Problem Sheet 6 [1]

Based on Chapters 4 and 5.1-5.2 of the course book.

Optional Revision Problems

Exercise 1. A book has n typos. Two proofreaders, Prue and Frida, independently read the book. Prue catches each typo with probability p_1 and misses it with probability $q_1 = 1 - p_1$, independently, and likewise for Frida, who has probabilities p_2 of catching and $q_2 = 1 - p_2$ of missing each typo. Let X_1 be the number of typos caught by Prue, X_2 be the number caught by Frida, and X be the number caught by at least one of the two proofreaders.

1. Find the distribution of X.
2. For this part only, assume that $p_1 = p_2$. Find the conditional distribution of X_1 given that $X_1 + X_2 = t$.

Exercise 2. Let X, Y, Z be discrete r.v.s (random variables) such that X and Y have the same conditional distribution given Z, i.e., for all a and z we have

$$P(X = a|Z = z) = P(Y = a|Z = z).$$

Show that X and Y have the same distribution (unconditionally, not just when given Z).

Week 6 Exercises

Exercise 3. Find the mean and variance of a Discrete Uniform r.v. on $1, 2, \ldots, n$.
Hint: See the math appendix for some useful facts about sums.

Exercise 4. A certain small town, whose population consists of 100 families, has 30 families with 1 child, 50 families with 2 children, and 20 families with 3 children. The birth rank of one of these children is 1 if the child is the firstborn, 2 if the child is the secondborn, and 3 if the child is the thirdborn.

1. A random family is chosen (with equal probabilities), and then a random child within that family is chosen (with equal probabilities). Find the PMF, mean, and variance of the child's birth rank.
2. A random child is chosen in the town (with equal probabilities). Find the PMF, mean, and variance of the child's birth rank.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

Exercise 5. Let $X \sim Bin(100, 0.9)$. For each of the following parts, construct an example showing that it is possible, or explain clearly why it is impossible. In this problem, Y is a random variable on the same probability space as X; note that X and Y are not necessarily independent.

1. Is it possible to have $Y \sim Pois(0.01)$ with $P(X \geq Y) = 1$?

2. **Optional Challenging Exercise:** Is it possible to have $Y \sim Bin(100, 0.5)$ with $P(X \geq Y) = 1$?

Exercise 6. Ten million people enter a certain lottery. For each person, the chance of winning is one in ten million, independently.

1. Find a simple, good approximation for the PMF of the number of people who win the lottery.

2. Congratulations! You won the lottery. However, there may be other winners. Assume now that the number of winners other than you is $W \sim Pois(1)$, and that if there is more than one winner, then the prize is awarded to one randomly chosen winner. Given this information, find the probability that you win the prize (simplify).

Exercise 7. In a group of 90 people, find a simple, good approximation for the probability that there is at least one pair of people such that they share a birthday and their biological mothers share a birthday (i.e. both of the two people have their birthdays at *date* x and both of the moms at *date* y). Assume that no one among the 90 people is the biological mother of another one of the 90 people, nor do two of the 90 people have the same biological mother. Express your answer as a fully simplified fraction in the form a/b, where a and b are positive integers and $b \leq 100$.

Make the usual assumptions as in the birthday problem (e.g. see Example 1.4.10 in the book). To simplify the calculation, you can use the approximations $365 \approx 360$ and $89 \approx 90$, and the fact that $e^x \approx 1 + x$ for $x \approx 0$.

See examples 4.7.5 to 4.7.7 for some hints/motivation.