
Problem Sheet 1 [1]

Exercise 1. A *round-robin tournament* is being held with 2^n tennis players; this means that every player will play against every other player exactly once.

1. How many games are played in total?
2. How many possible outcomes are there for the tournament (the outcome lists out who won and who lost for each game)?

Exercise 2. A *knock-out tournament* is being held with 2^n tennis players. This means that for each round the winners move on to the next round and the losers are eliminated, until only one person remains. For example, if initially there are $2^4 = 16$ players, then there are 8 games in the first round, then the 8 winners move on to round 2, then the 4 winners move on to round 3, then the 2 winners move on to round 4, the winner of which is declared the winner of the tournament. (There are various systems for determining who plays whom within a round, but these do not matter for this problem.)

1. How many rounds are there?
2. Count how many games in total are played, by adding up the numbers of games played in each round.
3. Count how many games in total are played, this time directly thinking about it without doing almost any calculations.

   Hint: How many players need to be eliminated?

Exercise 3. Give a story proof that

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$$

for all positive integers n.

Hint: Consider choosing a committee of size n with a single president or the committee, from two groups of size n each, where only one of the two groups has people eligible to become the president of the committee.

Exercise 4. Show that for any events A and B,

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A \cup B) \leq P(A) + P(B).$$

For each of these three inequalities, give a simple criterion for when the inequality is actually an equality (e.g., give a simple condition such that $P(A \cap B) = P(A \cup B)$ if and only if the condition holds).

Hint: If stuck, read Section 1.6 from the course book.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

Exercise 5. Let A and B be events. The *difference* between $B - A$ is defined to be the set of all elements of B that are not in A (equivalently denoted as $B \setminus A$). Show that if $A \subseteq B$, then

$$P(B - A) = P(B) - P(A),$$

directly using the axioms of probability.

Exercise 6. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

Hint: "Assign" 1 class to all 5 days, and consider how the remaining ones can be distributed.

Exercise 7. A widget inspector inspects 12 widgets and finds that exactly 3 are defective. Unfortunately, the widgets then get all mixed up and the inspector has to find the 3 defective widgets one by one.

1. Find the probability that the inspector will now have to test at least 9 widgets.
2. Find the probability that the inspector will now have to test at least 10 widgets.