MATH-232: Midterm 2022 - Solutions B

May 15, 2022

Exercise 1. At the end of a video game, the player must kill a monster sampled randomly as follows:

- 4 times out of 10, it is a *dragon*,
- 1 time out of 10, it is a *troll*,
- the rest of the time, it is a *giant*.

When the monster dies, the player gets a chance to get a *ruby*:

- the dragon gives a ruby 1 time out of 3,
- the troll gives a ruby 1 time out of 6,
- the giant always gives a ruby.

We assume that the game is very easy; thus the players always succeed at killing the monster. Two different games are independent.
Alice plays the game. We denote D (respectively T, G) the event "the monster is a dragon" (respectively a troll, a giant). We denote R the event "Alice wins a ruby", and $p = \mathbb{P}(R)$.

1. Compute p.

2. Alice won a ruby! What is the probability that the monster was a troll?

3. Bob decides to play 8 games. We denote S the number of rubies that he gets.

  (a) What is the law of S?
  (b) What is the expectation of S?
  (c) What is the probability that Bob wins exactly 3 rubies?
  (d) What is the probability that Bob wins at least 1 ruby?

4. Charlie decides to play until he wins one ruby. We denote X the number of games Charlie plays.

  (a) What is the law of X?
  (b) What is the expectation of X?
  (c) What is the probability that Charlie does exactly 5 games?

Solution. (8 points)

1. (1.5pt) Using the law of total probability we write,

$$\begin{aligned} p = \mathbb{P}(R) &= \mathbb{P}(R|D)\mathbb{P}(D) + \mathbb{P}(R|T)\mathbb{P}(T) + \mathbb{P}(R|G)\mathbb{P}(G) \\ &= \frac{1}{3} \cdot \frac{4}{10} + \frac{1}{6} \cdot \frac{1}{10} + 1 \cdot \frac{5}{10} \\ &= \frac{1}{10} \cdot \frac{8+1+30}{6} \\ &= \frac{39}{60} = \frac{13}{20}. \end{aligned}$$

2. (1pt) Using Bayes rule,

$$\mathbb{P}(T|R) = \frac{\mathbb{P}(R|T)\mathbb{P}(T)}{p} = \frac{1}{6} \cdot \frac{1}{10} \cdot \frac{60}{39} = \frac{1}{39}.$$

3. (3pt)
   - a) (1pt) Since S is the sum of 8 independent events, with probability of success p, it follows that $S \sim$Binomial$(p, 8)$
   - b) (0.5pt) $\mathbb{E}[S] = 8p$
   - c) (1pt) $\mathbb{P}(S = 3) = \binom{8}{3} p^3 (1-p)^5$
   - d) (0.5pt) $\mathbb{P}(S \geq 1) = 1 - \mathbb{P}(S = 0) = 1 - (1-p)^8$

4. (2.5pt)
   - a) (1pt) Since X denotes the number of trials to get the first success of independent events of probability p, $X \sim$Geom(p)
   - b) (0.5pt) $\mathbb{E}[X] = 1/p = 60/39$
   - c) (1pt) $\mathbb{P}(X = 5) = (1-p)^4 p$.

□

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0\,, \\ -9x\exp(3x) & \text{if } x < 0\,. \end{cases}$$

1. Justify that f is a probability density.

2. What is the expectation of a random variable X with density f?

Solution. (2.5 points)

1. (1.5pt) $f(x) \geq 0$ for all x. One can check by integration by part that $\int_{-\infty}^{\infty} f(x)dx = 1$.
2. (1pt)

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{0} -9x^2 \exp(3x)dx && (1) \\ &= -3x^2 \exp(3x)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} 6x\exp(3x)dx && (2) \\ &= -\frac{2}{3} \cdot \int_{-\infty}^{0} -9x\exp(3x)dx && (3) \\ &= -\frac{2}{3} && (4) \end{aligned}$$

□

Exercise 3. Let (X, Y) be a joint random variable whose probability mass function is given by the following table:

Y X	-1	1
0	0.6	0.2
1	0.1	0.1

1. What are the marginal laws of X and Y?

2. Are X and Y independent? (As always, justify your answer.)

3. Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$.

