Probability and Statistics — MATH-235
Autumn 2025 — Mats J. Stensrud

Problem Sheet 1

Exercise 1. A *round-robin tournament* is being held with 2^n tennis players; this means that every player will play against every other player exactly once.

1. How many games are played in total?

2. How many possible outcomes are there for the tournament (the outcome lists out who won and who lost for each game)?

Solution 1.

1. Each of the 2^n players has to play against each of the $2^n - 1$ other players, so in total we would expect $2^n \cdot (2^n - 1)$ tennis matches. However, we double counted each of the pairings, as the match "Player A" vs "Player B" is the same as the match "Player B" vs "Player A". Therefore, $\frac{2^n \cdot (2^n - 1)}{2}$ games are played in total.

   A possible, not entirely formal, approach is to start with small numbers, and try to generalize from there:

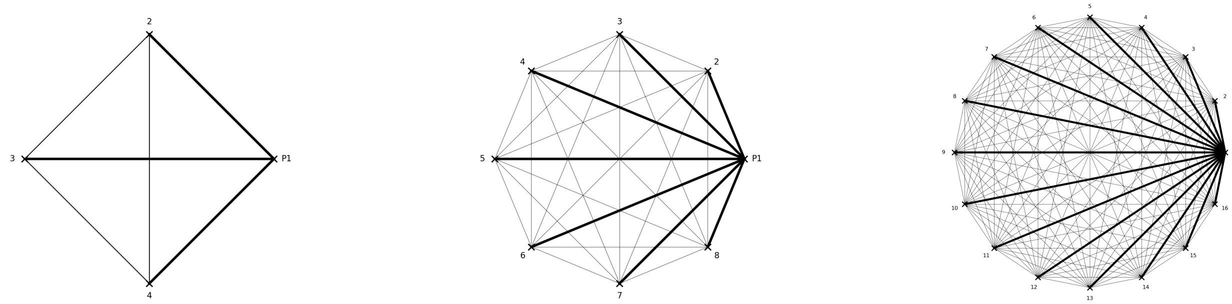

Figure 1: Number of games for $n = 2, 3, 4$. The games played by Player 1 are highlighted.

2. Each of the games can have two outcomes: Either the player who starts serving wins, or the player who starts as a receiver wins. In the previous exercise we derived that there are $\frac{2^n \cdot (2^n - 1)}{2}$ games on the tournament. For each game we can pick between the two possible outcomes, so following from the multiplication rule, there can be $2^{\frac{2^n \cdot (2^n - 1)}{2}}$ different outcomes to the tournament.

Exercise 2. A *knock-out tournament* is being held with 2^n tennis players. This means that for each round the winners move on to the next round and the losers are eliminated, until only one person remains. For example, if initially there are $2^4 = 16$ players, then there are 8 games in the first round, then the 8 winners move on to round 2, then the 4 winners move on to round 3, then the 2 winners move on to round 4, the winner of which is declared the winner of the tournament. (There are various systems for determining who plays whom within a round, but these do not matter for this problem.)

1. How many rounds are there?
2. Count how many games in total are played, by adding up the numbers of games played in each round.
3. Count how many games in total are played, this time directly thinking about it without doing almost any calculations.

   Hint: How many players need to be eliminated?

Solution 2. 1. In each of the games, one player loses and the other wins. In other words, in each of the rounds, half of the current players are eliminated, until only one player, the winner of the tournament remains. Therefore, the question can be reduced to finding x, such that $2^n \cdot \left(\frac{1}{2}\right)^x = 1 = 2^0$. It follows that $x = n$, i.e. there are n rounds, congruent with the example provided.

