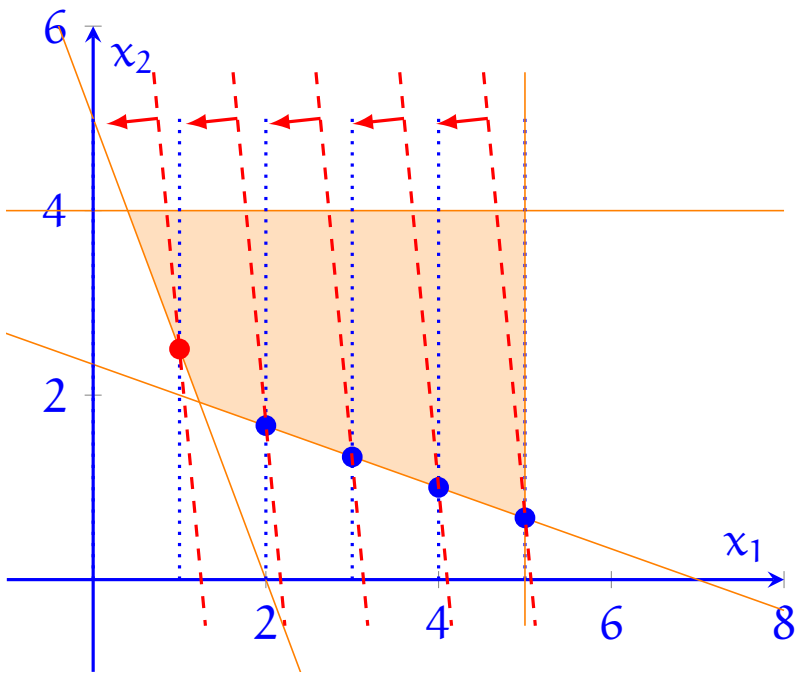


Figure 24: Graphical method for the mixed integer optimization problem

Question 08

1. Consider an integer minimization problem, and its linear relaxation. The optimal value of the linear relaxation is always a lower bound on the optimal value of the integer optimization problem.

   ☐ True.

   ☐ False.

2. Consider an integer minimization problem, and its linear relaxation. The optimal value of the linear relaxation is always strictly lower than the optimal value of the integer optimization problem.

   ☐ True.

   ☐ False.

Solution 08

1. True. Because the optimal solution of the integer optimization problem is feasible for the linear relaxation. As we have a minimization problem, the optimal value of the relaxation is a lower bound on the optimal value of the integer problem.

2. False. We show a counter example. Consider the integer optimization problem:

   $$\min_{x \in \mathbb{R}} x$$

   subject to

   $$x \in \{0, 1\}.$$

   Its relaxation is

   $$\min_{x \in \mathbb{R}} x$$

   subject to

   $$0 \leq x \leq 1.$$

   The optimal solution of both problems is $x^* = 0$, with optimal value 0.

Question 09

The journal *Millenium* needs to schedule the staff for the printing workshop for the five days of the week. During each day, there are eight one-hour time slots. Four employees are available for the tasks. Each employee has reported his or her preference for each time slot and each day on a scale from 0 to 10, where 10 corresponds to the highest preference and 0 corresponds to unavailability (see Table 14 on the next page). The following constraints must be verified.

1. Each of the 40 slots must be covered by exactly one employee.
2. An employee cannot be assigned to a time slot if she/he is not available.
3. Every person must take a lunch break either between 12:00 and 13:00, or between 13:00 and 14:00.
4. Because of the noisy work environment, every person can work only two consecutive time slots. A break of at least one hour must be taken after that.
5. No one can work more than 20 hours per week.

Write an integer optimization problem to help *Millenium* schedule the workshop employees in order to maximize their satisfaction according to the stated preferences, while verifying the constraints.

Name	Slot	Mo	Tu	We	Th	Fr
Mikael	9-10	10	10	10	10	10
Mikael	10-11	9	9	9	9	9
Mikael	11-12	8	8	8	8	8
Mikael	12-13	1	1	1	1	1
Mikael	13-14	1	1	1	1	1
Mikael	14-15	1	1	1	1	1
Mikael	15-16	1	1	1	1	1
Mikael	16-17	1	1	1	1	1
Lisbeth	9-10	10	9	8	7	6
Lisbeth	10-11	10	9	8	7	6
Lisbeth	11-12	10	9	8	7	6
Lisbeth	12-13	10	3	3	3	3
Lisbeth	13-14	1	1	1	1	1
Lisbeth	14-15	1	2	3	4	5
Lisbeth	15-16	1	2	3	4	5
Lisbeth	16-17	1	2	3	4	5
Harriet	9-10	10	10	10	10	10
Harriet	10-11	9	9	9	9	9
Harriet	11-12	8	8	8	8	8
Harriet	12-13	0	0	0	0	0
Harriet	13-14	1	1	1	1	1
Harriet	14-15	1	1	1	1	1
Harriet	15-16	1	1	1	1	1
Harriet	16-17	1	1	1	1	1
Alexander	9-10	10	9	8	7	6
Alexander	10-11	10	9	8	7	6
Alexander	11-12	10	9	8	7	6
Alexander	12-13	10	3	3	3	3
Alexander	13-14	1	1	1	1	1
Alexander	14-15	1	2	3	4	5
Alexander	15-16	1	2	3	4	5
Alexander	16-17	1	2	3	4	5

Table 14: Preference of each worker for each time slot

Solution 09

We want to schedule the workshop employees in order to maximize their satisfaction to the stated preferences. As there are 4 employees and 40 times slots, we define $4 \cdot 40$ binary decision variables x_{ij}, for $i = 1, 2, 3, 4$ and $j = 1, ..., 40$, which are 1 if employee i works during the time slot j and 0 otherwise.

We denote by a_{ij} the reported preference of employee i for the time slot j. Since the goal is to maximize the satisfaction of the employees, we end up with the objective function

$$\max_{x \in \mathbb{R}^{4\times 40}} \sum_{i=1}^{4} \sum_{j=1}^{40} a_{ij} x_{ij}.$$

Now we model the constraints.

1. Each of the 40 slots must be covered by exactly one employee:

$$\sum_{i=1}^{4} x_{ij} = 1, \quad \forall j = 1, \ldots, 40.$$

2. An employee cannot be assigned to a time slot if she/he is not available:

$$a_{ij} - x_{ij} \geq 0, \quad \forall i = 1, 2, 3, 4, \quad \forall j = 1, \ldots, 40.$$

3. Every day, each person must take a lunch break either between 12:00 and 13:00, or between 13:00 and 14:00:

$$x_{i,4} + x_{i,5} \leq 1, \quad x_{i,12} + x_{i,13} \leq 1, \quad x_{i,20} + x_{i,21} \leq 1,$$
$$x_{i,28} + x_{i,29} \leq 1, \quad x_{i,36} + x_{i,37} \leq 1, \quad i = 1, ..., 4.$$

4. Because of the noisy work environment, every person can work only two consecutive time slots. Every day, a break of at least one hour must be taken after that:

$$\begin{aligned}
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 1, ..., 6, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 9, ..., 14, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 17, ..., 22, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 25, ..., 30, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 33, ..., 38.
\end{aligned}$$

5. No one can work more than 20 hours per week:

$$\sum_{j=1}^{40} x_{ij} \leq 20, \quad \forall i = 1, 2, 3, 4.$$

Thus, we end up with the optimization problem

$$\max_{x \in \mathbb{R}^{4\times 40}} \sum_{i=1}^{4} \sum_{j=1}^{40} a_{ij} x_{ij}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{4} x_{ij} &= 1, && \forall j = 1, \ldots, 40, \\
a_{ij} - x_{ij} &\geq 0, && \forall i = 1, 2, 3, 4, \forall j = 1, \ldots, 40, \\
x_{i,4} + x_{i,5} &\leq 1, && \forall i = 1, 2, 3, 4, \\
x_{i,12} + x_{i,13} &\leq 1, && \forall i = 1, 2, 3, 4, \\
x_{i,20} + x_{i,21} &\leq 1, && \forall i = 1, 2, 3, 4, \\
x_{i,28} + x_{i,29} &\leq 1, && \forall i = 1, 2, 3, 4, \\
x_{i,36} + x_{i,37} &\leq 1, && \forall i = 1, 2, 3, 4, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 1, ..., 6, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 9, ..., 14, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 17, ..., 22, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 25, ..., 30, \\
x_{ij} + x_{i,j+1} + x_{i,j+2} &\leq 2, && \forall i = 1, 2, 3, 4, \forall j = 33, ..., 38, \\
\sum_{j=1}^{40} x_{ij} &\leq 20, && \forall i = 1, 2, 3, 4 \\
x_{ij} &\in \{0, 1\}, && \forall i = 1, 2, 3, 4, \forall j = 1, \ldots, 40.
\end{aligned}$$

The optimal schedule is as follows.

	Mon	Tue	Wed	Thu	Fri
09-10	Mikael	Mikael	Mikael	Mikael	Mikael
10-11	Lisbeth	Mikael	Mikael	Mikael	Harriet
11-12	Lisbeth	Lisbeth	Lisbeth	Harriet	Harriet
12-13	Alexander	Alexander	Lisbeth	Alexander	Lisbeth
13-14	Lisbeth	Lisbeth	Mikael	Lisbeth	Harriet
14-15	Mikael	Lisbeth	Lisbeth	Lisbeth	Lisbeth
15-16	Mikael	Alexander	Lisbeth	Alexander	Lisbeth
16-17	Lisbeth	Lisbeth	Alexander	Lisbeth	Alexander

We verify that the constraints are verified.

(a) Each of the 40 slots must be covered by exactly one employee. This is obviously verified.

(b) An employee cannot be assigned to a time slot if she/he is not available. It applies only to Harriet, who is not available between 12 and 13 every day. And, indeed, she has not been scheduled during this time slot.

(c) Every day, each person must take a lunch break either between 12:00 and 13:00, or between 13:00 and 14:00. It can be verified that nobody works during the two consecutive slots from 12 to 14.

(d) Every person can work only two consecutive time slots. Indeed, the number of consecutive time slots for the same person is not larger than two.

(e) No one can work more than 20 hours per week. Indeed,

- Mikael works 11 hours,
- Lisbeth works 18 hours,
- Harriet works 4 hours,
- Alexander works 7 hours.

Question 10

The Nespresso factory in Orbe must deliver capsules to Lausanne, Neuchatel, Fribourg, Bern and Sierre. The distance between each city is reported in Table 15. How does the company have to organize the delivery of the capsules to satisfy the demand of each city, while minimizing the distance traveled? Write an integer optimization problem to answer this question.

