Problem Sheet 5 [1]

Optional Revision Problems

Exercise 1. People are arriving at a party one at a time. While waiting for more people to arrive they entertain themselves by comparing their birthdays. Let X be the number of people needed to obtain a birthday match, i.e., before the X-th person arrives no two people have the same birthday, but when person X arrives there is a match. Find the PMF of X.

Exercise 2. Let X be an r.v. (random variable) with CDF F, and $Y = \mu + \sigma X$, where μ and σ are real numbers with $\sigma > 0$. (Then Y is called a location-scale transformation of X.) Find the CDF of Y, in terms of F.

Exercise 3.

1. Show that $p(n) = \left(\frac{1}{2}\right)^{n+1}$ for $n = 0, 1, 2, \ldots$ is a valid PMF for a discrete r.v.
2. Find the CDF of a random variable with a PMF from part 1.

Exercise 4. In a chess tournament, n games are being played, independently. Each game ends in a win for one player with probability 0.4 and ends in a draw (tie) with probability 0.6. Find the PMFs of the number of games ending in a draw, and of the number of players whose games end in draws.

Week 5 exercises

Exercise 5.

1. A fair die is rolled. Find the expected value of the roll.
2. Four fair dice are rolled. Find the expected total of the rolls.

Exercise 6. Consider the St. Petersburg paradox (Example 4.3.14 in the course book), except that you receive \$n rather than $\$2^n$ if the game lasts for n rounds. What is the fair value (i.e. the amount a rational player would be willing to offer to play this and the amount that a rational bettor would be willing to accept) of this game? What if the payoff is $\$n^2$?

Hint: Try to use the trick from Example 4.3.6

Exercise 7. Are there discrete random variables X and Y such that $E(X) > 100E(Y)$ but Y is greater than X with probability at least 0.99?

Exercise 8. Let $X \sim Bin(n, p)$ and $Y \sim NBin(r, p)$. Using a story about a sequence of Bernoulli trials, prove that $P(X < r) = P(Y > n - r)$.

Exercise 9. A discrete random variable X has the *memoryless property* if it has the distribution $P(X \geq j + k | X \geq j) = P(X \geq k)$ for all nonnegative integers j, k.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

1. If X has a memoryless distribution with CDF F and PMF $p_i = P(X = i)$, find an expression for $P(X \geq j + k)$ in terms of $F(j), F(k), p_j, p_k$.

2. Name a discrete distribution which has the memoryless property. Justify your answer with a clear interpretation in words or with a computation.

Exercise 10. A coin with probability p of Heads is flipped n times. The sequence of outcomes can be divided into *runs* (blocks of H's or blocks of T's), e.g., HHHHTTHTTTH becomes $\boxed{\text{HHH}}\ \boxed{\text{TT}}\ \boxed{\text{H}}\ \boxed{\text{TTT}}\ \boxed{\text{H}}$, which has 5 runs. Find the expected number of runs.

Hint: Start by finding the expected number of tosses (other than the first) where the outcome is different from the previous one.