Similarly to the previous exercise, you can start with smaller number and try to observe the pattern:

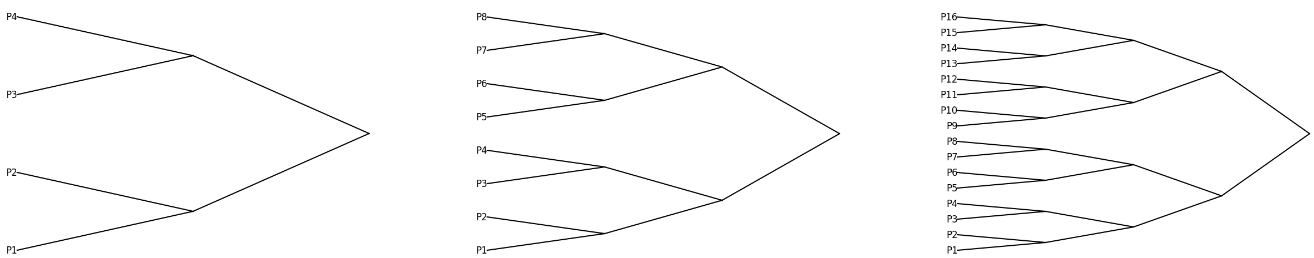

Figure 2: Number of games and rounds for $n = 2, 3, 4$ in a knockout tournament.

2. We start with 2^n players in the first round, and we create pairs for each of the matches. This means that in the first round there are $2^n/2 = 2^{n-1}$ matches, and 2^{n-1} players continue to the second round. In the second round we create $2^{n-1}/2 = 2^{n-2}$ pairs, and from these matches 2^{n-2} players qualify to the third round. Following this pattern, we have $2^{n-n} = 1$ match in the n-th round. That is, $\sum_{i=1}^{n} 2^{n-i} = \frac{2^n-1}{2-1} = 2^n - 1$ games are played in the tournament. (Please consult the Math Appendix of the course notes, if you are unfamiliar with the sums of Geometric series.)

   While in Exercise 1 the number of games grows quadratically with the number of players, here the relationship is linear. This makes a reasonable argument for tennis tournaments adapting the *knock-out* format, as the $128 = 2^7$ tennis players on the main draw of Grand Slams only need to play $128 - 1 = 127$ games, while following the *round robin* format, this number would be $128 \cdot 127/2 = 8128$.

3. Every match results in the elimination of exactly one player, hence, by counting the number of eliminations, we can count the number of games. Once a player is eliminated, they cannot reenter the tournament, that implies all the eliminated players are eliminated exactly once. Moreover, in the tournament every player is eliminated, except for the winner of the tournament. I.e. the number of eliminations is equal to $2^n - 1$, hence there are $2^n - 1$ games in total.

Exercise 3. Give a story proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

for all positive integers n.

Hint: Consider choosing a committee of size n with a single president or the committee, from two groups of size n each, where only one of the two groups has people eligible to become the president of the committee.

Solution 3. Using the equality

$$\binom{n}{k} = \binom{n}{n-k}$$

we can rewrite the left-hand side as

$$\sum_{k=1}^{n} k\binom{n}{k}\binom{n}{n-k}$$

This formulation can be interpreted that we have two groups of size n. From the first group, which group consists of "president eligible" people, we pick k members for the committee $\binom{n}{k}$, and from the k members we can pick a president $\binom{k}{1} = k$, this constitutes the $k\binom{n}{k}$ part. Then from the other group of n individuals, we can pick the remaining $n-k$ members for the committee, $\binom{n}{n-k}$ in different ways. Finally, we sum up over the different compositions of the two groups, i.e. 1 person from the president eligible group, $n-1$ from the non-eligible group; 2 people from the eligible group, $n-2$ from the non-eligible group; 3 people from the eligibles, $n-3$ from the non-eligible, etc.

Alternatively, let us first pick the committee president from the first group of n people, clearly, we have n many options for that. Then combine the remaining $n-1$ people from the first group with the n people from the second, as they will have equivalent roles. From this larger group of $2n-1$, we have to pick $n-1$ committee members, that we can do $\binom{2n-1}{n-1}$ different ways. Thus following this order, we can pick our committee members $n\binom{2n-1}{n-1}$ in different ways, that establish the identity.

