MATH-251(a) - Numerical analysis

Examples of exam questions

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The subjects that are examined through the following questions are those related to linear and nonlinear systems. You can answer a subquestion even if you did not answer all preceding subquestions. Feel free to contact `fabio.matti@epfl.ch` or `guillaume.olikier@epfl.ch` to ask questions.

Question 1 (approximate solutions to an overdetermined linear system)**.** Let $A := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ and $b \in \mathbb{R}^3$ such that $b_1 \geq b_2 \geq b_3$.

1. Prove that there exists $x \in \mathbb{R}$ such that $Ax = b$ if and only if $b_1 = b_2 = b_3$.

   If $b_1 = b_2 = b_3$, then $Ab_1 = b$. Otherwise, $b_1 > b_2$ or $b_2 > b_3$, thus, if $x \in \mathbb{R}$ satisfies $Ax = b$, then $x = b_1 = b_2 = b_3$, which is impossible.

2. For every $p \in \{1, 2, \infty\}$, solve the optimization problem $\min_{x\in\mathbb{R}} \|Ax - b\|_p$.

   For all $x \in \mathbb{R}$,

$$\|Ax - b\|_p = \begin{cases} \sum_{i=1}^{3} |x - b_i| & \text{if } p = 1, \\ \sqrt{\sum_{i=1}^{3} (x - b_i)^2} & \text{if } p = 2, \\ \max_{i\in\{1,2,3\}} |x - b_i| & \text{if } p = \infty. \end{cases}$$

   By definition of the absolute value, for all $x \in \mathbb{R}$,

$$\|Ax - b\|_1 = \begin{cases} b_1 + b_2 + b_3 - 3x & \text{if } x \leq b_3, \\ b_1 + b_2 - b_3 - x & \text{if } b_3 \leq x \leq b_2, \\ b_1 - b_2 - b_3 + x & \text{if } b_2 \leq x \leq b_1, \\ 3x - b_1 - b_2 - b_3 & \text{if } x \geq b_1. \end{cases}$$

   Thus, the function $\mathbb{R} \to \mathbb{R} : x \mapsto \|Ax - b\|_1$ is strictly decreasing on $(-\infty, b_2]$ and strictly increasing on $[b_2, \infty)$. Hence, b_2 is its unique minimizer and the global minimum is $b_1 - b_3$.

   A real number minimizes the function $\mathbb{R} \to \mathbb{R} : x \mapsto \|Ax - b\|_2$ if and only if it minimizes the squared function. For all $x \in \mathbb{R}$,

$$\|Ax - b\|_2^2 = \sum_{i=1}^{3} (x - b_i)^2 = 3x^2 - 2(b_1 + b_2 + b_3)x + (b_1^2 + b_2^2 + b_3^2).$$

   Thus, $(b_1 + b_2 + b_3)/3$ is the unique global minimizer of the function $\mathbb{R} \to \mathbb{R} : x \mapsto \|Ax - b\|_2$ and the global minimum is $\sqrt{(2/3)(b_1^2 + b_2^2 + b_3^2 - b_1b_2 - b_1b_3 - b_2b_3)}$.

   By inspection, for all $x \in \mathbb{R}$,

$$\|Ax - b\|_\infty = \begin{cases} b_1 - x & \text{if } x \leq (b_1 + b_3)/2, \\ x - b_3 & \text{if } x \geq (b_1 + b_3)/2. \end{cases}$$

   Thus, the function $\mathbb{R} \to \mathbb{R} : x \mapsto \|Ax - b\|_\infty$ is strictly decreasing on $(-\infty, (b_1 + b_3)/2]$ and strictly increasing on $[(b_1 + b_3)/2, \infty)$. Hence, $(b_1 + b_3)/2$ is its unique minimizer and the global minimum is $(b_1 - b_3)/2$.

Question 2 (global convergence of the Newton iteration). Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{e}^x - x - 2$.

1. Prove that f has exactly two zeros, one negative, denoted $\underline{x}$, and one positive, denoted $\overline{x}$.

   Since f is continuous, $f(-2) > 0 > f(-1)$, and $f(1) < 0 < f(2)$, the intermediate value theorem ensures that f has a zero on $(-2, -1)$ and a zero on $(1, 2)$. Furthermore, for all $x \in \mathbb{R}$, $f'(x) = \mathrm{e}^x - 1$. Thus, for every $x \in \mathbb{R}$, the sign of $f'(x)$ is the sign of x. Hence, f is strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, \infty)$, and its global minimum is $f(0) = -1$. Therefore, f has at most one zero on $(-\infty, 0)$ and at most one zero on $(0, \infty)$.

2. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto x - \frac{f(x)}{f'(x)}$. Deduce from the sign of f that:

   (a) for all $x \in (-\infty, \underline{x})$, $x < g(x) < \underline{x}$;

   (b) for all $x \in (\underline{x}, 0)$, $g(x) < \underline{x}$;

   (c) for all $x \in (0, \overline{x})$, $g(x) > \overline{x}$;

   (d) for all $x \in (\overline{x}, \infty)$, $\overline{x} < g(x) < x$.

   For all $x \in \mathbb{R} \setminus \{0\}$, $g'(x) = \frac{f(x) f''(x)}{f'(x)^2} = f(x) \frac{\mathrm{e}^x}{f'(x)^2}$, hence the sign of $g'(x)$ is the sign of $f(x)$, which can be deduced immediately from the preceding point. Thus, for all $x \in (-\infty, \underline{x}) \cup (\overline{x}, \infty)$, $g'(x) > 0$ and, for all $x \in (\underline{x}, \overline{x}) \setminus \{0\}$, $g'(x) < 0$. Therefore, g is strictly increasing on $(-\infty, \underline{x}]$ and on $[\overline{x}, \infty)$, and strictly decreasing on $[\underline{x}, 0)$ and on $(0, \overline{x}]$. Furthermore, for all $x \in \mathbb{R} \setminus \{0\}$, $x - g(x) = \frac{f(x)}{f'(x)}$, hence $x - g(x) < 0$ if $x < \underline{x}$ and $x - g(x) > 0$ if $x > \overline{x}$. The result follows since $\underline{x}$ and $\overline{x}$ are fixed points of g.

3. Deduce that, for every $x_0 \in \mathbb{R} \setminus \{\underline{x}, 0, \overline{x}\}$, the Newton iteration generates a sequence $(x_i)_{i \in \mathbb{N}}$ that converges to $\underline{x}$ if $x_0 < 0$ and to $\overline{x}$ if $x_0 > 0$.

   Given $x_0 \in \mathbb{R} \setminus \{\underline{x}, 0, \overline{x}\}$, the Newton iteration is $x_{i+1} := g(x_i)$ for all $i \in \mathbb{N}$ such that $x_i \neq 0$. Consider the case where $x_0 > 0$; the case where $x_0 < 0$ is similar. If $x_0 > \overline{x}$, then, by 2d, the Newton iteration generates a strictly decreasing sequence $(x_i)_{i \in \mathbb{N}}$ in $(\overline{x}, x_0]$, which converges since it is bounded from below. By continuity of g, the limit is a fixed point of g. As the fixed points of g are exactly the zeros of f, the limit is $\overline{x}$. If $x_0 \in (0, \overline{x})$, then $x_1 := g(x_0) > \overline{x}$ and the same reasoning applies.

4. Prove that, for all $x \in (\overline{x}, \infty)$, $g(x) - \overline{x} < \frac{3}{4}(x - \overline{x})^2$.

   Let $x \in (\overline{x}, \infty)$. By Taylor's theorem, there exists $a \in (\overline{x}, x)$ such that $f(\overline{x}) = f(x) + f'(x)(\overline{x} - x) + \frac{1}{2} f''(a)(x - \overline{x})^2$. Thus, since $f(\overline{x}) = 0$, $g(x) - \overline{x} = \frac{f''(a)}{2f'(x)}(x - \overline{x})^2$. Since $f''(a) = \mathrm{e}^a < \mathrm{e}^x$, $\frac{f''(a)}{f'(x)} < \frac{\mathrm{e}^x}{\mathrm{e}^x - 1} = 1 + \frac{1}{\mathrm{e}^x - 1}$. Since the function $(0, \infty) \to \mathbb{R} : y \mapsto 1 + \frac{1}{\mathrm{e}^y - 1}$ is strictly decreasing, $\frac{\mathrm{e}^x}{\mathrm{e}^x - 1} < \frac{\mathrm{e}^{\overline{x}}}{\mathrm{e}^{\overline{x}} - 1} = \frac{\overline{x} + 2}{\overline{x} + 1}$. Since the function $(-1, \infty) \to \mathbb{R} : y \mapsto \frac{y+2}{y+1}$ is strictly decreasing and $\overline{x} > 1$, $\frac{\overline{x}+2}{\overline{x}+1} < \frac{3}{2}$. The result follows.

