CORRECTION

SCIPER : __ __ __ __ __ __

| | | | | | |
|---|---|---|---|---|---|
| ☐0 | ☐0 | ☐0 | ☐0 | ☐0 | ☐0 |
| ☐1 | ☐1 | ☐1 | ☐1 | ☐1 | ☐1 |
| ☐2 | ☐2 | ☐2 | ☐2 | ☐2 | ☐2 |
| ☐3 | ☐3 | ☐3 | ☐3 | ☐3 | ☐3 |
| ☐4 | ☐4 | ☐4 | ☐4 | ☐4 | ☐4 |
| ☐5 | ☐5 | ☐5 | ☐5 | ☐5 | ☐5 |
| ☐6 | ☐6 | ☐6 | ☐6 | ☐6 | ☐6 |
| ☐7 | ☐7 | ☐7 | ☐7 | ☐7 | ☐7 |
| ☐8 | ☐8 | ☐8 | ☐8 | ☐8 | ☐8 |
| ☐9 | ☐9 | ☐9 | ☐9 | ☐9 | ☐9 |

Name : ..............................................

Signature :

# Midterm Exam
# MATH-232 Probability and Statistics
## Instructor: Prof. Emmanuel Abbe
## Spring 2023
## Duration : 20.04.2023 from 13:15 to 14:30.

**Do not turn the page before the start of the exam. This document is double-sided and has 12 pages. Do not unstaple.**

- First and foremost, write your name on this paper and also write your SCIPER number using the checkbox provided above.
- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is **not permitted** during the exam.
- A cheat sheet is provided on the last pages of this booklet.
- For the **multiple choice** questions, you will receive:
  - $+3$ points if your answer is correct,
  - $0$ points if you give no answer or more than one,
  - $-1$ points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- The multiple choice questions are shuffled, and hence are not in the order of difficulty.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

## First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer. No justifications are needed for this part.

**Question 1** There are 3 different permutations of the word BOB (that give different words): BOB, OBB, and BBO. Which of the following words has the largest number of different permutations?

- [ ] HAMID
- [ ] ARIANA
- [x] APOLLO
- [ ] HANNAH

**Question 2** Let $X \sim \text{Uniform}(1,3)$ and $Y$ such that $Y|X \sim \exp(X)$, i.e., $f_{Y|X}(y|x) = xe^{-yx}$ for $y \in (0,\infty), x \in (1,3)$. What is the value of $\mathbb{E}[X^2Y]$?

- [ ] 4
- [x] 2
- [ ] 1
- [ ] 9

**Question 3** Assume that in a society people's weights follow a normal distribution with mean $\mu = 71$ and variance $\sigma^2 = 9$. The percentage of people with weights less than 65 is closest to

- [x] 2.28%
- [ ] 15.87%
- [ ] 25.2%
- [ ] 0.13%

**Question 4** Assume that $\theta \sim \mathcal{N}(0,1)$ (standard Normal distribution). Consider $X = \sin(\theta)$ and $Y = \cos(\theta)$. Which of the following statements is correct?

- [x] $X$ and $Y$ are uncorrelated and dependent
- [ ] $X$ and $Y$ are correlated and dependent
- [ ] $X$ and $Y$ are correlated and independent
- [ ] $X$ and $Y$ are uncorrelated and independent

**Question 5** A man tells the truth only 75% of times (in other words, he lies with probability 0.25). He has a biased coin that gives heads also with probability 0.75. In an experiment, the man flips his coin and says that the result is heads; what is the probability that the coin was really heads?

- [ ] $\frac{9}{16}$
- [ ] $\frac{10}{16}$
- [x] $\frac{9}{10}$
- [ ] $\frac{3}{4}$

**Question 6** Let $X_n$ have Binomial distribution $X_n \sim \text{Bin}(n,p)$ such that $np = \lambda$ is constant. As $n \to \infty$, the probability $\mathbb{P}(X_n = 0)$ tends to

- [ ] $e^{-\lambda} - e^{-2\lambda}$
- [ ] $\lambda e^{-\lambda}$
- [x] $e^{-\lambda}$
- [ ] $1 - e^{-\lambda}$

**Question 7** Assume that $\mathbb{P}(C) = 0.5$, $\mathbb{P}(D) = 0.3$, and $\mathbb{P}(C|D) = 0.5$. What is the value of $\mathbb{P}(D^c|C^c)$? ($C^c$ and $D^c$ represent the complements of $C$ and $D$ respectively.)