Note: The president of the committee must be picked from the first group, hence we must select at least one person from the first group (who will consequently be the president). Thus for the sake of the story proof, we can pick all members from the first group ($k = n$), but we cannot pick all the members from the second group ($k = 0$), although the equality would not change.

Exercise 4. Show that for any events A and B,

$$P(A) + P(B) - 1 \overset{(1)}{\leq} P(A \cap B) \overset{(2)}{\leq} P(A \cup B) \overset{(3)}{\leq} P(A) + P(B).$$

For each of these three inequalities, give a simple criterion for when the inequality is actually an equality (e.g., give a simple condition such that $P(A \cap B) = P(A \cup B)$ if and only if the condition holds).

Solution 4. From Theorem 1.6.2 of the lecture notes, it follows that

$$P(A) + P(B) - P(A \cup B) = P(A \cap B).$$

[From Figure 3, this corresponds to $(I + II) + (II + III) - (II) = (I + II + III)$.]

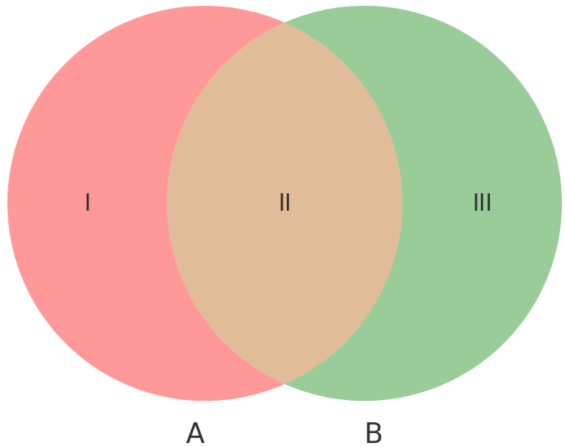

Figure 3: Illustration of A and B.

By the same Theorem, $P(A \cup B) \leq 1$, hence

$$P(A) + P(B) - 1 \leq P(A) + P(B) - P(A \cup B) = P(A \cap B),$$

thus (1) holds.

By definition of the union and the intersection $A \cup B \supseteq A \cap B$, therefore by Theorem 1.6.2

$$P(A \cap B) \leq P(A \cup B),$$

that shows (2). [See Figure 3, $(II) \leq (I + II + III)$]

$P(A)+P(B)-P(A \cap B) = P(A \cup B)$ and $P(A \cap B) \geq 0$. Thus (3) holds, $P(A)+P(B) \geq P(A \cup B)$.

$$P(A) + P(B) - 1 = P(A) + P(B) - P(A \cup B) = P(A \cap B)$$

if and only if $P(A \cup B) = 1$.

$$P(A \cap B) = P(A \cup B)$$

if and only if $A \cap B = A \cup B \iff A = B$.

$$P(A \cup B) = P(A) + P(B)$$

if and only if $P(A \cap B) = 0$.

Exercise 5. Let A and B be events. The *difference* between $B - A$ is defined to be the set of all elements of B that are not in A (equivalently denoted as $B \setminus A$). Show that if $A \subseteq B$, then

$$P(B - A) = P(B) - P(A),$$

directly using the axioms of probability.

Solution 5. Since $A \subseteq B$, $B = (B - A) \cup A$, where A and $B - A$ are disjoint sets. Then by the axioms of probability,

$$P(B) = P(B - A) + P(A),$$

and the result follows.

If the solution is unclear, please try to draw the diagram.

Exercise 6. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

Hint: "Assign" 1 class to all 5 days, and consider how the remaining ones can be distributed.

