Probability and Statistics
Autumn 2025

MATH-235
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Problem Sheet 10 [1]

Based on Chapters 7.3, 9.1-9.4, 9.5, and 10.1 of the course book.

Optional Revision Problems

Exercise 1. Let X and Y have joint PDF $f_{X,Y}(x, y) = x + y$, for $0 < x < 1$ and $0 < y < 1$.

1. Check that this is a valid joint PDF.
2. Find the marginal PDFs of X and Y.
3. Are X and Y independent?
4. Find the conditional PDF of Y given $X = x$.

1 Week 10 Exercises

Exercise 2.
1. Let X and Y be Bernoulli r.v.s, possibly with different parameters. Show that if X and Y are uncorrelated, then they are independent.
2. Give an example of three Bernoulli r.v.s such that each pair of them is uncorrelated, yet the three r.v.s are dependent.

Exercise 3. Show that $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$

Exercise 4. Show that for any two random variables X and Y,

$$-1 \leq \text{Corr}(X, Y) \leq 1.$$

Hint: Consider the variance of $X + Y$ and $X - Y$, respectively.

Exercise 5. Optional Exercise You get to choose between two envelopes, each of which contains a check for some positive amount of money. Unlike in the two-envelope paradox, it is not given that one envelope contains twice as much money as the other envelope. Instead, assume that the two values were generated independently from some distribution on the positive real numbers, with no information given about what that distribution is.

After picking an envelope, you can open it and see how much money is inside (call this value x), and then you have the option of switching. As no information has been given about the distribution, it may seem impossible to have better than a 50% chance of picking the better envelope. Intuitively, we may want to switch if x is "small" and not switch if x is "large", but how do we define "small"

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

and "large" in the grand scheme of all possible distributions? [The last sentence was a rhetorical question.]

Consider the following strategy for deciding whether to switch. Generate a threshold $T \sim Expo(1)$, and switch envelopes if and only if the observed value x is less than the observed value of T. Show that this strategy succeeds in picking the envelope with more money with probability strictly greater than 1/2.

Hint: Let t be the value of T (generated by a random draw from the $Expo(1)$ distribution). First explain why the strategy works very well if t happens to be in between the two envelope values, and does no harm in any case (i.e., there is no case in which the strategy succeeds with probability strictly less than 1/2).

Exercise 6. Let $X \sim Expo(\lambda)$. Find $E(X|X < 1)$ in two different ways:

1. By calculus, working with the conditional PDF of X given $X < 1$.

   Hint: For the conditional PDF use the formula under Definition 9.1.1 in the book.

2. Without calculus, by expanding $E(X)$ using the law of total expectation.

   Hint: Use the memoryless property of the exponential, in particular that if X is Exponential, then $E(X|X \geq a) = E(X) + a$ (if you don't see this, look at Example 9.1.8, that describes this property for the discrete case).

Exercise 7. A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

Hint: If you get a conditional expectation somewhere in your solution, try to think about what its distribution could be, instead of "brute-force" calculating the expectation.

Exercise 8. Show that $E((Y - E(Y|X))^2|X) = E(Y^2|X) - (E(Y|X))^2$, so these two expressions for $Var(Y|X)$ agree.

Hint for the variance: Adding a constant (or something acting as a constant) does not affect variance.

Exercise 9. Show that if $E(Y|X) = c$ is a constant, then X and Y are not correlated.

Hint: Use Adam's law to find $E(Y)$ and $E(XY)$.

Exercise 10. Joe will read $N \sim Pois(\lambda)$ books next year. Each book has a $G \sim Pois(\mu)$ number of pages, with book lengths independent of each other and independent of N.

1. Find the expected number of book pages denoted by T, that Joe will read next year.

   Hint: $T \neq N \cdot G$, just consider whether it is possible for Joe to read a prime number of pages in total, even if he read more than one book.

2. **Ignore, unless "Eve's law is covered"** ~~Find the variance of the number of book pages Joe will read next year.~~