Solution: (4 points)

1. (1pt)

$$f_Y(y) = \begin{cases} 0.7 & \text{if } y = -1\,, \\ 0.3 & \text{if } y = 1 \end{cases} \tag{5}$$

$$f_X(x) = \begin{cases} 0.8 & \text{if } x = 0\,, \\ 0.2 & \text{if } x = 1 \end{cases} \tag{6}$$

2. (0.5pt) No. For instance,

$$f_{X,Y}(0,-1) = 0.6 \tag{7}$$

$$f_X(0) \cdot f_Y(-1) = 0.8 \cdot 0.7 = 0.56. \tag{8}$$

   Since, $f_{X,Y}(0,-1) \neq f_X(0) \cdot f_Y(-1)$, it follows that X and Y are not independent.

3. (2.5pt)

$$\mathbb{E}[X] = 0.2 \cdot 1 = 0.2 \tag{9}$$

$$\mathbb{E}[Y] = 0.3 \cdot 1 + 0.7 \cdot -1 = -0.4 \tag{10}$$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = -0.2 \tag{11}$$

$$\mathbb{E}[X \cdot Y] = -1 \cdot 0.1 + 1 \cdot 0.1 = 0. \tag{12}$$

□

Exercise 4. Let $F : \mathbb{R} \to [0,1]$ be the function defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ \frac{x^2}{4} & \text{if } 0 \leq x \leq 2\,, \\ 1 & \text{if } x > 2\,. \end{cases}$$

1. Justify that F is a cumulative distribution function.

2. Justify that the law associated to F is continuous.

3. Compute the density f associated to this law.

4. Let X be a random variable with cumulative distribution function F.

  (a) Compute $\mathbb{P}(X > 1/2)$.

  (b) Compute $\mathbb{E}[X]$.

  (c) Compute the variance of X.

Solution. (5 points)

1. (1pt) One can verify that F is such that

   - $\lim_{x \to -\infty} F(x) = 0$;
   - $\lim_{x \to \infty} F(x) = 1$;
   - F is non-decreasing;

- F is continuous on the right.

Alternatively, one can check that $f(x) = \frac{d}{dx}F(x)$ is a valid density function.

2. (0.5pt) Since $F(x)$ is continuous, it follows that the law associated to F is continuous.
3. (1pt)

$$f(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ x/2 & \text{if } 0 \le x \le 2\,, \\ 0 & \text{if } x > 1\,. \end{cases} \tag{13}$$

4. (2.5pt) One can compute that:
   a) (0.5pt) $\mathbb{P}(X > 1/2) = 15/16$
   b) (1pt) $\mathbb{E}[X] = 4/3$
   c) (1pt) $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2/9$.

□

Exercise 5. *This exercise is significantly harder than the previous ones, but not worth many points. Try it once you have finished the rest of the midterm.*
We remind that:

- *an exponential random variable with rate* λ *has density* $f(x) = \lambda e^{-\lambda x} I(x > 0)$,
- *a gamma random variable with shape parameter* α *and rate* λ *has density*

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I(x > 0)\,,$$

- *if* n *is a non-negative integer,* $\Gamma(n+1) = n!$, *and*
- *a Poisson random variable with rate* λ *has probability mass function* $f(n) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, \ldots$.

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent random variables of exponential law with rate parameter 1. We denote $T_0 = 0$ and for all $n \ge 1$, $T_n = X_1 + \cdots + X_n$. For all $t \ge 0$, we denote $N_t = \max\{n \ge 0 \,|\, T_n \le t\}$.

1. (a) Compute the cumulative distribution function of T_2. Deduce that T_2 is a gamma random variable with shape parameter 2 and rate 1. (Like everywhere else, a proof is required.)

(b) Let $n \ge 1$. Using the same method, compute the law of T_n.

2. Let $t > 0$. Compute the law of N_t.

3. Let $n \ge 1$.

(a) Compute the joint law of $(T_1, \ldots, T_n)$.

(b) Compute the conditional law of $(T_1, \ldots, T_n)$ given that $N_t = n$.