Solution 6. *Direct Counting Method:* There are two general ways that Alice can have class every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1 day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is $\binom{5}{2}\binom{6}{2}^2 6^3$ (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 class for the other days). The number of possibilities for the latter is $\binom{5}{1}\binom{6}{3}6^4$. So the probability is

$$\frac{\binom{5}{2}\binom{6}{2}^2 6^3 + \binom{5}{1}\binom{6}{3}6^4}{\binom{30}{7}} = \frac{114}{377}$$

Inclusion-Exclusion Method: We will use inclusion-exclusion to find the probability of the complement, which is the event that she has at least one day with no classes. Let $B_i = A_i^c$ denote the event that Alice has no class on day i. Then

$$P(B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5) = \sum_i P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$$

(terms with the intersection of 4 or more B_i's are not needed since Alice must have classes on at least 2 days, as $7 > 6$). We have

$$P(B_1) = \frac{\binom{24}{7}}{\binom{30}{7}},\ P(B_1 \cap B_2) = \frac{\binom{18}{7}}{\binom{30}{7}},\ P(B_1 \cap B_2 \cap B_3) = \frac{\binom{12}{7}}{\binom{30}{7}},$$

Since for $P(B_1)$, she cannot have classes on Monday, so she should select all 7 classes from the remaining $30 - 6 = 24$ options. For $P(B_1 \cap B_2)$, she cannot have classes on Monday and Tuesday, so she should select all 7 classes from the remaining $30 - 2 \cdot 6 = 18$ options, etc. The probability of the other intersections follow similarly. So

$$P(B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5) = 5\frac{\binom{24}{7}}{\binom{30}{7}} - \binom{5}{2}\frac{\binom{18}{7}}{\binom{30}{7}} + \binom{5}{3}\frac{\binom{12}{7}}{\binom{30}{7}} = \frac{263}{377},$$

since she can select 1 day to not to have classes 5 different ways, 2 days from the 5, to not to have classes $\binom{5}{2}$ ways, and 3 days $\binom{5}{3}$ ways.

Therefore (by De Morgan's laws from the course notes),

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{114}{377}.$$

Exercise 7. A widget inspector inspects 12 widgets and finds that exactly 3 are defective. Unfortunately, the widgets then get all mixed up and the inspector has to find the 3 defective widgets one by one.

1. Find the probability that the inspector will now have to test at least 9 widgets.

2. Find the probability that the inspector will now have to test at least 10 widgets.

Solution 7. 1. The event that at least 9 widgets need to be inspected to find the 3 defective ones, is equivalent to the event that less than or equal to 2 defective widgets were found among the first 8. Let A_i denote the event that i defective widgets were found among the first 8. Then the probability of interest can be written as

$$P(A_0 \cup A_1 \cup A_2) = P(A_0) + P(A_1) + P(A_2)$$

as these events are disjoint. The number of possibilities for selecting 0 defectives among the first 8 is $\binom{9}{8}\binom{3}{0}$ as we have to select 8 from the non-defectives, and 0 from the 3 defectives. Similarly, the number of possibilities for the other two events are $\binom{9}{7}\binom{3}{1}$ and $\binom{9}{6}\binom{3}{2}$, respectively. Finally, the total number of possibilities for selecting the first 8 widgets from the 12, is $\binom{12}{8}$. So the probability is

$$\frac{\binom{9}{8}\binom{3}{0} + \binom{9}{7}\binom{3}{1} + \binom{9}{6}\binom{3}{2}}{\binom{12}{8}} = \frac{41}{55}$$

2. By identical reasoning, the probability of interest is

$$P(B_0) + P(B_1) + P(B_2)$$

where B_i denotes the event, that i many defective widgets were found among the first 9 inspected. The number of possibilities for selecting 0, 1, and 2 defectives among the first 9 are $\binom{9}{9}\binom{3}{0}$, $\binom{9}{8}\binom{3}{1}$, and $\binom{9}{7}\binom{3}{2}$ respectively, while the total number of possibilities for selecting 9 widgets out of the 12 is $\binom{12}{9}$. So the probability is

$$\frac{\binom{9}{9}\binom{3}{0} + \binom{9}{8}\binom{3}{1} + \binom{9}{7}\binom{3}{2}}{\binom{12}{9}} = \frac{34}{55}.$$