Problem Sheet 3 [1]

Exercise 1. In deterministic logic, the statement "A implies B" is equivalent to *contrapositive*, "not B implies not A". In this problem we will consider analogous statements in probability, the logic of uncertainty. Let A and B be events with probabilities not equal to 0 or 1.

1. Show that if $P(B|A) = 1$, then $P(A^c|B^c) = 1$
2. Show however that the result in 1. does not hold in general, if "=" is replaced by "≈". In particular, find an example where $P(B|A)$ is very close to 1, but $P(A^c|B^c)$ is very close to 0.

   Hint: What happens if A and B are independent?

Exercise 2. A family has 3 children, creatively named A, B, and C.

1. Discuss intuitively (but clearly) whether the event "A is older than B" is independent of the event "A is older than C".
2. Find the probability that A is older than B, given that A is older than C.

Exercise 3. A family has two children. Let C be a characteristic that a child can have, and assume that each child has characteristic C with probability p, independently of each other and of gender. Under the assumptions of binary gender that have equal probability for each child ($P(boy) = P(girl)$) and independence between the genders of the children, show that the probability that both children are girls given that at least one is a girl with characteristic C is $\frac{2-p}{4-p}$. Note that this is 1/3 if $p = 1$ and approaches 1/2 from below as $p \to 0$.

Exercise 4. Suppose that there are two types of drivers: good drivers and bad drivers. Let G be the event that a certain person is a good driver, A be the event that they get into a car accident next year, and B be the event that they get into a car accident the following year. Let $P(G) = g$ and $P(A|G) = P(B|G) = p_1$, $P(A|G^c) = P(B|G^c) = p_2$, with $p_1 < p_2$. Suppose, that given the information of whether or not the person is a good driver, A and B are independent (for simplicity and to avoid being morbid, assume that the accidents being considered are minor and wouldn't make the person unable to drive).

1. Explain intuitively whether or not A and B are independent.
2. Find $P(G|A^c)$.
3. Find $P(B|A^c)$.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

Exercise 5. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on which, your chances of winning an individual game are 90%, 50%, or 30%, respectively.

1. What is your probability of winning the first game?

2. Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game (assume that, given the skill level of your opponent, the outcomes of the games are independent)?

3. Explain the distinction between assuming that the outcomes of the games are independent and assuming that they are conditionally independent given the opponent's skill level. Which of these assumptions seems more reasonable, and why?

Exercise 6.

1. Suppose that in the population of college applicants, being good at baseball is independent of having a good math score on a certain standardized test (with respect to some measure of "good"). A certain college has a simple admissions procedure: admit an applicant if and only if the applicant is good at baseball or has a good math score on the test.

   Give an intuitive explanation of why it makes sense that among students that the college admits, having a good math score is *negatively associated* with being good at baseball, i.e., conditioning on having a good math score decreases the chance of being good at baseball.

2. Show that if A and B are independent and $C = A \cup B$, then A and B are conditionally dependent given C (as long as $P(A \cap B) > 0$ and $P(A \cup B) < 1$), with

   $$P(A|B, C) < P(A|C).$$

   This phenomenon is known as *Berkson's paradox*, especially in the context of admissions to a school, hospital, etc.

Exercise 7. Consider the following conversation from an episode of *The Simpsons*:

> **Lisa**: *Dad, I think he's an ivory dealer! His boots are ivory, his hat is ivory, and I'm pretty sure that check is ivory.*
> **Homer**: *Lisa, a guy who has lots of ivory is less likely to hurt Stampy than a guy whose ivory supplies are low.*

Here Homer and Lisa are debating the question of whether or not the man (named Blackheart) is likely to hurt Stampy the Elephant if they sell Stampy to him. They clearly disagree about how to use their observations about Blackheart to learn about the probability (conditional on the evidence) that Blackheart will hurt Stampy.

1. Define clear notation for the various events of interest here.

2. Express Lisa's and Homer's arguments (Lisa's is partly implicit) as conditional probability statements in terms of your notation from 1.

3. Assume it is true that someone who has a lot of a commodity will have less desire to acquire more of the commodity. Explain what is wrong with Homer's reasoning that the evidence about Blackheart makes it less likely that he will harm Stampy.

Exercise 8. Monty Hall is trying out a new version of his game. In this version, instead of there always being 1 car and 2 goats, the prizes behind the doors are generated independently, with each door having probability p of having a car and $q = 1 - p$ of having a goat. In detail: There are three doors, behind each of which there is one prize: either a car or a goat. For each door, there is probability p that there is a car behind it and $q = 1 - p$ that there is a goat, independent of the other doors.

The contestant chooses a door. Monty, who knows the contents of each door, then opens one of the two remaining doors. In choosing which door to open, Monty will always reveal a goat if possible. If both of the remaining doors have the same kind of prize, Monty chooses randomly (with equal probabilities). After opening a door, Monty offers the contestant the option of switching to the other unopened door.

The contestant decides in advance to use the following strategy: first choose door 1. Then, after Monty opens a door, switch to the other unopened door.

1. Find the unconditional probability that the contestant will get a car.
2. **Optional, challenging exercise:** Monty now opens door 2, revealing a goat. Given this information, find the conditional probability that the contestant will get a car.