	Orbe	Lausanne	Neuchatel	Fribourg	Berne	Sierre
Orbe	0	33	50	63	86	140
Lausanne	33	0	73	76	106	111
Neuchatel	50	73	0	42	49	179
Fribourg	63	76	42	0	35	143
Berne	86	106	49	35	0	173
Sierre	140	111	179	143	173	0

Table 15: Distance between each pair of cities

Solution 10

This problem is a traveling salesman problem. The decision variables are binary variables x_{ij}, $i, j \in \{1, ..., 6\}$, $i \neq j$. Therefore, there are 30 decision variables. Each decision variable is equal to 1 if Nespresso delivers capsules to city j just after city i. We define by a_{ij} the distance from city i to city j (note that the distance from city j to city i is the same).

Since we want to minimize the total distance traveled, the objective function is

$$\min_{x \in \mathbb{R}^{6\times 6}} \sum_{i \neq j} a_{ij} x_{ij}.$$

Then we model the constraints.

- Each city must have exactly one successor, that is

$$\sum_{j \neq i} x_{ij} = 1, \quad \forall i \in \{1, ..., 6\}.$$

- Each city must have exactly one predecessor, that is

$$\sum_{j \neq i} x_{ji} = 1, \quad \forall i \in \{1, ..., 6\}.$$

- Finally, we need to define the subtour elimination constraints. We introduce integer variables y_i, $\forall i \in \{2, ..., 6\}$, denoting the position of each city within the route. Since Nespresso is starting its delivery from Orbe, there is no variable y_{Orbe}. The subtour elimination constraints are defined as:

$$x_{ij}(n-1) \leq y_j - y_i + n - 2, \qquad \forall i, j \in \{2, ..., 6\}.$$

  As $n = 6$, we have

$$5x_{ij} \leq y_j - y_i + 4, \qquad \forall i, j \in \{2, ..., 6\}.$$

  We observe that if Nespresso delivers city j right after city i, we have $x_{ij} = 1$ and

$$y_j \geq y_i + 1.$$

If $x_{ij} = 0$, the constraint becomes

$$y_i - y_j \leq 4 \qquad \forall i, j \in \{2, ..., 6\},$$

which is always verified. Indeed, the cities that are further apart in the tour are at position $y_j = 1$ and $y_i = 5$, and the difference cannot be larger than 4.

The optimization problem is

$$\min_{x \in \mathbb{R}^{30}, y \in \mathbb{R}^5} \sum_{i \neq j} a_{ij} x_{ij}$$

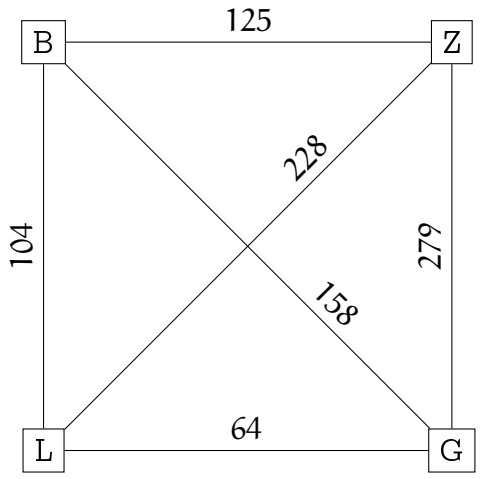
subject to

$$\begin{aligned}
\sum_{j \neq i} x_{ij} &= 1, \quad \forall i \in \{1, ..., 6\}, \\
\sum_{j \neq i} x_{ji} &= 1, \quad \forall i \in \{1, ..., 6\}, \\
5x_{ij} + y_i - y_j &\leq 4, \quad \forall i, j \in \{2, ..., 6\}, \\
x_{ij} &\in \{0, 1\}, \quad \forall i, j \in \{1, ..., 6\}, i \neq j \\
y_i &\geq 0, \quad \forall i \in \{2, ..., 6\}.
\end{aligned}$$

The optimal tour is: Orbe $\rightarrow$ Lausanne $\rightarrow$ Sierre $\rightarrow$ Fribourg $\rightarrow$ Berne $\rightarrow$ Neuchatel $\rightarrow$ Orbe, for a total of 421 km.

Question 11

Consider the traveling salesman problem with 4 cities. A salesman living in Lausanne (L) must visit 3 customers during the day: one in Geneva (G), one in Bern (B), and one in Zurich (Z). Starting from home, he has to plan a tour, that is a sequence of customers to visit, in order to minimize the travel distance. We model the problem with a graph represented in the figure below.



Solve this problem using the branch & bound algorithm. The calculation of the lower bound can be done using the following reasoning.

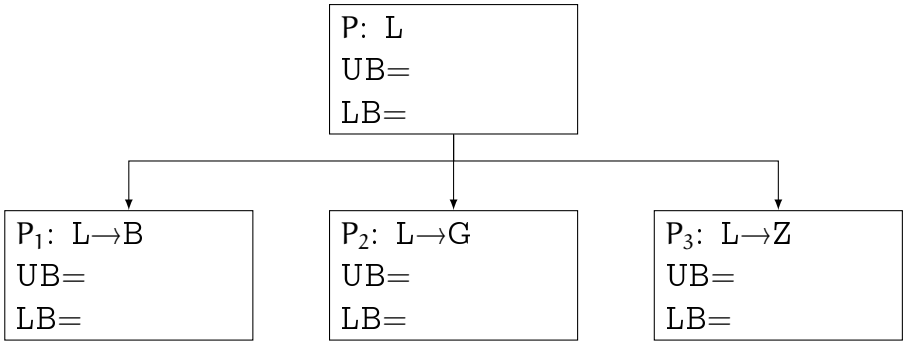
Each city in the tour involves two different arcs, one to reach the city, and one to leave it. A lower bound on the length of the two arcs incident to a city is obtained by considering the two shortest arcs incident to that city. If we compute this lower bound for all nodes and add them up, we have double counted each arc (once for each of the two incident cities). Therefore, a lower bound on the total path is obtained by dividing this quantity by two.

- For Lausanne, the two shortest incident arcs have length 64 and 104.
- For Geneva, the two shortest incident arcs have length 64 and 158.
- For Bern, the two shortest incident arcs have length 104 and 125.
- For Zürich, the two shortest incident arcs have length 125 and 228.

Therefore, a lower bound on the optimal tour is

$$\frac{64 + 104 + 64 + 158 + 104 + 125 + 125 + 228}{2} = 486.$$

Hint: build a tree to keep track of the progress of the method.



Solution 11

In order to calculate an upper bound, we need a feasible solution. Here, any tour can be selected. For instance, consider the tour L $\rightarrow$ G $\rightarrow$ B $\rightarrow$ Z $\rightarrow$ L. Its length is 575. Therefore, we initialize

$$x^* = L \rightarrow G \rightarrow B \rightarrow Z \rightarrow L,$$
$$UB = 575.$$

The tour starts in Lausanne. We create subproblems by branching on the city visited just after Lausanne:

- P_1: L $\rightarrow$ B,
- P_2: L $\rightarrow$ G,
- P_3: L $\rightarrow$ Z.

All these subproblems are feasible. We calculate a lower bound for each of them, using the method described in the problem description. In the following tables, the numbers in bold correspond to an arc already selected. For problem P_1, the lower bound is 486, based on the following table:

	Arc 1	Arc 2	Sum
L	64	**104**	168
G	64	158	222
B	**104**	125	229
Z	125	228	353
			972

For problem P_2, the bound is also 486.

	Arc 1	Arc 2	Sum
L	**64**	104	168
G	**64**	158	222
B	104	125	229
Z	125	228	353
			972

For problem P_3, the bound is 548.

	Arc 1	Arc 2	Sum
L	64	**228**	292
G	64	158	222
B	104	125	229
Z	125	**228**	353
			1096

Note that none of these lower bounds is higher than the best upper bound $UB = 575$. Therefore, we have to branch. The current set of problems is

$$P, P_1, P_2, P_3.$$

Consider problem P_1, where it has been decided that Bern will be visited after Lausanne. We create two subproblems, corresponding to the next city:

- P_{11}: $L \to B \to Z$.
- P_{12}: $L \to B \to G$,

The current set of problems is

$$P, P_1, P_2, P_3, P_{11}, P_{12}.$$

The lower bound for problem P_{11} is 486, based on the following table:

	Arc 1	Arc 2	Sum
L	64	**104**	168
G	64	158	222
B	**104**	**125**	229
Z	**125**	228	353
			972

This bound is not higher than the best upper bound $UB = 575$, and the problem cannot be discarded. But the problem is trivial to solve. Indeed, once the first three cities have been chosen, the last one is obvious. Therefore, the solution of P_{11} is $L \to B \to Z \to G \to L$, with length 572. As it is better than the current upper bound, this is the new candidate for optimality, and we set

$$x^* = L \to B \to Z \to G \to L,$$
$$UB = 572.$$

As we have solved problem P_{11}, the current set of problems is

$$P, P_1, P_2, P_3, P_{12}.$$

The lower bound for problem P_{12} is 502.5, based on the following table:

	Arc 1	Arc 2	Sum
L	64	**104**	168
G	64	**158**	222
B	**104**	**158**	262
Z	125	228	353
			1005

This bound is not higher than the best upper bound $UB = 572$, and the problem cannot be discarded. But the problem is trivial to solve. Indeed, once the first three cities have been chosen, the last one is obvious. Therefore, the solution of P_{12} is L $\rightarrow$ B $\rightarrow$ G $\rightarrow$ Z $\rightarrow$ L, with length 769. As it is higher than the solution of P_{11}, P_{12} can be discarded and the solution of P_1 is the solution of P_{11}. The current set of problems is

$$P, P_2, P_3.$$

Consider now problem P_2, where it has been decided that Geneva will be visited after Lausanne. We create two subproblems, corresponding to the next city:

- P_{21}: L $\rightarrow$ G$\rightarrow$ Z,
- P_{22}: L $\rightarrow$ G$\rightarrow$ B.

The current set of problems is

$$P, P_2, P_3, P_{21}, P_{22}.$$

The lower bound on problem P_{21} is 572, based on the following table:

	Arc 1	Arc 2	Sum
L	**64**	104	168
G	**64**	**279**	343
B	104	125	229
Z	125	**279**	404
			1144

This is equal to the current upper bound UB. Therefore, there is no point pursuing this problem as it will not lead to a better solution than the one we already have. We discard it, and the set of problem is

$$P, P_2, P_3, P_{22}.$$

The lower bound on P_{22} is 502.5.

	Arc 1	Arc 2	Sum
L	**64**	104	168
G	**64**	**158**	222
B	104	**158**	262
Z	125	228	353
			1005

This bound is not higher than the best upper bound $UB = 572$, and the problem cannot be discarded. But the problem is trivial to solve. Indeed, once the first three cities have been chosen, the last one is obvious. Therefore, the solution of P_{22} is L $\rightarrow$ G $\rightarrow$ B $\rightarrow$ Z $\rightarrow$ L, with length 575. As P_{21} has been discarded, it is the solution of P_2. But as it is higher than the current upper bound $UB = 572$, P_2 is discarded. The current set of problems is

$$P, P_3.$$

Consider now problem P_3, where it has been decided that Zürich will be visited after Lausanne. We create two subproblems, corresponding to the next city:

- P_{31}: L $\rightarrow$ Z$\rightarrow$ B,
- P_{32}: L $\rightarrow$ Z$\rightarrow$ G.