Solution. (5 points)

1. (2pt)

a) (1pt) The CDF of T_2 can be computed as follows:

$$\begin{aligned}
F_{T_2}(t) = \mathbb{P}(X_1 + X_2 \leq t) = & \int_0^t dx_1 \int_0^{t-x_1} dx_2 f_{(X_1,X_2)}(x_1, x_2) && (14)\\
\overset{\text{(indep.)}}{=} & \int_0^t dx_1 \int_0^{t-x_1} dx_2 f_{X_1}(x_1) f_{X_2}(x_2) && (15)\\
= & \int_0^t dx_1 e^{-x_1} \int_0^{t-x_1} dx_2 e^{-x_2} && (16)\\
= & \int_0^t dx_1 e^{-x_1}(1 - e^{-(t-x_1)}) && (17)\\
= & \int_0^t dx_1 (e^{-x_1} - e^{-t}) && (18)\\
= & 1 - e^{-t}(t+1). && (19)
\end{aligned}$$

Deriving the above with respect to t, we obtain

$$\frac{d}{dt} F_{T_2}(t) = -e^{-t} + e^{-t}(t+1) = te^{-t} \tag{20}$$

that is the pdf of a Gamma(2,1) random variable.

b) (1pt) We show by induction that T_n ~Gamma$(n, 1)$. The base case is given by point a). The inductive step is as follows: assume that T_n ~Gamma$(n, 1)$, then

$$\begin{aligned}
F_{T_{n+1}}(t) = \mathbb{P}(T_n + X_{n+1} \leq t) = & \int_0^t dy \int_0^{t-y} dx f_{(T_n, X_{n+1})}(y, x) && (21)\\
= & \int_0^t dy \int_0^{t-y} dx f_{T_n}(y) f_{X_{n+1}}(x) && (22)\\
= & \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} \int_0^{t-y} dx e^{-x} && (23)\\
= & \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} (1 - e^{-(t-y)}) && (24)\\
= & \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} - e^{-t} \int_0^t dy \frac{1}{(n-1)!} y^{n-1}. && (25)
\end{aligned}$$

Deriving the above with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt} F_{T_{n+1}}(t) = & \frac{1}{(n-1)!} t^{n-1} e^{-t} - e^{-t} \frac{1}{(n-1)!} t^{n-1} + e^{-t} \int_0^t dy \frac{1}{(n-1)!} y^{n-1} && (26)\\
= & \frac{1}{n!} t^n e^{-t}, && (27)
\end{aligned}$$

that is the pdf of a Gamma distribution with shape parameter $n + 1$ and scale parameter 1.

2. (2pt) For all $n \in \mathbb{N}$, we have $\{N_t = n\} = \{T_n \leq t, T_{n+1} > t\}$, since the sequence $(T_n)_{n \geq 0}$ is increasing. Note that T_n and T_{n+1} are not independent, since they both depend on $X_1, ...X_n$.

However, T_n and X_{n+1} are independent. Thus, for $n \geq 1$, we get

$$\begin{aligned}
\mathbb{P}(N_t = n) &= \mathbb{P}(T_n \leq t, T_n + X_{n+1} > t) && (28)\\
&= \mathbb{P}(T_n \leq t, X_{n+1} > t - T_n) && (29)\\
&= \int_0^t dy \int_{t-y}^{\infty} dx f_{(T_n, X_{n+1})}(y, x) && (30)\\
&= \int_0^t dy f_{T_n}(y) \int_{t-y}^{\infty} dx f_{X_{n+1}}(x) && (31)\\
&= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} \int_{t-y}^{\infty} dx e^{-x} && (32)\\
&= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} e^{-(t-y)} && (33)\\
&= e^{-t} \int_0^t \frac{1}{(n-1)!} y^{n-1} = \frac{1}{n!} t^n e^{-t}, && (34)
\end{aligned}$$

thus $N_t \sim$Poisson(t).