- ■ 0.7
- □ 0.4
- □ 0.6
- □ 0.8

**Question 8** For random variables $X$ and $Y$, we have $\mathrm{Var}(X) = 2$, $\mathrm{Var}(Y) = 8$ and $\mathrm{corr}(X, Y) = -0.5$. What is $\mathrm{cov}(2X + Y, Y - 3X)$?

- ■ $-2$
- □ 2
- □ 0
- □ 4
- □ $-4$

**Question 9** The random variable $X$ has the density function

$$f_X(x) = \begin{cases} cx & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases},$$

where $c$ is a constant which makes $f_X$ a valid density. What is $\mathrm{Var}(X)$?

- □ $\frac{4}{3}$
- □ $\frac{1}{18}$
- □ $\frac{3}{4}$
- ■ $\frac{2}{9}$

# MATH-232 Midterm Solutions to the Open Problems

Spring 2023

## Question 10 (6 points)

Assume that $X$ is a random variable that only takes non-negative values, i.e., $X \geq 0$, and assume that its expected value, $\mathbb{E}[X]$, exists. For any $a > 0$ prove the following inequality:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

(The above is true for both continuous and discrete random variables, however, you can write the proof for the case that you prefer.)

### Solution

We prove the statement when $X$ is a discrete random variable taking non-negative values in $\Omega_X$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x_i \in \Omega_X} x_i \, \mathbb{P}(X = x_i) && \text{by definition.} && (1)\\
&= \sum_{\substack{0 \leq x_i < a \\ x_i \in \Omega_X}} x_i \, \mathbb{P}(X = x_i) + \sum_{\substack{x_i \geq a \\ x_i \in \Omega_X}} x_i \, \mathbb{P}(X = x_i) && \text{splitting the sum.} && (2)\\
&\geq \sum_{\substack{x_i \geq a \\ x_i \in \Omega_X}} x_i \, \mathbb{P}(X = x_i) && \text{dropping a non-negative term as } X \geq 0. && (3)\\
&\geq \sum_{\substack{x_i \geq a \\ x_i \in \Omega_X}} a \, \mathbb{P}(X = x_i) && \text{as the } x_i\text{'s in the sum are} \geq a. && (4)\\
&\geq a \sum_{\substack{x_i \geq \\ x_i \in \Omega_X}} \mathbb{P}(X = x_i) && a \text{ is a constant.} && (5)\\
&\geq a \, \mathbb{P}(X \geq a) && \text{summing over a partition.} && (6)
\end{aligned}$$

Which gives the desired result by dividing both terms of the inequality by $a > 0$.
For the case when $X$ is a continuous variable, the proof is similar replacing the sum with integrals.

# Question 11 (9 points)

## Problem Statement

Raphael and his $n-1$ friends sit around the table and play the following game: at the beginning (step 1) Raphael has a card and passes the card to one of his friends randomly (each of his $n-1$ friends receive the card with equal probability). In each of the next steps, the person who has the card passes the card randomly to someone at the table. The game finishes when Raphael receives the card (when the card returns to Raphael). Note that no one can pass the card to himself/herself.
a)[4 points] What is the expected number of times the card is passed through the game until it ends? (The game needs at least two passes to finish.)
b)[5 points] (This part is relatively harder than the rest of the exam, you may want to leave it for the end.) What is the expected number of people that never receive the card during the game?

You may want to use the following formula for this question:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

## Problem Solution

### a)

The game always lasts more than 1 round since Raphael cannot pass the card to himself at the beginning. Let $X$ be the random variable that tracks the number of additional rounds (after the first) for which the game continues. Hence, the total number of rounds will be given as $X+1$, and the expected number of rounds becomes $\mathbb{E}[X+1] = 1 + \mathbb{E}[X]$ by linearity of expectation.