The current set of problems is

$$P, P_3, P_{31}, P_{32}.$$

The lower bound of P_{31} is 548, based on the table below.

	Arc 1	Arc 2	Sum
L	64	**228**	292
G	64	158	222
B	104	**125**	229
Z	**125**	**228**	353
			1096

This bound is not higher than the best upper bound $UB = 572$, and the problem cannot be discarded. But the problem is trivial to solve. Indeed, once the first three cities have been chosen, the last one is obvious. Therefore, the solution of P_{31} is $L \to Z \to B \to G \to L$, with length 575. As it is higher that the current upper bound $UB = 572$, the problem is discarded. The current set of problems is

$$P, P_3, P_{32}.$$

The lower bound on problem P_{32} is 685.5.

	Arc 1	Arc 2	Sum
L	64	**228**	292
G	64	**279**	343
B	104	125	229
Z	**279**	**228**	507
			1371

As it is higher that the current upper bound $UB = 572$, the problem is discarded. The solution of P_3 is the solution of P_{31}. P_3 is discarded. And the solution of P is the solution of P_1:

$$x^* = L \to B \to Z \to G \to L,$$
$$UB = 572.$$

The process is illustrated as a tree in Figure 25 on the next page, where each node corresponds to a subproblem, and its lower bound is indicated. The nodes in gray correspond to the optimal solution.

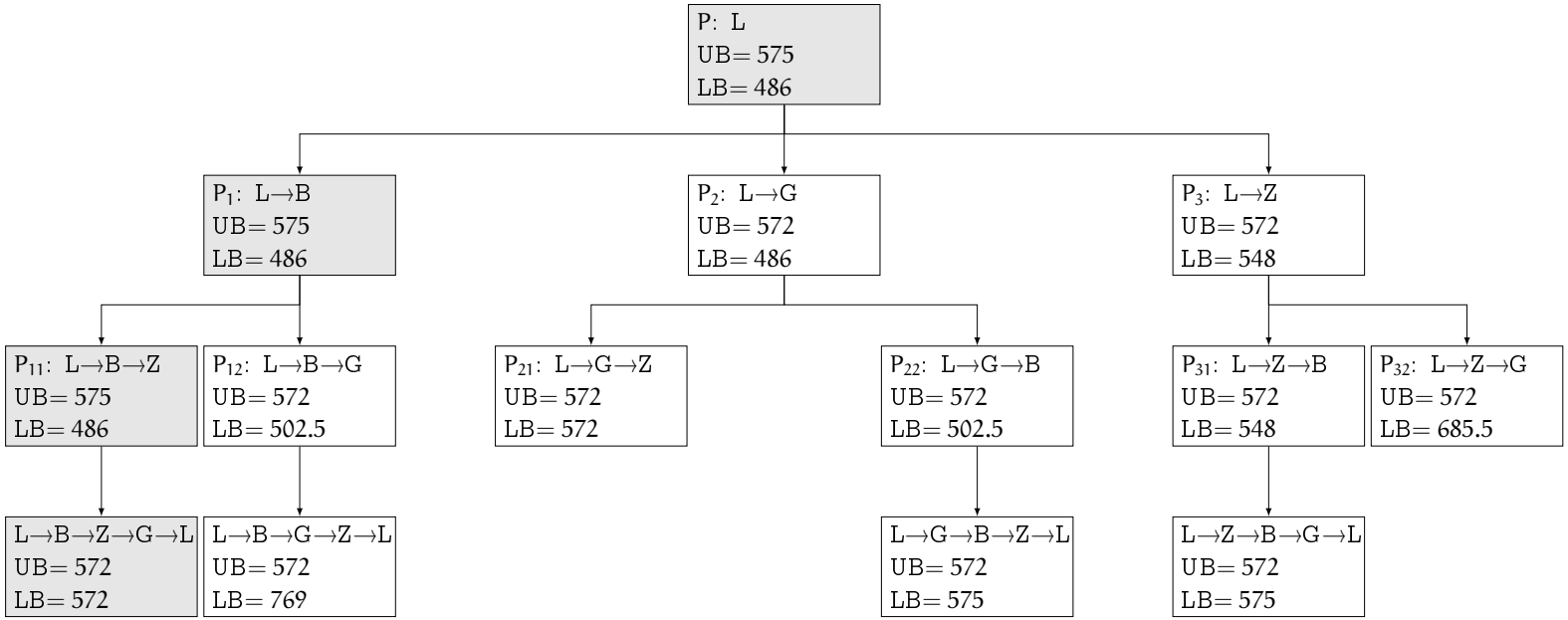


Figure 25: Branch and bound tree

Question 12

Consider the following integer optimization problem.

$$\min_{x \in \mathbb{N}^2} -13x_1 - 8x_2$$

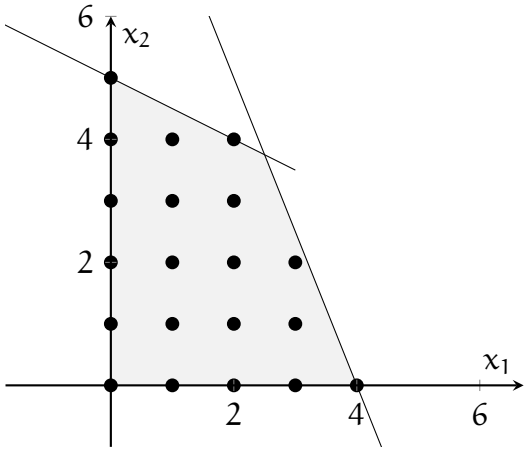
subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 10, \\ 5x_1 + 2x_2 &\leq 20, \\ x_1, x_2 &\in \mathbb{N}. \end{aligned}$$

Solve the problem using the branch and bound algorithm.

Solution 12

The feasible set is illustrated in the following figure.



As (0,0) is feasible, we initialize the upper bound to UB$= 0$. We call P_0 the original problem, and we solve its relaxation:

$$R(P_0) : \min_{x \in \mathbb{N}^2} -13x_1 - 8x_2$$

subject to

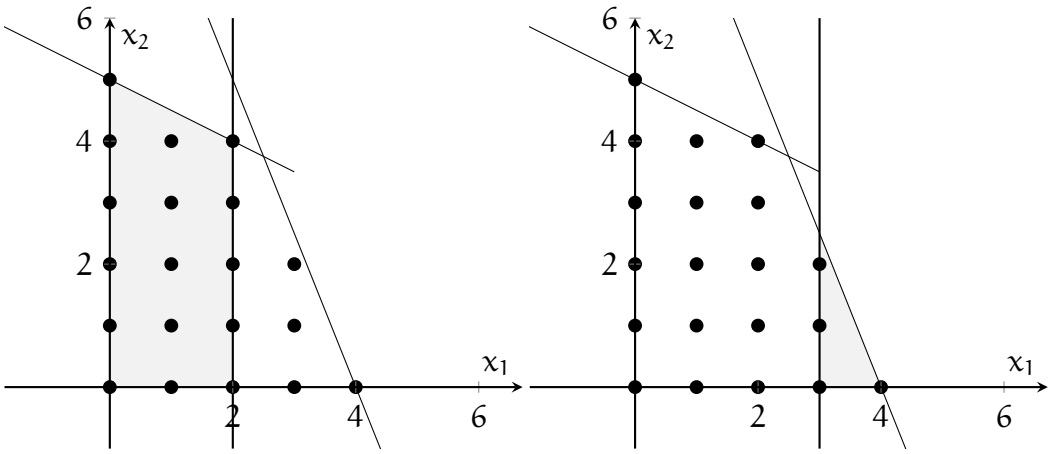
$$\begin{aligned} x_1 + 2x_2 &\leq 10, \\ 5x_1 + 2x_2 &\leq 20, \\ x_1, x_2 &\geq 0. \end{aligned}$$

The optimal solution is

$$x_0^* = (2.5, 3.75)$$

with value $f(x_0^*) = -62.5$. We decide to branch on x_1. We create two subproblems, incorporating a different constraint on x_1:

- P_1: $x_1 \leq 2$,
- P_2: $x_1 \geq 3$.



(a) Feasible set for P_1 (b) Feasible set for P_2

Figure 26: Feasible sets for the first iteration

We now have $\mathcal{P} = \{P_1, P_2\}$:

P_1			P_2		
$\min -13x_1 - 8x_2$			$\min -13x_1 - 8x_2$		
subject to			subject to		
$x_1 + 2x_2$	$\leq$	10	$x_1 + 2x_2$	$\leq$	10
$5x_1 + 2x_2$	$\leq$	20	$5x_1 + 2x_2$	$\leq$	20
x_1, x_2	$\geq$	0	x_1, x_2	$\geq$	0
x_1, x_2	$\in$	$\mathbb{N}$	x_1, x_2	$\in$	$\mathbb{N}$
x_1	$\leq$	2	x_1	$\geq$	3

The feasible sets are represented in Figure 26.

We consider first P_1, and solve its relaxation $R(P_1)$. The optimal solution is

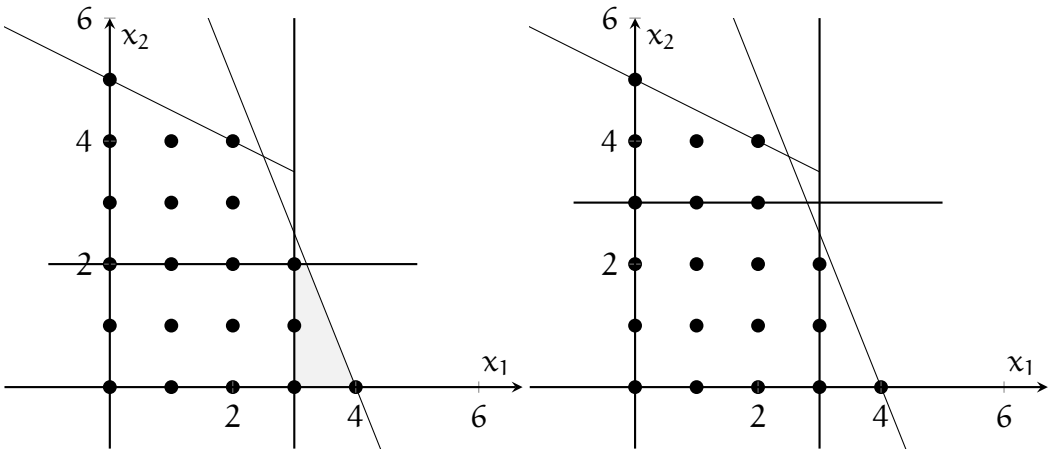
$$x_1^* = (2, 4)$$

with value $f(x_1^*) = -58$. It is a feasible solution of the original problem. Therefore, we update the best upper bound to UB$= -58$. As P_1 has been solved, the set of problems is

$$P_0, P_2.$$

We consider P_2, and solve its relaxation $R(P_2)$. The optimal solution is

$$x_2^* = (3, 2.5)$$



(a) Feasible set for P_{21} (b) Feasible set for P_{22}

Figure 27: Feasible sets for the second iteration

with value $f(x_2^*) = -59$. This is a lower bound on the optimal value of P_2. As it is better than the best upper bound (UB$= -58$), we continue to investigate P_2 by branching. We create two subproblems, incorporating a different constraint on x_2, which has a non integer value:

- P_{21}: $x_2 \leq 2$,
- P_{22}: $x_2 \geq 3$.

The set of problems is now $\{P_0, P_2, P_{21}, P_{22}\}$

P_{21}			P_{22}		
$\min -13x_1 - 8x_2$			$\min -13x_1 - 8x_2$		
subject to			subject to		
$x_1 + 2x_2$	$\leq$	10	$x_1 + 2x_2$	$\leq$	10
$5x_1 + 2x_2$	$\leq$	20	$5x_1 + 2x_2$	$\leq$	20
x_1, x_2	$\geq$	0	x_1, x_2	$\geq$	0
x_1, x_2	$\in$	$\mathbb{N}$	x_1, x_2	$\in$	$\mathbb{N}$
x_1	$\geq$	3	x_1	$\geq$	3
x_2	$\leq$	2	x_2	$\geq$	3

The feasible sets are represented in Figure 27.

We consider P_{21}, and solve its relaxation $R(P_{21})$. The optimal solution is

$$x_{21}^* = (3.2, 2)$$

with value $f(x_{21}^*) = -57.6$. As the solution is not integral, this is a lower bound on the optimal solution of P_{21}. But as it is larger than the best upper bound UB$= -58$, problem P_{21} can be discarded, and the set of problems is

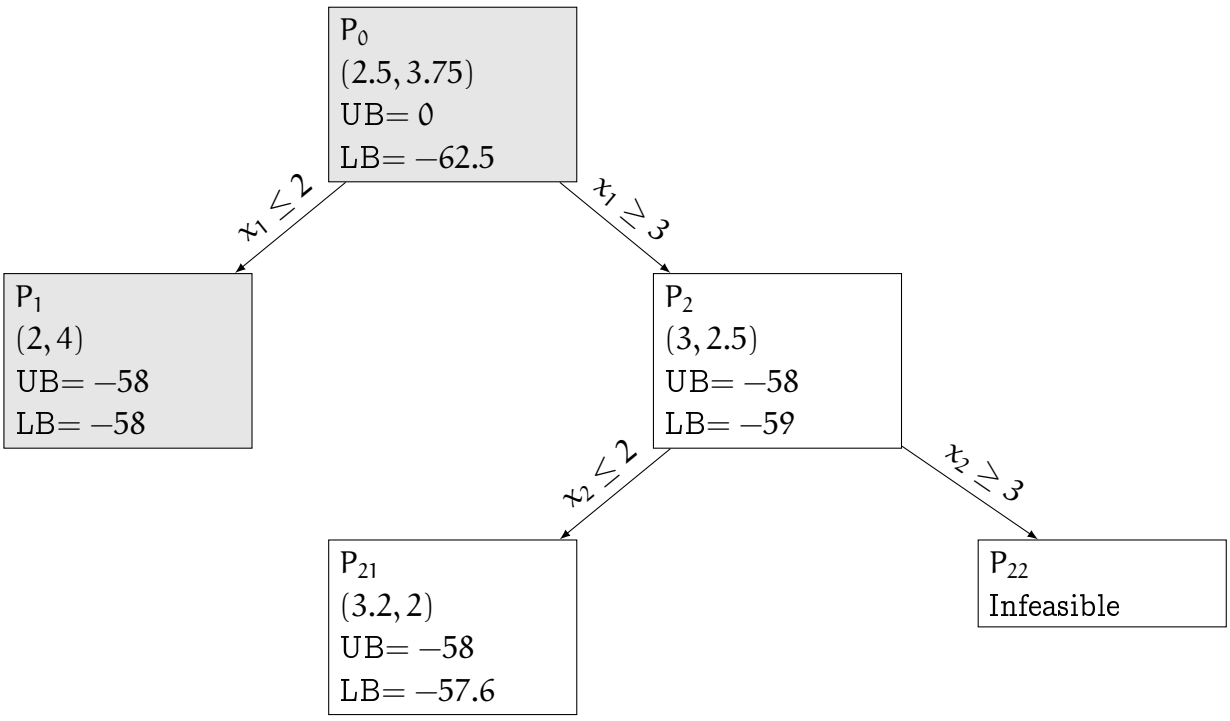
$$\{P_0, P_2, P_{22}\}.$$

Consider problem P_{22}. We observe that it is infeasible, and can be discarded from the set. Therefore, P_2 can also be discarded from the set, and the solution of the original problem is

$$x^* = (2, 4)$$

with value $f(x^*) = -58$.

The process in illustrated as the tree below, where each node corresponds to a subproblem, and the solution of the relaxation and the lower bound are indicated. The nodes in gray correspond to the optimal solution.



Question 13

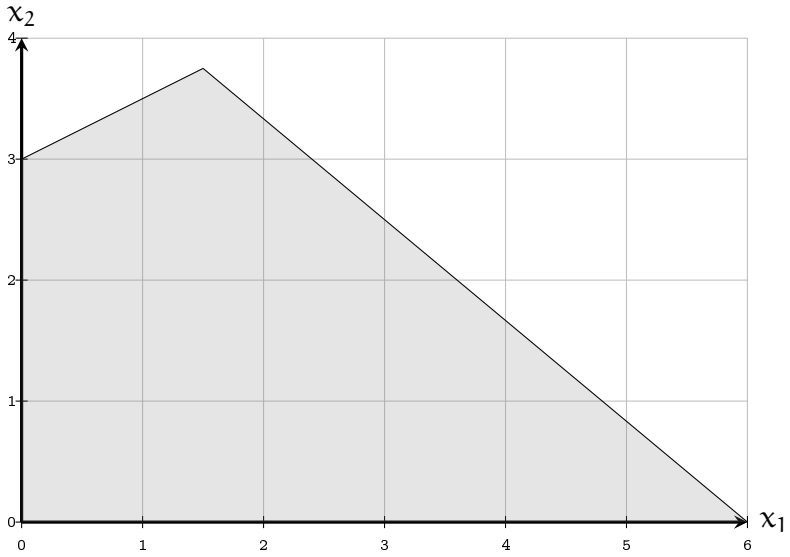
Consider the following integer linear optimization problem (P_0):

$$\min -13/4x_1 - 8x_2$$

subject to

$$\begin{aligned} 5x_1 + 6x_2 &\leq 30, \\ -x_1 + 2x_2 &\leq 6, \\ x_1, x_2 &\in \mathbb{N}. \end{aligned}$$

The feasible set of problem P_0 is the intersection between the constraint polyhedron of the relaxation $R(P_0)$ (represented in gray below), and the lattice composed of all integers values of the variables.



- Solve problem P_0 using the Branch and Bound algorithm. Use the branching strategy suggested by the tree in the Figure 28 and solve the relaxations using the graphical method. For each node (representing a sub-problem) specify the upper bound (UB) and the lower bound (LB) on the objective function, as well as the optimal solution $x_1^\star$ and $x_2^\star$ of the relaxation of the corresponding sub-problem. Complete the tree in Figure 28. Write the optimal solution of the problem P_0, as well as the optimal value of the objective function.

- **Hint**: Use (1,1) as a promising feasible point to initialize the upper bound.

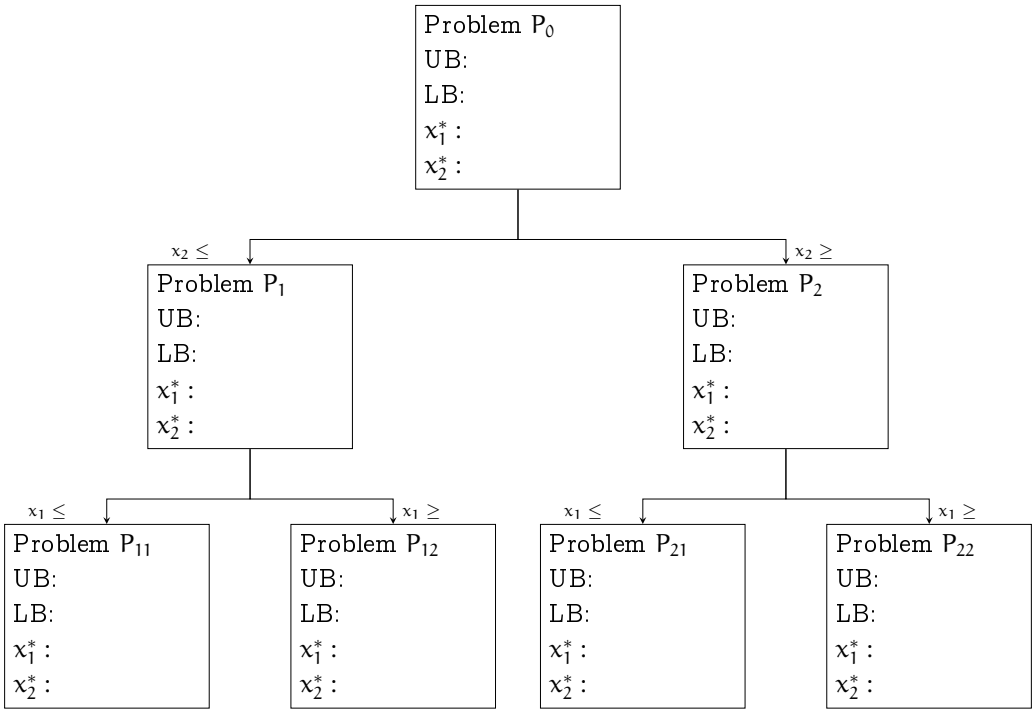


Figure 28: Branch and bound tree

Solution 13

At each node, the upper bound is the best feasible solution and the lower bound is the solution of the relaxation. As (1, 1) is a feasible solution, we use the value of the objective function at that point as the initial upper bound, that is: UB=-11.25. The solution of the first relaxed problem is: $x_1^* = 1.5, x_2^* = 3.75$ with an objective function equal to -34.875.

As we branch on x_2, we have the two following constraints :

$$x_2 \leq \lfloor 3.75 \rfloor = 3,$$
$$x_2 \geq \lceil 3.75 \rceil = 4.$$

The subproblem with additional constraint $x_2 \geq 4$ is infeasible, so this branch is terminated.