5. Deduce that, for every $x_0 \in (\overline{x}, \infty)$, the sequence $(x_i)_{i \in \mathbb{N}}$ generated by the Newton iteration satisfies $0 < x_i - \overline{x} \leq (\frac{3}{4})^{2^i - 1}(x_0 - \overline{x})^{2^i}$ for all $i \in \mathbb{N}$.

   Let us prove both inequalities by induction. They are true if $i = 0$. Furthermore, if they are true for $i \in \mathbb{N}$, then they are also true for $i + 1$. This follows from 2d for the first inequality, and from the preceding point for the second inequality: $x_{i+1} - \overline{x} < \frac{3}{4}(x_i - \overline{x})^2 \leq \frac{3}{4}(\frac{3}{4})^{2^{i+1} - 2}(x_0 - \overline{x})^{2^{i+1}} = (\frac{3}{4})^{2^{i+1} - 1}(x_0 - \overline{x})^{2^{i+1}}$.

6. Given $\varepsilon \in (0, \frac{4}{3})$ and $x_0 \in (\overline{x}, \overline{x} + \frac{3}{4}]$, deduce that $x_i - \overline{x} \leq \varepsilon$ for all integers $i \geq \log_2 \left(1 - \log_{\frac{4}{3}} \varepsilon\right) - 1$.

   Let $i \in \mathbb{N}$. By the preceding point, $x_i - \overline{x} \leq (\frac{3}{4})^{2^i - 1}(\frac{3}{4})^{2^i} = (\frac{3}{4})^{2^{i+1} - 1}$. Thus, $x_i - \overline{x} \leq \varepsilon$ if $(\frac{3}{4})^{2^{i+1} - 1} \leq \varepsilon$, i.e., $i \geq \log_2 \left(1 - \log_{\frac{4}{3}} \varepsilon\right) - 1$.

Question 3 (divergence of the Jacobi iteration for a symmetric positive-definite matrix)**.** Let

$$A := \begin{bmatrix} 29 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & \frac{1}{5} \end{bmatrix}.$$

1. Compute the characteristic polynomial of A. Let p be the associated polynomial function.

   The characteristic polynomial of A is $\det(\lambda I_3 - A) = \lambda^3 - \frac{176}{5}\lambda^2 + 175\lambda - 3$.

2. Prove that $p(\lambda) \leq -3$ for all $\lambda \in (-\infty, 0]$.

   This follows from the signs of the coefficients of the polynomial.

3. Deduce that A is symmetric positive-definite.

   Since A is symmetric, its eigenvalues, which are the zeros of p, are real. By the preceding point, the zeros of p are positive. Thus, A is symmetric positive-definite.

4. Define $M := \operatorname{diag}(29, 6, \frac{1}{5})$ and $N := M - A$. Compute $M^{-1}N$.

   Since $M^{-1} = \operatorname{diag}(\frac{1}{29}, \frac{1}{6}, 5)$, a straightforward computation shows that

$$M^{-1}N = -\begin{bmatrix} 0 & \frac{2}{29} & \frac{1}{29} \\ \frac{1}{3} & 0 & \frac{1}{6} \\ 5 & 5 & 0 \end{bmatrix}.$$

5. Compute the characteristic polynomial of $M^{-1}N$. Let q be the associated polynomial function.

   The characteristic polynomial of $M^{-1}N$ is $\det(\lambda I_3 - M^{-1}N) = \lambda^3 - \frac{179}{174}\lambda + \frac{10}{87}$.

6. Prove that q has a zero on $(-2, -1)$.

   Since q is continuous (as a polynomial function) and $q(-2) < 0 < q(-1)$, the intermediate value theorem ensures that q has a zero on $(-2, -1)$.

7. Deduce that the Jacobi iteration for A does not converge for every initial iterate in $\mathbb{R}^3$.

   By the preceding point, the spectral radius of $M^{-1}N$ is greater than 1. Thus, by a theorem seen in a lecture, the Jacobi iteration for A does not converge for every initial iterate in $\mathbb{R}^3$.