Now, notice that $X$ follows the geometric distribution $\text{Geom}(p)$ with success probability $p = 1/(n-1)$. Indeed, conditioning on the event that the game has not ended at round $t \geq 2$, the game ends at round $t+1$ with probability $1/(n-1)$ since Raphael is one possible choice from a total of $n-1$ possible recipients:

$$\mathbb{P}(X+1 = t+1 | X+1 > t) = \mathbb{P}(X = t | X > t-1) = \frac{1}{n-1}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}(X = t) &= \mathbb{P}(X = t | X > t-1)\, \mathbb{P}(X \neq t-1 | X > t-2) \ldots \mathbb{P}(X \neq 2 | X > 1)\, \mathbb{P}(X \neq 1) \\
&= \frac{1}{n-1} \left( \frac{n-2}{n-1} \right)^{t-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{t=1}^{\infty} t\, \mathbb{P}(X = t) \\
&= \sum_{t=1}^{\infty} t \frac{1}{n-1} \left( \frac{n-2}{n-1} \right)^{t-1} \\
&= \frac{1}{n-1} \sum_{t=1}^{\infty} t \left( \frac{n-2}{n-1} \right)^{t-1} \\
&= \frac{1}{n-1} \frac{d}{dx} \left( \frac{1}{1-x} \right) \Big|_{x=\frac{n-2}{n-1}} \\
&= \frac{1}{n-1} \frac{1}{(1 - \frac{n-2}{n-1})^2} \\
&= n-1.
\end{aligned}$$

So, we conclude that the expected number of rounds is $\mathbb{E}[X+1] = n$.

**b)**

Let Raphael be denoted as Player 1 from now on. We introduce the indicator random variables $Y_2, \dots, Y_n$ such that $Y_i = 1$ if Player $i$ did not receive the card during the game and $Y_i = 0$ otherwise. Hence, the random variable $Y = Y_2 + \cdots + Y_n$ tracks the total number of players who did not receive the card. We are interested in computing $\mathbb{E}[Y] = \mathbb{E}[Y_2] + \cdots + \mathbb{E}[Y_n] = (n-1)\,\mathbb{E}[Y_i]$ for any $i \in \{2, 3, \dots, n\}$ by symmetry.

Let $X$ be the random variable from a) which keeps track of the number of additional turns. Now, notice that if we condition on the game having $t+1$ turns, we get that

$$\mathbb{P}(Y_i = 1 | X = t) = \frac{n-2}{n-1}\left(\frac{n-3}{n-2}\right)^{t-1}.$$

In the above expression, the term $(n-2)/(n-1)$ represents the probability that Player $i$ did not receive the card on the first turn, and $(n-3)/(n-2)$ stands for the conditional probability that Player $i$ did not receive the card on turn $j \in \{2, 3, \dots, t\}$ given that Player 1 also did not receive the card on this turn. Indeed, conditioned on the card returning to Player 1 on round $t+1$, on each round $j \in \{2, 3, \dots, t\}$, the card has $n-2$ equally likely recipients.

Recall that $\mathbb{P}(X = t) = \frac{1}{n-1}\left(\frac{n-2}{n-1}\right)^{t-1}$. Hence,

$$\begin{aligned}
\mathbb{P}(Y_i = 1) &= \sum_{t=1}^{\infty} \mathbb{P}(Y_i = 1 | X = t)\,\mathbb{P}(X = t) \\
&= \sum_{t=1}^{\infty} \frac{n-2}{n-1}\left(\frac{n-3}{n-2}\right)^{t-1} \frac{1}{n-1}\left(\frac{n-2}{n-1}\right)^{t-1} \\
&= \frac{n-2}{(n-1)^2} \sum_{t=1}^{\infty} \left(\frac{n-3}{n-1}\right)^{t-1} \\
&= \frac{n-2}{(n-1)^2} \frac{1}{1 - \frac{n-3}{n-1}} \\
&= \frac{n-2}{2(n-1)}.
\end{aligned}$$

Therefore, the expected number of players who did not receive the card is $\mathbb{E}[Y] = \frac{n}{2} - 1$.