The solution of the subproblem with additional constraint $x_2 \leq 3$ is $x_1^* = 2.4$, $x_2^* = 3$, with an objective function equal to -31.8. The upper bound stays the same since we didn't obtain an integer solution. As the lower bound is lower than the upper bound and the optimal solution of $R(P_1)$ is not integer, we will branch further on x_1, which is the only fractional variable.

In the first subproblem P_{11}, we include the constraint:

$$x_1 \leq \lfloor 2.4 \rfloor = 2.$$

In the second subproblem P_{12}, we include the constraint:

$$x_1 \geq \lceil 2.4 \rceil = 3.$$

The solution of the subproblem with additional constraint $x_2 \leq 3$ and $x_1 \leq 2$ is $x_1^* = 2$, $x_2^* = 3$, with an objective function equals to -30.5, which is better than the previous upper bound value. Therefore, we will update UB=-30.5. This problem has integer solution, so no further branching is required. Since the lower bound is less or equal to the upper bound, this solution is a potential solution of the original problem.

We now treat P_{12}, the subproblem with additional constraint $x_2 \leq 3$ and $x_1 \geq 3$. The solution of P_{12} is $x_1^* = 3$, $x_2^* = 2.5$, with an objective function equal to -29.75. Since the lower bound is greater than the upper bound, we can discard this solution.

As there is no more node left to investigate in the tree, the optimal objective function value is -30.5 with the optimal solution: (2,3).

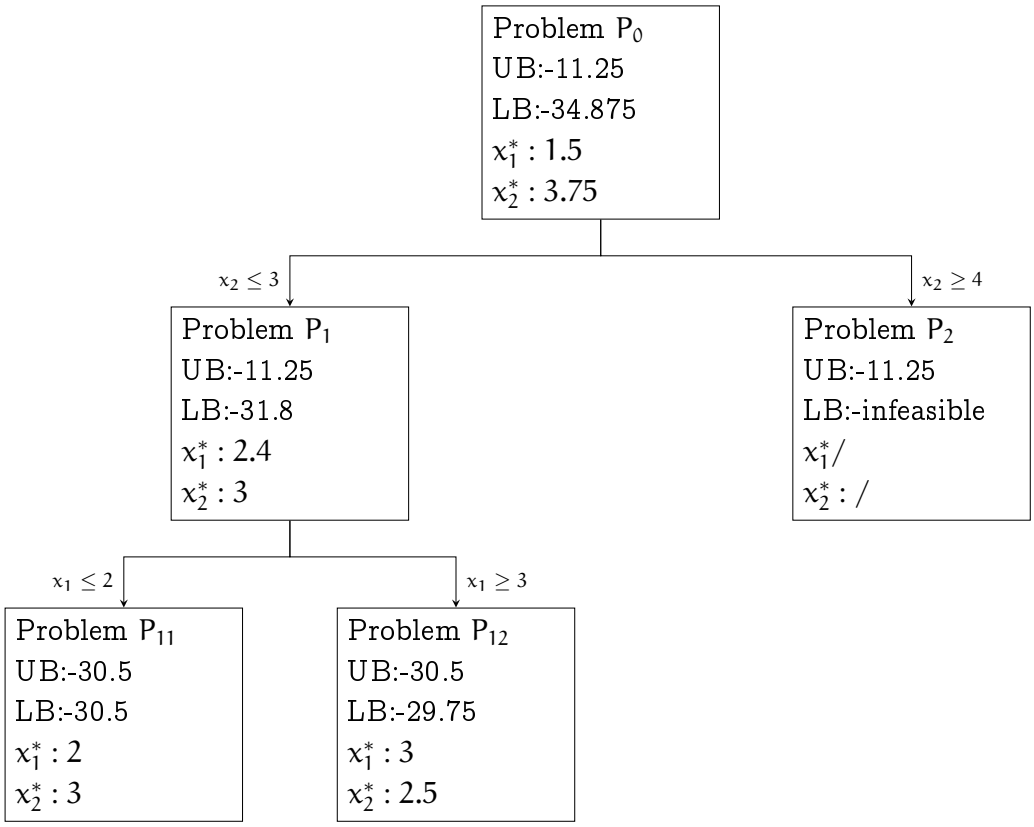


Figure 29: Branch and bound tree

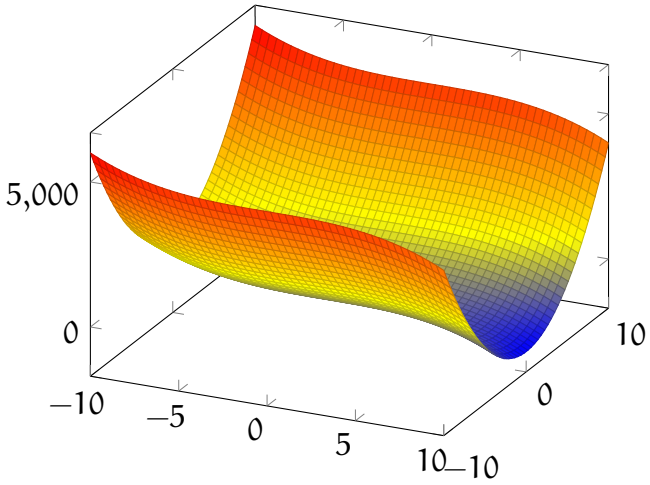
9 Non linear optimization

Question 01

Consider the function

$$f(x_1, x_2) = 50x_1^2 - x_2^3$$

illustrated in the figure below.



1. Show that the point

$$x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

satisfies the necessary optimality conditions.

2. Show that this point is neither a local minimum nor a local maximum.
Hint: Consider the directions

$$d_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } d_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Solution 01

1. The gradient is

$$\nabla f(x) = \begin{pmatrix} 100x_1 \\ -3x_2^2 \end{pmatrix}, \ \nabla f(x^*) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, the first order necessary condition is verified.

The second derivative matrix is

$$\nabla^2 f(x) = \begin{pmatrix} 100 & 0 \\ 0 & -6x_2 \end{pmatrix}, \ \nabla^2 f(x^*) = \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is positive semidefinite. Indeed, the eigenvalues are read on the diagonal, as it is a diagonal matrix. And they are non negative. Therefore, the second order necessary condition is verified.

2. Note that the value of the objective function at x^* is

$$f(x^*) = 0.$$

Consider the direction

$$d_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the point x_1 obtained by following this direction from x^* with a step α:

$$x_1 = x^* + \alpha d_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}.$$

The value of the objective function at x_1 is

$$f(x_1) = -\alpha^3.$$

Therefore, for any $\alpha > 0$, we have

$$f(x_1) < f(x^*),$$

and x^* cannot be a local minimum.

Consider now the direction

$$d_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and the point x_2 obtained by following this direction from x^* with a step α:

$$x_2 = x^* + \alpha d_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}.$$

The value of the objective function at x_2 is

$$f(x_2) = \alpha^3.$$

Therefore, for any $\alpha > 0$, we have

$$f(x_2) > f(x^*),$$

and x^* cannot be a local maximum.

Question 02

Consider the affine function

$$f : \mathbb{R}^n \to \mathbb{R} : x \to c^T x + b = \sum_{i=1}^{n} c_i x_i + b,$$

where $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$. For what values of x, b and c are the necessary optimality conditions verified?

Solution 02

The gradient of f is

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c,$$

for any $x \in \mathbb{R}^n$. As it is a constant vector, the second derivative matrix is zero, for any $x \in \mathbb{R}^n$:

$$\nabla^2 f(x) = 0.$$

We first note that the value of b is irrelevant to determine the necessary optimality conditions. Adding a constant to an objective function does not change the optimal solution. If $c = 0$, that is, if

$$c_1 = \ldots = c_n = 0,$$

then the necessary optimality conditions are verified for all $x \in \mathbb{R}^n$. If $c \neq 0$, that is, if there is at least one i such that $c_i \neq 0$, then the first order necessary optimality conditions are not verified . The second order necessary conditions are always verified, as the null matrix is positive semidefinite.

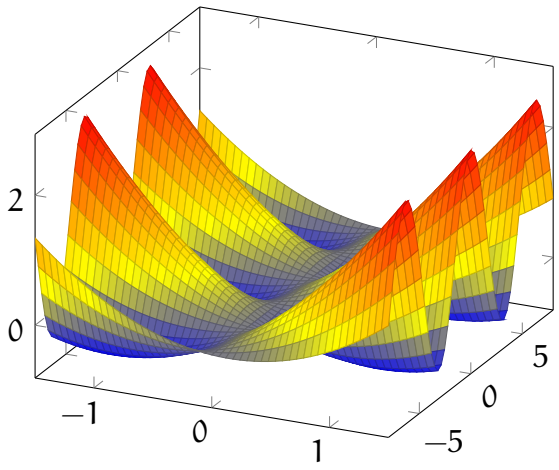
Note that, if $c = 0$, the linear function is constant, with value b. And any $x \in \mathbb{R}^n$ is a (global) optimum. If not, the function is not constant. As there is no constraint, it is not bounded and there is no optimum.

Question 03

Consider the function

$$f(x_1, x_2) = \frac{1}{2}x_1^2 + x_1 \cos x_2$$

illustrated in the figure below.



Use the optimality conditions to identify the minima of this function.

Solution 03

The gradient of the function is

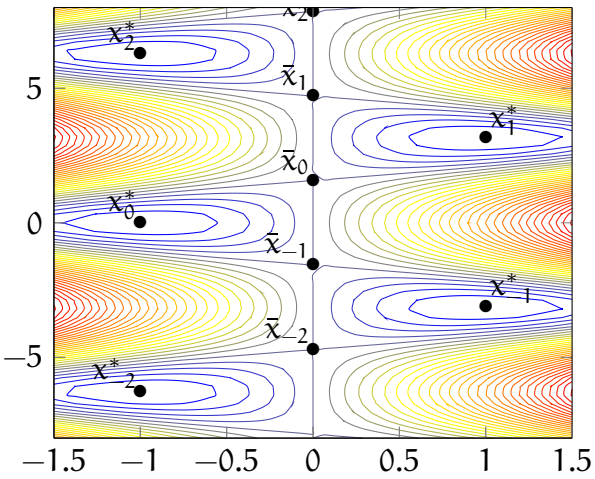
$$\nabla f(x_1, x_2) = \begin{pmatrix} x_1 + \cos x_2 \\ -x_1 \sin x_2 \end{pmatrix}.$$

This gradient is zero for

$$x_k^* = \begin{pmatrix} (-1)^{k+1} \\ k\pi \end{pmatrix}, \; k \in \mathbb{Z},$$

and for

$$\bar{x}_k = \begin{pmatrix} 0 \\ \frac{\pi}{2} + k\pi \end{pmatrix}, \; k \in \mathbb{Z}.$$



The second derivative matrix is

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 1 & -\sin x_2 \\ -\sin x_2 & -x_1 \cos x_2 \end{pmatrix}.$$

By evaluating this matrix at x_k^*, we obtain for any $k \in \mathbb{Z}$

$$\nabla^2 f(x_k^*) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since this matrix is positive definite, each point x_k^* satisfies the sufficient optimality conditions and is a local minimum of the function.