3. (1pt)

   a) (0.5pt) First, consider the case where $n = 2$.

$$\begin{aligned}
f_{(T_1,T_2)}(t_1, t_2) &= f_{(X_1, X_1+X_2)}(t_1, t_2) && (35)\\
&= f_{X_1}(t_1) f_{X_1+X_2|X_1}(t_2|t_1) && (36)\\
&= f_{X_1}(t_1) f_{X_2|X_1}(t_2 - t_1|t_1). && (37)
\end{aligned}$$

Since X_1 and X_2 are independent we can write $f_{X_2|X_1}(t_2 - t_1|t_1) = f_{X_2}(t_2 - t_1)$, and plug in the density functions of X_1 and X_2

$$\begin{aligned}
f_{(T_1,T_2)}(t_1, t_2) &= f_{X_1}(t_1) f_{X_2}(t_2 - t_1) && (38)\\
&= e^{-t_1} \mathbb{1}(t_1 > 0) e^{-(t_2 - t_1)} \mathbb{1}(t_2 > t_1) && (39)\\
&= e^{-t_2} \mathbb{1}(0 < t_1 < t_2). && (40)
\end{aligned}$$

We now prove by induction that for general n,

$$f_{(T_1,...,T_n)}(t_1, ..., t_n) = e^{-t_n} \mathbb{1}(0 < t_1 < t_2 < ... < t_n) \quad (41)$$

The base case is given by the above. For the inductive step, assume that (41) holds for n. Then,

$$\begin{aligned}
f_{(T_1,...,T_n,T_{n+1})}(t_1, ..., t_n, t_{n+1}) &= f_{(T_1,...,T_n,T_n+X_{n+1})}(t_1, ..., t_n, t_{n+1}) && (42)\\
&= f_{(T_1,...,T_n)}(t_1, ..., t_n) f_{T_n+X_{n+1}|(T_1,...,T_n)}(t_{n+1}|t_1, ..., t_n) && (43)\\
&= f_{(T_1,...,T_n)}(t_1, ..., t_n) f_{X_{n+1}|(T_1,...,T_n)}(t_{n+1} - t_n|t_1, ..., t_n). && (44)
\end{aligned}$$

Observe one more time that X_{n+1} is independent from $T_1, ..., T_n$, which gives

$$\begin{aligned}
f_{(T_1,...,T_n,T_{n+1})}(t_1, ..., t_n, t_{n+1}) &= f_{(T_1,...,T_n)}(t_1, ..., t_n) f_{X_{n+1}}(t_{n+1} - t_n) && (45)\\
&= e^{-t_n} \mathbb{1}(0 < t_1 < t_2 < ... < t_n) e^{-(t_{n+1} - t_n)} \mathbb{1}(t_{n+1} > t_n) && (46)\\
&= e^{-t_{n_1}} \mathbb{1}(0 < t_1 < t_2 < ... < t_{n+1}), && (47)
\end{aligned}$$

which concludes the proof.

b) (0.5pt) Recall that $\{N_t = n\} = \{T_n \leq t, T_{n+1} > t\}$. We first compute the joint distribution of $(T_1, ..., T_n, N_t)$:

$$
\begin{aligned}
f_{(T_1,...,T_n,N_t)}(t_1, ...t_n, n) &= \int_0^\infty dt_{n+1} f_{(T_1,...,T_n,T_{n+1},N_t)}(t_1, ...t_n, t_{n+1}, n) && (48)\\
&= \int_0^\infty dt_{n+1} f_{(T_1,...,T_n,T_{n+1})}(t_1, ...t_n, t_{n+1}) \mathbb{1}(t_n \leq t, t_{n+1} > t) && (49)\\
&= \int_0^\infty dt_{n+1} e^{-t_{n+1}} \mathbb{1}(0 < t_1 < ... < t_{n+1}) \mathbb{1}(t_n \leq t, t_{n+1} > t) && (50)\\
&= \int_t^\infty dt_{n+1} e^{-t_{n+1}} \mathbb{1}(0 < t_1 < ... < t_n \leq t) && (51)\\
&= e^{-t} \mathbb{1}(0 < t_1 < t_2 < ... < t_n \leq t). && (52)
\end{aligned}
$$

Applying the formula for the conditional probability we get

$$
\begin{aligned}
f_{(T_1,...,T_n)|N_t}(t_1, ..., t_n|n) &= \frac{f_{(T_1,...,T_n,N_t)}(t_1, ..., t_n, n)}{\mathbb{P}(N_t = n)}\\
&= \frac{e^{-t} \mathbb{1}(0 < t_1 < t_2 < ... < t_n \leq t)}{e^{-t} t^n / n!}\\
&= n! t^{-n} \mathbb{1}(0 < t_1 < t_2 < ... < t_n \leq t).
\end{aligned}
$$

□