EPFL
Probability and Statistics for IC
2022, Spring semester

Do not turn this page until instructed

Probability and Statistics: Midterm

April 12, 2022

B

Duration: The exam will start at 13:15 and end at 14:45 (unless special arrangements).

Family name:

First name:

SCIPER number:

Exercise	Points	Indicative marks
1		/8 points
2		/2.5 points
3		/4 points
4		/5 points
5		/5 points
Total:		/24.5 points

PROTOCOL:

- If caught cheating, you will get a 0 and we will report to the section.
- No personal documents, cheat sheets or calculators are allowed during the exam.
- Justify all your answers! Unjustified answers will not get full score even if correct. However, partial reasoning might get a partial points.
- Try to simplify numerical expressions but no need to give exact decimal expressions (e.g., you can leave factorial expressions).
- There are 5 problems. After the problems statements, you have some blank pages to write your solutions (pages 4-14).

Exercise 1. At the end of a video game, the player must kill a monster sampled randomly as follows:

- 4 times out of 10, it is a *dragon*,
- 1 time out of 10, it is a *troll*,
- the rest of the time, it is a *giant*.

When the monster dies, the player gets a chance to get a *ruby*:

- the dragon gives a ruby 1 time out of 3,
- the troll gives a ruby 1 time out of 6,
- the giant always gives a ruby.

We assume that the game is very easy; thus the players always succeed at killing the monster. Two different games are independent.
Alice plays the game. We denote D (respectively T, G) the event "the monster is a dragon" (respectively a troll, a giant). We denote R the event "Alice wins a ruby", and $p = \mathbb{P}(R)$.

1. Compute p.

2. Alice won a ruby! What is the probability that the monster was a troll?

3. Bob decides to play 8 games. We denote S the number of rubies that he gets.

(a) What is the law of S?
(b) What is the expectation of S?
(c) What is the probability that Bob wins exactly 3 rubies?
(d) What is the probability that Bob wins at least 1 ruby?

4. Charlie decides to play until he wins one ruby. We denote X the number of games Charlie plays.

(a) What is the law of X?
(b) What is the expectation of X?
(c) What is the probability that Charlie does exactly 5 games?

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0\,, \\ -9x\exp(3x) & \text{if } x < 0\,. \end{cases}$$

1. Justify that f is a probability density.

2. What is the expectation of a random variable X with density f?

Exercise 3. Let (X, Y) be a joint random variable whose probability mass function is given by the following table:

X \ Y	-1	1
0	0.6	0.2
1	0.1	0.1

1. What are the marginal laws of X and Y?

2. Are X and Y independent? (As always, justify your answer.)

3. Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$.

Exercise 4. Let $F : \mathbb{R} \to [0, 1]$ be the function defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ \frac{x^2}{4} & \text{if } 0 \leq x \leq 2\,, \\ 1 & \text{if } x > 2\,. \end{cases}$$

1. Justify that F is a cumulative distribution function.

2. Justify that the law associated to F is continuous.

3. Compute the density f associated to this law.

4. Let X be a random variable with cumulative distribution function F.

   (a) Compute $\mathbb{P}(X > 1/2)$.

   (b) Compute $\mathbb{E}[X]$.

   (c) Compute the variance of X.

Exercise 5. *This exercise is significantly harder than the previous ones, but not worth many points. Try it once you have finished the rest of the midterm.*
We remind that:

- *an exponential random variable with rate* λ *has density* $f(x) = \lambda e^{-\lambda x} I(x > 0)$,
- *a gamma random variable with shape parameter* α *and rate* λ *has density*

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I(x > 0)\,,$$

- *if* n *is a non-negative integer,* $\Gamma(n + 1) = n!$, *and*
- *a Poisson random variable with rate* λ *has probability mass function* $f(n) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, \ldots$.

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent random variables of exponential law with rate parameter 1. We denote $T_0 = 0$ and for all $n \geq 1$, $T_n = X_1 + \cdots + X_n$. For all $t \geq 0$, we denote $N_t = \max\{n \geq 0 \,|\, T_n \leq t\}$.

1. **(a)** Compute the cumulative distribution function of T_2. Deduce that T_2 is a gamma random variable with shape parameter 2 and rate 1. (Like everywhere else, a proof is required.)

   (b) Let $n \geq 1$. Using the same method, compute the law of T_n.

2. Let $t > 0$. Compute the law of N_t.

3. Let $n \geq 1$.

   (a) Compute the joint law of $(T_1, \ldots, T_n)$.

   (b) Compute the conditional law of $(T_1, \ldots, T_n)$ given that $N_t = n$.