By evaluating the second derivative matrix at $\bar{x}_k$, we obtain for any $k \in \mathbb{Z}$

$$\nabla^2 f(\bar{x}_k) = \begin{pmatrix} 1 & (-1)^{k+1} \\ (-1)^{k+1} & 0 \end{pmatrix}.$$

Regardless of k, this matrix is not positive semidefinite. Indeed, if k is even,

$$\nabla^2 f(x_k^*) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}.$$

If k is odd,

$$\nabla^2 f(x_k^*) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In both cases, the eigenvalues are -0.61803 and 1.61803.

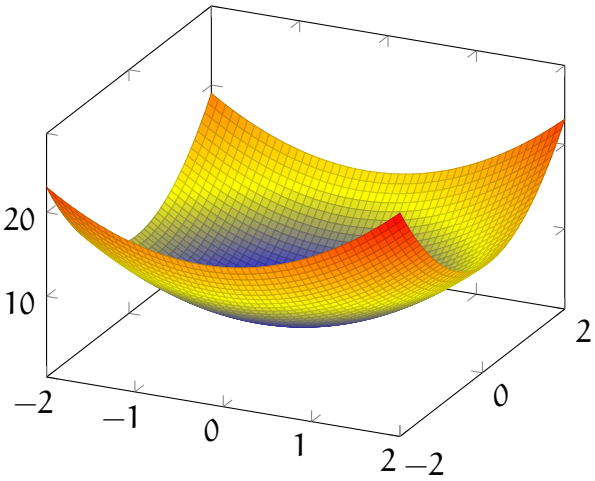
Therefore, there is no $\bar{x}_k$ that satisfies the necessary optimality conditions. None of them can then be a local minimum.

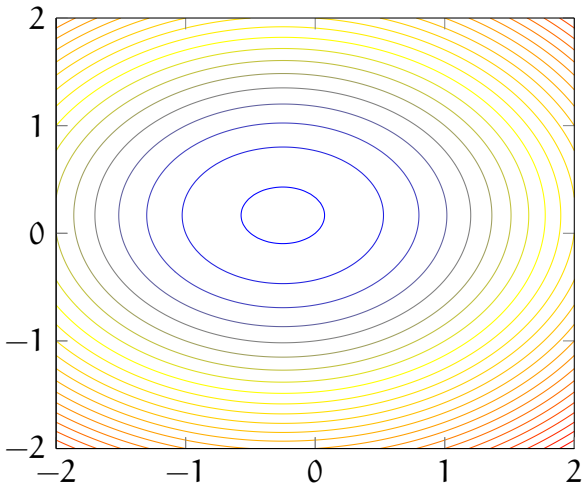
Question 04

Consider the function

$$f(x_1, x_2) = 2x_1^2 + 3x_2^2 + x_1 - x_2 + 3$$

illustrated in the figures below.





Show that

$$x^* = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$$

is a global minimum of f. Is it the only one?

Solution 04

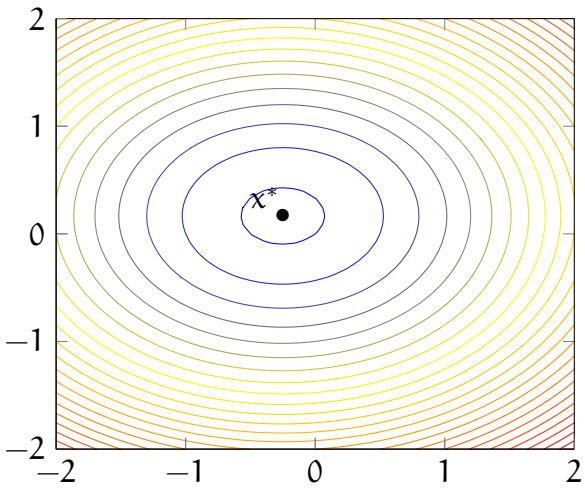
First, we show that the point

$$x^* = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{6} \end{pmatrix}.$$

is a local minimum of the function f. The gradient of the function is

$$\nabla f(x_1, x_2) = \begin{pmatrix} 4x_1 + 1 \\ 6x_2 - 1 \end{pmatrix},$$

which is zero at x^*.



The second derivative matrix is

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix},$$

which is positive definite at x^*. Therefore, the point

$$x^* = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$$

satisfies the sufficient optimality conditions and is a local minimum.

The second derivative matrix is positive definite for all $x \in \mathbb{R}^2$, then f is strictly convex. Therefore, x^* is the unique global minimum of the function.

Question 05

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{2}x_1^2 + \frac{101}{2}x_2^2 + x_1x_2.$$

1. Consider a change of variables

   $$x' = L_k^T x,$$

   where

   $$L_k L_k^T = \nabla^2 f(x_k).$$

   Write the function in the new variables

   $$\tilde{f}(x') = f(L_k^{-T} x').$$

2. Calculate the first and second derivatives of $\tilde{f}$.

3. Consider the point

   $$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Calculate the corresponding point in the new variables x_0'. Apply one iteration of the steepest descent algorithm on $\tilde{f}$ from that point, that is

   $$x'_{k+1} = x'_k - \alpha \nabla \tilde{f}(x'_k),$$

   where the step size is

   $$\alpha = \frac{\nabla \tilde{f}(x'_k)^T \nabla \tilde{f}(x'_k)}{\nabla \tilde{f}(x'_k)^T \nabla^2 \tilde{f}(x'_k) \nabla \tilde{f}(x'_k)}.$$

4. Finally, identify the corresponding point in the original variables.

Solution 05

The derivatives of the function are

$$\nabla f(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 + 101x_2 \end{pmatrix}, \ \nabla^2 f(x) = \begin{pmatrix} 1 & 1 \\ 1 & 101 \end{pmatrix}.$$

1. The matrix L_k defining the change of variables is the Cholesky factor of the second derivatives matrix, that is

$$L_k = \begin{pmatrix} 1 & 0 \\ 1 & 10 \end{pmatrix}, \ \forall k,$$

as

$$L_k L_k^T = \begin{pmatrix} 1 & 1 \\ 1 & 101 \end{pmatrix}.$$

Therefore, the change of variables is $x' = L_k^T x$, that is

$$x' = \begin{pmatrix} 1 & 1 \\ 0 & 10 \end{pmatrix} x$$

or

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 10x_2. \end{aligned}$$

We write the change of variables in the opposite direction as $x = L_k^{-T} x'$, that is

$$x = \begin{pmatrix} 1 & -\frac{1}{10} \\ 0 & \frac{1}{10} \end{pmatrix} x'$$

or

$$\begin{aligned} x_1 &= x_1' - \frac{1}{10} x_2', \\ x_2 &= \frac{1}{10} x_2'. \end{aligned}$$

The function $\tilde{f}$ is therefore defined as

$$\begin{aligned} \tilde{f}(x') &= f(x_1' - \frac{1}{10}x_2', \frac{1}{10}x_2') \\ &= \frac{1}{2}\left(x_1' - \frac{1}{10}x_2'\right)^2 + \frac{101}{2}\left(\frac{1}{10}x_2'\right)^2 + \left(x_1' - \frac{1}{10}x_2'\right)\left(\frac{1}{10}x_2'\right), \\ &= \frac{1}{2}x_1'^2 + \frac{1}{2}x_2'^2. \end{aligned}$$

2. The derivatives of $\tilde{f}$ are

$$\nabla\tilde{f}(x_1', x_2') = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \ \nabla^2\tilde{f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. We write the point in the new variables:

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x_0{}' = \begin{pmatrix} 2 \\ 10 \end{pmatrix},$$

so that

$$\nabla\tilde{f}(x_0{}') = \begin{pmatrix} 2 \\ 10 \end{pmatrix}.$$

The step to perform along the steepest descent direction is

$$\alpha = \frac{\nabla\tilde{f}(x_0{}')^T\nabla\tilde{f}(x_0{}')}{\nabla\tilde{f}(x_0{}')^T\nabla^2\tilde{f}(x_0{}')\nabla\tilde{f}(x_0{}')}.$$

As $\nabla^2\tilde{f}(x_0{}')$ is the identity matrix, $\alpha = 1$. Actually, this would be true for any x_0. Therefore, we obtain

$$x_1' = x_0' - \alpha\nabla\tilde{f}(x_0{}') = \begin{pmatrix} 2 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4. In the original variables, we have

$$x_1 = x_1' - \frac{1}{10}x_2' = 0,$$
$$x_2 = \frac{1}{10}x_2' = 0.$$

It happens to be the optimal solution of the problem.

Question 06

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^2} f(x) = 4x_1^2 - 4x_1 + x_2^2 + 2x_2,$$

and the point $x_0 = (0, 0)^T$.

1. Calculate Newton's direction at x_0.
2. Verify that it is a descent direction.
3. Consider the first Wolfe condition with $\beta_1 = 0.1$. What are the values of the step α that verify the condition?

Solution 06

1. Newton's direction is a solution of the linear system

$$\nabla^2 f(x_k) d_k = -\nabla f(x_k).$$

The derivatives of the function are

$$\nabla f(x) = \begin{pmatrix} 8x_1 - 4 \\ 2x_2 + 2 \end{pmatrix}, \ \nabla^2 f(x) = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore, Newton's equations at $x_0 = (0, 0)^T$ are

$$\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} -4 \\ 2 \end{pmatrix},$$

that is

$$\begin{aligned} 8d_1 &= 4 \\ 2d_2 &= -2, \end{aligned}$$

and Newton's direction is

$$d_N = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}.$$

2. The directional derivative of the function along d_N at x_0 is

$$\nabla f(x_0)^T d_N = -4.$$

It is negative, so d_N is a descent direction.