# Cheat sheet

## Basic formulas

- Properties of binomial coefficients
  (a) Pascal's triangle $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
  (b) Vandermonde's formula $\sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j} = \binom{m+n}{r}$.
  (c) Negative binomial series $(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i, |x| < 1$.
  (d) $\lim_{n\to\infty} n^{-r}\binom{n}{r} = \frac{1}{r!}$, where $r \in \mathbb{N}$ is fixed.
- Inclusion-exclusion formula:

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1\leq i_1 < i_2 < \ldots < i_r \leq n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r})$$

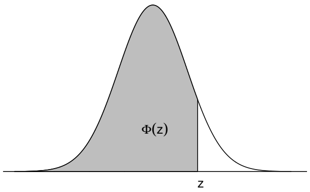
- For random variables $X$ and $Y = g(X)$, where $g$ is a monotone increasing or decreasing function with differentiable inverse $g^{-1}$, we have

$$f_Y(y) = |\frac{dg^{-1}(y)}{dy}| f_X(g^{-1}(y)).$$

## Distributions

| Distribution | PMF/PDF | Expected Value | Variance |
|---|---|---|---|
| Bernoulli<br>Bern($p$) | $\mathbb{P}(X=1)=p$<br>$\mathbb{P}(X=0)=1-p$ | $p$ | $p(1-p)$ |
| Binomial<br>Bin($n,p$) | $\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$<br>$k\in\{0,1,2,\dots n\}$ | $np$ | $np(1-p)$ |
| Geometric<br>Geom($p$) | $\mathbb{P}(X=k)=(1-p)^{k-1}p$<br>$k\in\{1,\,2,\,\dots\}$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| Negative Binomial<br>NegBin($r,p$) | $\mathbb{P}(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$<br>$x\in\{r,r+1,r+2,\dots\}$ | $\frac{r}{p}$ | $\frac{r(1-p)}{p^2}$ |
| Hypergeometric<br>HyperGeom($w,b,n$) | $\mathbb{P}(X=k)=\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$<br>$k\in\{0,1,2,\dots,n\}$ | $\mu=\frac{nw}{b+w}$ | $\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$ |
| Poisson<br>Pois($\lambda$) | $\mathbb{P}(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$<br>$k\in\{0,\,1,\,2,\,\dots\}$ | $\lambda$ | $\lambda$ |
| Uniform<br>U($a,b$) | $f(x)=\frac{1}{b-a}$<br>$x\in[a,b]$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| Exponential<br>exp($\lambda$) | $f(x)=\lambda e^{-\lambda x}$<br>$x\in(0,\infty)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ |
| Normal<br>$\mathcal{N}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$<br>$x\in(-\infty,\infty)$ | $\mu$ | $\sigma^2$ |
| Gamma<br>Gamma($\alpha,\lambda$) | $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$<br>$x\in(0,\infty)$ | $\frac{\alpha}{\lambda}$ | $\frac{\alpha}{\lambda^2}$ |

**Standard normal distribition $\Phi(z)$**




For $z < 0$ we use symmetry: $\mathbb{P}(Z \leq z) = \Phi(z) = 1 - \Phi(-z)$, $z \in \mathbb{R}$.

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| 0.1 | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56750 | .57142 | .57535 |
| 0.2 | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| 0.3 | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| 0.4 | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| 0.5 | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| 0.6 | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| 0.7 | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| 0.8 | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| 0.9 | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| 1.0 | .84134 | .84375 | .84614 | .84850 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| 1.1 | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| 1.2 | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| 1.3 | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| 1.4 | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92786 | .92922 | .93056 | .93189 |
| 1.5 | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| 1.6 | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| 1.7 | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| 1.8 | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| 1.9 | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| 2.0 | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| 2.1 | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| 2.2 | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| 2.3 | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| 2.4 | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| 2.5 | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| 2.6 | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| 2.7 | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| 2.8 | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| 2.9 | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| 3.0 | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| 3.1 | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| 3.2 | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| 3.3 | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| 3.4 | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| 3.5 | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| 3.6 | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| 3.7 | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| 3.8 | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| 3.9 | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |