MATH-251(a) - Numerical analysis

Examples of exam questions on numerical methods for initial-value problems

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In the following questions, $n \in \mathbb{N} \setminus \{0\}$, $t_0 \in \mathbb{R}$, and $u_0 \in \mathbb{R}^n$ are given.

Question 1 (order of convergence of the backward Euler method)**.** Let $\tau, r \in (0, \infty)$ and $f \in C^1([t_0, t_0 + \tau] \times B[u_0, r], \mathbb{R}^n)$. Let $L \coloneqq \mathrm{Lip}^2_{[t_0,t_0+\tau]\times B[u_0,r]} f \in (0, \infty)$. Assume that $u : [t_0, t_0 + \tau] \to B[u_0, r]$ is the unique solution to the IVP

$$\begin{cases} u'(t) = f(t, u(t)) \text{ for all } t \in [t_0, t_0 + \tau], \\ u(t_0) = u_0. \end{cases}$$

Given an integer $m > \tau L$, define $h \coloneqq \tau/m$. The backward Euler method, which iterates

$$u_{i+1} = u_i + hf(t_{i+1}, u_{i+1}),$$

generates a sequence $(u_1, \ldots, u_m)$ such that, for all $i \in \{1, \ldots, m\}$, $u_i \in B[u_0, \frac{i}{m} r]$. Prove that there exists $c \in (0, \infty)$ such that

$$\|u(t_0 + \tau) - u_m\| \le \frac{c}{L}\left(\exp\left(\frac{\tau L}{1 - hL}\right) - 1\right) h.$$

The only result from Section 7.1.3 of the lecture notes that you can use is the following.

Lemma. Let $a, b \in (0, \infty)$ and $(v_i)_{i\in\mathbb{N}}$ be a sequence in $[0, \infty)$ such that $v_0 = 0$ and, for all $i \in \mathbb{N}$,

$$v_{i+1} \le (1 + a)v_i + b.$$

Then, for all $i \in \mathbb{N}$,

$$v_i \le \frac{b}{a}(\exp(ai) - 1).$$

Define the *local discretization error* as

$$\Delta_h : [t_0, t_0 + \tau - h] \to \mathbb{R}^n : t \mapsto \frac{u(t + h) - u(t)}{h} - f(t + h, u(t + h)).$$

By Proposition 7.1.5 of the lecture notes, $u \in C^2([t_0, t_0 + \tau], B[u_0, r])$. Thus, by Taylor's theorem (Theorem B.3.3 of the lecture notes), for all $t \in [t_0, t_0 + \tau - h]$,

$$\|\Delta_h(t)\| \le \frac{1}{2} \max_{s\in[t,t+h]} \|u''(s)\|h \le \underbrace{\frac{1}{2} \max_{s\in[t_0,t_0+\tau]} \|u''(s)\|}_{=:c}\, h.$$

Let $i \in \{0, \ldots, m - 1\}$. Then,

$$\begin{aligned} u(t_{i+1}) - u_{i+1} &= (\Delta_h(t_i)h + u(t_i) + hf(t_{i+1}, u(t_{i+1}))) - (u_i + hf(t_{i+1}, u_{i+1})) \\ &= (u(t_i) - u_i) + h\left(f(t_{i+1}, u(t_{i+1})) - f(t_{i+1}, u_{i+1})\right) + \Delta_h(t_i)h. \end{aligned}$$

Thus,

$$\begin{aligned} \|u(t_{i+1}) - u_{i+1}\| &\le \|u(t_i) - u_i\| + h\,\|f(t_{i+1}, u(t_{i+1})) - f(t_{i+1}, u_{i+1})\| + \|\Delta_h(t_i)\|h \\ &\le \frac{\|u(t_i) - u_i\| + ch^2}{1 - hL}. \end{aligned}$$

Therefore, by the lemma,

$$\|u(t_i) - u_i\| \le \frac{c}{L}\left(\exp\left(\frac{hLi}{1 - hL}\right) - 1\right) h.$$

In particular,

$$\|u(t_0 + \tau) - u_m\| \le \frac{c}{L}\left(\exp\left(\frac{\tau L}{1 - hL}\right) - 1\right) h.$$

Question 2 (absolute stability of the second-order Taylor method)**.** Given $f \in \mathrm{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $h \in (0, \infty)$, the second-order Taylor method iterates

$$u_{i+1} \coloneqq u_i + hf(t_i, u_i) + \frac{h^2}{2}\left(\partial_1 f(t_i, u_i) + \partial_2 f(t_i, u_i) f(t_i, u_i)\right).$$

Determine the region of absolute stability of that method.
Given $\lambda \in \mathbb{C}$ such that $\Re(\lambda) < 0$, we consider $f : (t, x) \mapsto \lambda x$. Then, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\partial_1 f(t, x) = 0, \qquad \partial_2 f(t, x) = \lambda I_n.$$

The iteration becomes

$$u_{i+1} \coloneqq \phi(h\lambda) u_i$$

with

$$\phi : \mathbb{C} \to \mathbb{C} : z \mapsto 1 + z + \frac{z^2}{2}.$$

The region of absolute stability is

$$\{z \in \mathbb{C} \mid |\phi(z)| < 1\},$$

the same as that of Heun's method.