3. Consider the point

$$x_\alpha = x_0 + \alpha d_N = \begin{pmatrix} \alpha/2 \\ -\alpha \end{pmatrix},$$

where $\alpha \geq 0$ is the step performed along d_N. The point x_α verifies the first Wolfe condition if

$$\begin{aligned} f(x_\alpha) &\leq f(x_0) + \alpha\beta_1 \nabla f(x_0)^T d_N, \\ 2\alpha^2 - 4\alpha &\leq 0 - 4\alpha\beta_1, \end{aligned}$$

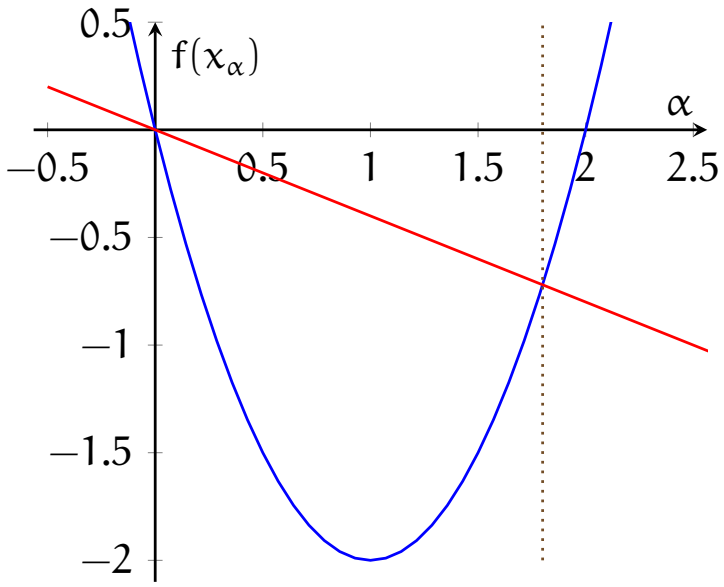
that is, if

$$\alpha \leq 2(1 - \beta_1).$$

If $\beta_1 = 0.1$, all steps α such that

$$\alpha \leq 1.8$$

verify the first Wolfe condition. It is illustrated by the figure below, where $f(x_\alpha)$ is represented in blue and the line $f(x_0) + \alpha\beta_1 \nabla f(x_0)^T d_N$ is represented in red. The slope of this line is $\beta_1 \nabla f(x_0)^T d_N = -0.4$. The value $\alpha = 1.8$ is represented by the dotted vertical line.



Question 07

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^2} f(x) = x_1^2 + x_1 x_2 + 2x_2^2,$$

and the point $x_0 = (1, 1)^T$.

1. Calculate Newton's direction at x_0.
2. Verify that it is a descent direction.
3. Consider the second Wolfe condition with $\beta_2 = 0.7$. What are the values of the step α that verify the condition?

Solution 07

1. Newton's direction is a solution of the linear system

$$\nabla^2 f(x_k) d_k = -\nabla f(x_k).$$

The derivatives of the function are

$$\nabla f(x) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 4x_2 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}. \tag{44}$$

Therefore, Newton's equations at $x_0 = (1, 1)^T$ are

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = - \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

that is

$$\begin{aligned} 2d_1 + d_2 &= -3 \\ d_1 + 4d_2 &= -5, \end{aligned}$$

and Newton's direction is

$$d_N = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

2. The directional derivative of the function along d_N at x_0 is

$$\nabla f(x_0)^T d_N = -8.$$

It is negative, so d_N is a descent direction.

3. Consider the point

$$x_\alpha = x_0 + \alpha d_N = \begin{pmatrix} 1 - \alpha \\ 1 - \alpha \end{pmatrix},$$

where $\alpha \geq 0$ is the step performed along d_N. Using (44), we have

$$\nabla f(x_\alpha) = \begin{pmatrix} 3 - 3\alpha \\ 5 - 5\alpha \end{pmatrix}.$$

Therefore, the point x_α verifies the second Wolfe condition if

$$\begin{aligned}\nabla f(x_\alpha)^T d_N &\geq \beta_2 \nabla f(x_0)^T d_N,\\ 8(\alpha - 1) &\geq -\beta_2 8,\end{aligned}$$

that is, if

$$\alpha \geq 1 - \beta_2.$$

If $\beta_2 = 0.7$, all steps α such that

$$\alpha \geq 0.3$$

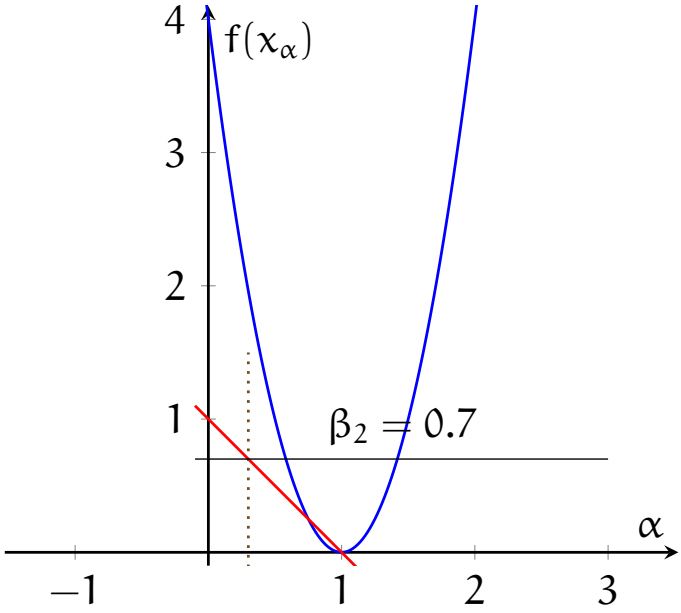
verify the second Wolfe condition. It is illustrated by the figure below, where $f(x_\alpha)$ is represented in blue and the line

$$\frac{\nabla f(x_\alpha)^T d_N}{\nabla f(x_0)^T d_N}$$

is represented in red. Note that, for that representation, we consider the second Wolfe written as

$$\frac{\nabla f(x_\alpha)^T d_N}{\nabla f(x_0)^T d_N} \leq \beta_2,$$

where the left hand side is plotted in red on the figure. The value $\alpha = 0.3$ is represented by the dotted vertical line.



Question 08

We are performing a line search along a direction d at a point x. The function evaluated at $x + \alpha d$ is defined as

$$g(\alpha) = f(x + \alpha d) = -1.833\alpha^3 + 8.5\alpha^2 - 10\alpha + 5.$$

Consider the step $\alpha = 3$. Consider the first Wolfe condition with $\beta_1 = 0.15$ and the second Wolfe condition with $\beta_2 = 0.8$.

1. Is the step $\alpha = 3$ too short, or too long?
2. Propose a value of α that verifies both Wolfe conditions.

Solution 08

1. As
$$g(\alpha) = f(x + \alpha d) = -1.833\alpha^3 + 8.5\alpha^2 - 10\alpha + 5,$$
the directional derivative is
$$g'(\alpha) = -3 \cdot 1.833\alpha^2 + 2 \cdot 8.5\alpha - 10.$$
Therefore, the second Wolfe condition for $\alpha = 3$ is
$$\frac{g'(3)}{g'(0)} \overset{?}{\leq} \beta_2,$$
$$-8.5/-10 = 0.85 \overset{?}{\leq} 0.8.$$

It is violated. Therefore, it is tempting to conclude that the step is too short. The first Wolfe condition for $\alpha = 3$ is
$$g(3) \overset{?}{\leq} g(0) + 3\beta_1 g'(0),$$
$$2.009 \overset{?}{\leq} 5 - 4.5 = 0.5.$$

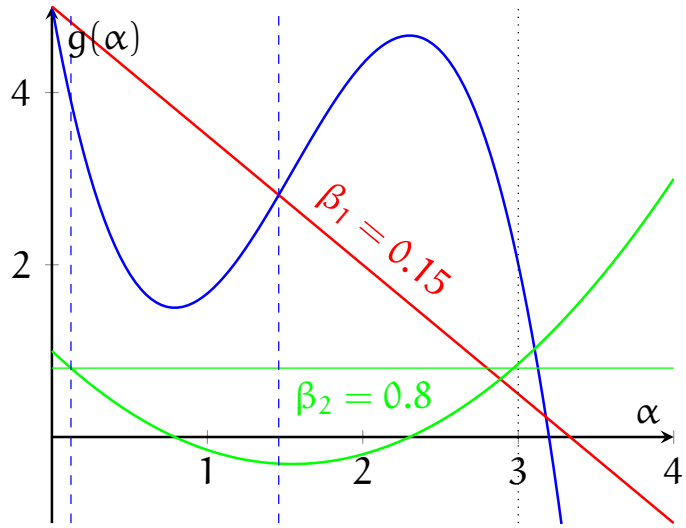
It is also violated. Therefore, it is tempting to conclude that the step is too long.

However, a step cannot be both too short and too long. The two conditions must be verified in the right order. The first Wolfe condition must be checked first. If it is violated, we conclude that the step is too long, and we do not need to consider the second condition. The second Wolfe condition is checked only when the step is not deemed too long by the first condition.

In the case described above, the step is too long, as it violates the first condition. And the algorithm should make it shorter.

In the next figure, the value of $g(\alpha)$ is plotted in blue, the line
$$g(0) + 3\beta_1 g'(0)$$
characterizing the first Wolfe condition is plotted in red, and the ratio
$$\frac{g'(\alpha)}{g'(0)}$$

characterizing the second Wolfe condition is plotted in green, as well as the threshold value $\beta_2 = 0.8$. The dotted vertical line corresponds to $\alpha = 3$.



2. All values of α such that

$$0.12251 \leq \alpha \leq 1.45912,$$

verify both conditions. The two thresholds are identified by the two dashed blue vertical lines on the picture above.

Question 09

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x) = \frac{9}{2}x_1^2 + \frac{1}{2}x_2^2.$$

Consider the point $x^+ = (1, 1)^T$.

1. Calculate the quadratic model of the function at x^+.
2. Calculate Newton's point at x^+.
3. Calculate Cauchy's point at x^+.
4. Perform one iteration of Newton's local method.

Solution 09

We start by calculating the derivatives of f:

$$\nabla f(x) = \begin{pmatrix} 9x_1 \\ x_2 \end{pmatrix}, \ \nabla^2 f(x) = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}.$$

1. The quadratic model of the function at x^+ is

$$\begin{aligned} m_{x^+}(x) &= f(x^+) + (x - x^+)^T \nabla f(x^+) + \frac{1}{2}(x - x^+)^T \nabla^2 f(x^+)(x - x^+), \\ &= 5 + 9(x_1 - 1) + (x_2 - 1) + \frac{1}{2}(x_1 - 1)(9x_1 - 9) + \frac{1}{2}(x_2 - 1)(x_2 - 1) \\ &= \frac{9}{2}x_1^2 + \frac{1}{2}x_2^2. \end{aligned}$$

   Actually, the objective function being already quadratic, the quadratic model is the function itself.

2. We solve the system of equations (Definition 10.3, Eq. (10.14) p. 243)

$$\begin{aligned} \nabla^2 f(x^+) d &= -\nabla f(x^+), \\ \begin{pmatrix} 9d_1 \\ d_2 \end{pmatrix} &= -\begin{pmatrix} 9 \\ 1 \end{pmatrix}. \end{aligned}$$

   The solution is $d_1 = -1$, $d_2 = -1$. Therefore, Newton's point is

$$x_N = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. We need to consider the objective function in the direction of the gradient, that is at a point

$$x_\alpha = \begin{pmatrix} 1 - 9\alpha \\ 1 - \alpha \end{pmatrix}.$$

   We have

$$\begin{aligned} f(x_\alpha) &= \frac{9}{2}(1 - 9\alpha)^2 + \frac{1}{2}(1 - \alpha)^2, \\ &= 365\alpha^2 - 82\alpha + 5. \end{aligned}$$

Its minimum is reached for

$$\alpha = \frac{41}{365} = 0.1123.$$

Note that, as f is convex, this value could have been obtained directly from Eq. (10.17) on page 244. Therefore, Cauchy's point (Definition 10.4, Eq.(10.15), p. 243) is

$$x_C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{41}{365} \begin{pmatrix} 9 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.01096 \\ 0.8877 \end{pmatrix}.$$

4. An iteration of Newton's local method calculates Newton's point. We have already calculated it:

$$x_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We note that the gradient is zero at that point. As the second derivative matrix is positive definite, we have found a minimum of the problem (it is actually the unique minimum). It illustrates an interesting property of Newton's local method: if the objective function is a convex quadratic function, Newton's local method converges in only one iteration.

Question 10

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x_1, x_2) = x_1^2 + 2x_2^2,$$

the point $x^0 = (9, 1)$, and the direction

$$d = \begin{pmatrix} -18 \\ -4 \end{pmatrix}.$$

Apply the line search algorithm with $\alpha_0 = 0.05$, $\beta_1 = 0.01$, $\beta_2 = 0.8$ and $\lambda = 20$.

Solution 10

The gradient of the function is

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix}.$$

Therefore,

$$\nabla f(x^0) = \begin{pmatrix} 18 \\ 4 \end{pmatrix},$$

and

$$\nabla f(x^0)^T d = -340 < 0,$$

and d is a descent direction (it happens to be the steepest descent direction). A point obtained by making a step α in the direction d is

$$x^+(\alpha) = \begin{pmatrix} 9 - 18\alpha \\ 1 - 4\alpha \end{pmatrix}.$$

We initialize $\alpha_\ell = 0$, and $\alpha_r = +\infty$. We try first $\alpha_0 = 0.05$, so that

$$x^+(0.05) = \begin{pmatrix} 8.1 \\ 0.8 \end{pmatrix}.$$

The first Wolfe condition is

$$f(x^+(\alpha_0)) \overset{?}{\leq} f(x^0) + \alpha_0 \beta_1 \nabla f(x^0)^T d,$$

$$66.89 \overset{?}{\leq} 83 - 0.17 = 82.83.$$

The first Wolfe condition is verified. The second Wolfe condition is

$$\frac{\nabla f(x^+(\alpha_0))^T d}{\nabla f(x^0)^T d} \overset{?}{\leq} \beta_2,$$

$$\frac{-304.4}{-340} = 0.895 \overset{?}{\leq} 0.8.$$

As Wolfe 2 is not verified, the step is too short. As α_r is still at $+\infty$, we multiply the step by $\lambda = 20$. We now have $\alpha_\ell = 0.05$ and $\alpha_1 = \lambda\alpha_0 = 1$, so that

$$x^+(1) = \begin{pmatrix} -9 \\ -3 \end{pmatrix}.$$

The first Wolfe condition is

$$f(x^+(\alpha_1)) \overset{?}{\leq} f(x^0) + \alpha_1\beta_1\nabla f(x^0)^T d,$$
$$99 \overset{?}{\leq} 83 - 3.4 = 79.6.$$

Wolfe 1 is violated. The step is too long. We set $\alpha_r = 1$, and

$$\alpha_2 = \frac{\alpha_\ell + \alpha_r}{2} = \frac{0.05 + 1}{2} = 0.525.$$

The first Wolfe condition is

$$f(x^+(\alpha_2)) \overset{?}{\leq} f(x^0) + \alpha_2\beta_1\nabla f(x^0)^T d,$$
$$2.6225 \overset{?}{\leq} 83 - 31.785 = 81.215.$$

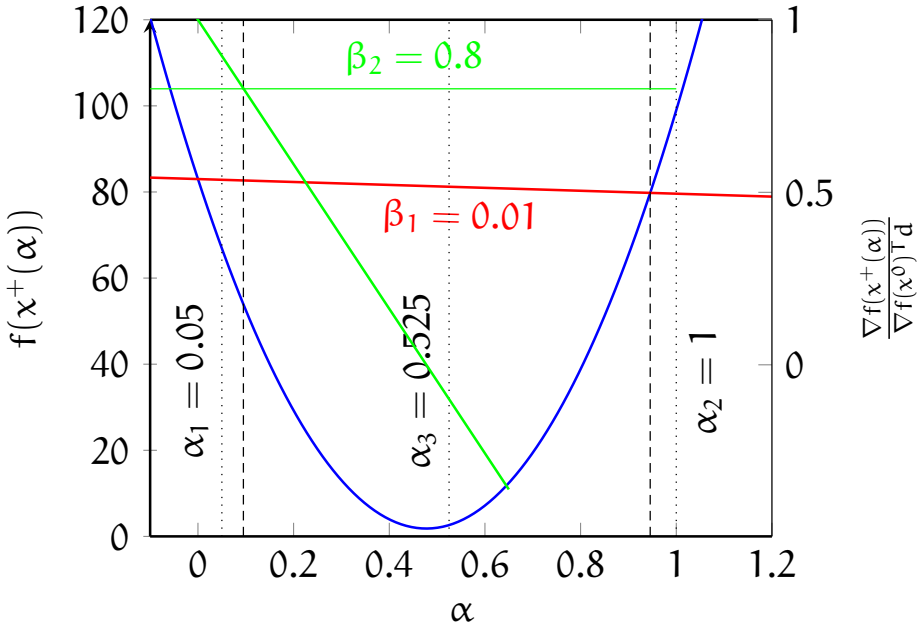
The condition is verified. We check Wolfe 2.

$$\frac{\nabla f(x^+(\alpha_2))^T d}{\nabla f(x^0)^T d} \overset{?}{\leq} \beta_2,$$
$$\frac{33.8}{-340} = -0.0994 \overset{?}{\leq} 0.8.$$

It is also verified. Therefore, the step $\alpha_2 = 0.525$ verifies both conditions and is selected by the algorithm.

All the quantities involved are plotted in the following figure. The x-axis corresponds to the length α of the step. The left y-axis corresponds to the value of the function $f(x^+(\alpha))$, plotted in blue. The line $f(x^0) + \alpha\beta_1\nabla f(x^0)^T d$ involved in the first Wolfe condition is plotted in red, with a value of $\beta_1 = 0.01$. The right axis corresponds to the quantity $\nabla f(x^+(\alpha))/\nabla f(x^0)^T d$ involved in the second Wolfe condition, and plotted in green. The threshold $\beta_2 = 0.8$ is also plotted in green. The three tested values of α are represented by the dotted line.

The line for the first Wolfe condition crosses the function at a value of $\alpha \approx 0.95$ (black dashed line on the right). The step $\alpha_1 = 1$ therefore violates Wolfe 1. The line for the second Wolfe condition reaches the threshold at $\alpha \approx 0.095$ (black dashed line on the left). The step $\alpha_0 = 0.05$ therefore violates Wolfe 2. And the step $\alpha_2 = 0.525$ verifies both.


120
100
80
60
40
20
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
α
$f(x^+(\alpha))$
$\frac{\nabla f(x^+(\alpha))}{\nabla f(x^0)^T d}$
$\beta_2 = 0.8$
$\beta_1 = 0.01$
$\alpha_1 = 0.05$
$\alpha_3 = 0.525$
$\alpha_2 = 1$

Question 11

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable nonlinear function, $x_k \in \mathbb{R}^n$ is a point and $d_k \in \mathbb{R}^n$ is a descent direction such that $\nabla f(x_k)^T d_k < 0$, and that f is bounded from below in the direction d_k. If we write the two Wolfe's conditions as:

$$f(x_k + \alpha d_k) \leq f(x_k) + \alpha \beta_1 \nabla f(x_k)^T d_k$$

and

$$\frac{\nabla f(x_k + \alpha d_k)^T d_k}{\nabla f(x_k)^T d_k} \leq \beta_2$$

Which condition should be satisfied by Wolfe's parameters β_1 and β_2 so that we can be sure that there exists a step size α that satisfies both Wolfe's conditions?

- Such as condition does not exist.
- $0 < \beta_1 < \beta_2 < 1$.
- $0 < \beta_2 < \beta_1 < 1$.
- $0 < \beta_1 = \beta_2 < 1$.

2. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable nonlinear function, $x_k \in \mathbb{R}^n$ is a point and $d_k \in \mathbb{R}^n$ is a descent direction. Consider α^* the step that minimizes the function along d_k, that is

$$f(x_k + \alpha^* d_k) \leq f(x_k + \alpha d_k), \quad \forall \alpha \geq 0$$

What can be said about α^*?

- α^* always verifies both Wolfe conditions.
- α^* always verifies the first Wolfe condition, but not necessarily the second.
- α^* verifies the second Wolfe condition, but not necessarily the first.
- Nothing can be said about the validity of the Wolfe conditions at α^* in the general case.

Solution 11

1. A sufficient condition for the Wolfe conditions to be compatible is that $0 < \beta_1 < \beta_2 < 1$. This is Theorem 11.9 in the book.

2. The second Wolfe condition is

$$\frac{\nabla f(x_k + \alpha^* d_k)^T d_k}{\nabla f(x_k)^T d_k} \leq \beta_2.$$

As α^* minimizes the function along d_k, the directional derivative at α^* is zero, that is

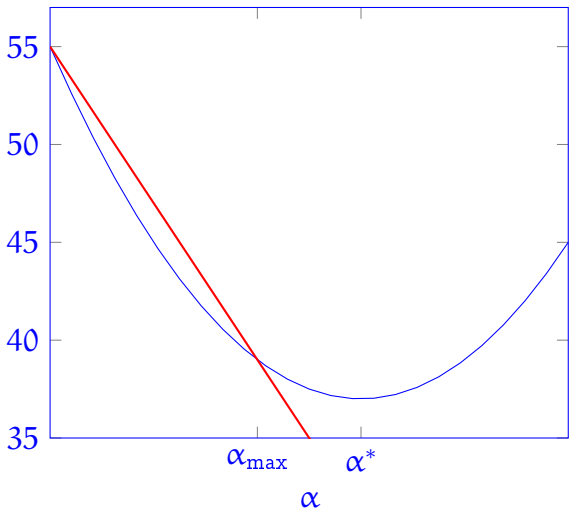
$$\nabla f(x_k + \alpha^* d_k)^T d_k = 0.$$

As $\beta_2 > 0$, the second Wolfe condition is verified.

The first Wolfe condition is not necessarily verified. If the value of β_1 is large, this condition allows only short steps. It is illustrated by the figure below, where

$$f(x_k + \alpha d_k) = 2\alpha^2 - 12\alpha + 55,$$

and $\beta_1 = \frac{2}{3}$. Therefore, the line defined by the first Wolfe condition is

$$-8\alpha + 55.$$

It crosses the curve at $\alpha_{max} = 2$, before its minimum $\alpha^* = 3$. Only steps shorter than α_{max} are accepted. And the step α^*, corresponding to the minimum, is rejected by the condition.