MATH-251(a) - Numerical analysis

Examples of exam questions

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The subjects that are examined through the following questions are those related to linear and nonlinear systems. You can answer a subquestion even if you did not answer all preceding subquestions. Feel free to contact `fabio.matti@epfl.ch` or `guillaume.olikier@epfl.ch` to ask questions.

Question 1 (approximate solutions to an overdetermined linear system). Let $A := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ and $b \in \mathbb{R}^3$ such that $b_1 \geq b_2 \geq b_3$.

1. Prove that there exists $x \in \mathbb{R}$ such that $Ax = b$ if and only if $b_1 = b_2 = b_3$.
2. For every $p \in \{1, 2, \infty\}$, solve the optimization problem $\min_{x \in \mathbb{R}} \|Ax - b\|_p$.

Question 2 (global convergence of the Newton iteration). Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{e}^x - x - 2$.

1. Prove that f has exactly two zeros, one negative, denoted $\underline{x}$, and one positive, denoted $\overline{x}$.
2. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto x - \frac{f(x)}{f'(x)}$. Deduce from the sign of f that:
   (a) for all $x \in (-\infty, \underline{x})$, $x < g(x) < \underline{x}$;
   (b) for all $x \in (\underline{x}, 0)$, $g(x) < \underline{x}$;
   (c) for all $x \in (0, \overline{x})$, $g(x) > \overline{x}$;
   (d) for all $x \in (\overline{x}, \infty)$, $\overline{x} < g(x) < x$.
3. Deduce that, for every $x_0 \in \mathbb{R} \setminus \{\underline{x}, 0, \overline{x}\}$, the Newton iteration generates a sequence $(x_i)_{i \in \mathbb{N}}$ that converges to $\underline{x}$ if $x_0 < 0$ and to $\overline{x}$ if $x_0 > 0$.
4. Prove that, for all $x \in (\overline{x}, \infty)$, $g(x) - \overline{x} < \frac{3}{4}(x - \overline{x})^2$.
5. Deduce that, for every $x_0 \in (\overline{x}, \infty)$, the sequence $(x_i)_{i \in \mathbb{N}}$ generated by the Newton iteration satisfies $0 < x_i - \overline{x} \leq (\frac{3}{4})^{2^i - 1}(x_0 - \overline{x})^{2^i}$ for all $i \in \mathbb{N}$.
6. Given $\varepsilon \in (0, \frac{4}{3})$ and $x_0 \in (\overline{x}, \overline{x} + \frac{3}{4}]$, deduce that $x_i - \overline{x} \leq \varepsilon$ for all integers $i \geq \log_2\left(1 - \log_{\frac{4}{3}} \varepsilon\right) - 1$.

Question 3 (divergence of the Jacobi iteration for a symmetric positive-definite matrix). Let

$$A := \begin{bmatrix} 29 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & \frac{1}{5} \end{bmatrix}.$$

1. Compute the characteristic polynomial of A. Let p be the associated polynomial function.
2. Prove that $p(\lambda) \leq -3$ for all $\lambda \in (-\infty, 0]$.
3. Deduce that A is symmetric positive-definite.
4. Define $M := \mathrm{diag}(29, 6, \frac{1}{5})$ and $N := M - A$. Compute $M^{-1}N$.
5. Compute the characteristic polynomial of $M^{-1}N$. Let q be the associated polynomial function.
6. Prove that q has a zero on $(-2, -1)$.
7. Deduce that the Jacobi iteration for A does not converge for every initial iterate in $\mathbb{R}^3$.