Exercise 1. Figure 1 shows a graph with 9 edges and 8 vertices, two of which are denoted by A and B. In an experiment, we delete each edge independently with probability $\frac{1}{2}$.

(a) Write down the sample space Ω for this experiment. What is the cardinality of Ω?

(b) With what probability is there a path connecting A and B?

(c) With what probability is there a path connecting A and B, if we also know that at most one edge is deleted?

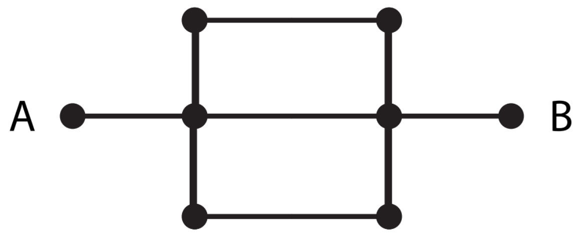

Figure 1: The graph for Exercise 1.

Exercise 2. According to the statistics department, about 1% of the students cheat during exams at EPFL. To detect cheating students more accurately, EPFL designed a new method based on an AI system. The detector can correctly detect a cheater with 90% probability, but unfortunately, the detector classifies a honest student as a cheater with 2% probability.

(a) The detector said your friend Alice was cheating on the exam. What is the probability that she actually cheated? Formulate the problem described above precisely by defining random variables and conditional probabilities.

(c) The experiment can be repeated n times for the same student, and the detector's answer is conditionally independent on the state of the student (cheater or honest). A student is declared a cheater if all of the n times the detector said he or she is cheating. How large does n need to be so that a student declared as a cheater has more than 99.99% chance of being actually a cheater?

Exercise 3. Consider rolling two fair dice, and consider the addition of their scores as the random variable X. Assume you have two options for gambling: 1. If $X = 7$, you will win 1 CHF, otherwise you will loose B_1 CHF; 2. If $X = 2$ or $X = 12$, you will win 1 CHF, otherwise you will loose B_2 CHF.

(a) For each option, compute the probability of winning. We call them P_1 and P_2.

(b) For each option, compute the expected value of the money you will win as functions of B_1 and B_2. We call them μ_1 and μ_2.

(c) For each option, compute the variance of the money you will win as functions of B_1 and B_2. We call them σ_1^2 and σ_2^2.

(d) Find B_2 as a function of B_1 such that $\mu_1 = \mu_2$. We call it B_2^*.

(e) *BONUS:* When $B_2 = B_2^*$, find σ_2^2 as a function of σ_1^2. Which one is smaller? In this case, if you want to maximize the probability of winning, which option is better for you? If you want to minimize the variance, which option is better for you?

Exercise 4.

(a) A component of a system breaks down independently with probability $p = 0.001$ each day. Compute the probability that the component is still functional after 1000 days.

(b) Compute $\mathrm{P}[X > E[X]]$, where $X \sim \mathrm{Exp}(\lambda)$ is an exponential random variable.

(c) *BONUS:* Compare the results of part a) and b) and explain it intuitively using a propery seen in class.