MATH-251(a) - Numerical analysis

Examples of exam questions on numerical methods for initial-value problems

December 18, 2025

In the following questions, $n \in \mathbb{N} \setminus \{0\}$, $t_0 \in \mathbb{R}$, and $u_0 \in \mathbb{R}^n$ are given.

**Question 1** (order of convergence of the backward Euler method)**.** Let $\tau, r \in (0, \infty)$ and $f \in \mathrm{C}^1([t_0, t_0 + \tau] \times B[u_0, r], \mathbb{R}^n)$. Let $L \coloneqq \mathrm{Lip}^2_{[t_0,t_0+\tau]\times B[u_0,r]} f \in (0, \infty)$. Assume that $u : [t_0, t_0 + \tau] \to B[u_0, r]$ is the unique solution to the IVP

$$\begin{cases} u'(t) = f(t, u(t)) \text{ for all } t \in [t_0, t_0 + \tau], \\ u(t_0) = u_0. \end{cases}$$

Given an integer $m > \tau L$, define $h \coloneqq \tau/m$. The backward Euler method, which iterates

$$u_{i+1} = u_i + hf(t_{i+1}, u_{i+1}),$$

generates a sequence $(u_1, \ldots, u_m)$ such that, for all $i \in \{1, \ldots, m\}$, $u_i \in B[u_0, \frac{i}{m} r]$. Prove that there exists $c \in (0, \infty)$ such that

$$\|u(t_0 + \tau) - u_m\| \leq \frac{c}{L} \left( \exp\left( \frac{\tau L}{1 - hL} \right) - 1 \right) h.$$

The only result from Section 7.1.3 of the lecture notes that you can use is the following.

**Lemma.** Let $a, b \in (0, \infty)$ and $(v_i)_{i \in \mathbb{N}}$ be a sequence in $[0, \infty)$ such that $v_0 = 0$ and, for all $i \in \mathbb{N}$,

$$v_{i+1} \leq (1 + a)v_i + b.$$

Then, for all $i \in \mathbb{N}$,

$$v_i \leq \frac{b}{a}(\exp(ai) - 1).$$

**Question 2** (absolute stability of the second-order Taylor method)**.** Given $f \in \mathrm{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $h \in (0, \infty)$, the second-order Taylor method iterates

$$u_{i+1} \coloneqq u_i + hf(t_i, u_i) + \frac{h^2}{2} \left( \partial_1 f(t_i, u_i) + \partial_2 f(t_i, u_i) f(t_i, u_i) \right).$$

Determine the region of absolute stability of that method.