TEST - solutions

24 April 2017

Ex. 1 **a)** [*2 marks*] Let r, g denote the outcomes of the red and green dice respectively. The sample space is $\Omega = \{(r,g) : r, g \in \{1,\ldots,6\}\}$ and by symmetry, each element of Ω has probability $\frac{1}{36}$.

b) [*3 marks*] We have $A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$, and

$$B = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,1),(4,2),(5,1),(5,2),(6,1),(6,2)\},$$

so

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}, \quad \Pr(B) = \frac{|B|}{|\Omega|} = \frac{12}{36} = \frac{1}{3}.$$

c) [*1 marks*] We have

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{|A \cap B|}{|A|} = \frac{5}{6}.$$

d) [*2 marks*] The total score is the random variable $X\{(r,g)\} = r + g$, and

$$\Pr(X = 2 \mid A) = \frac{\Pr[\{(1,1)\} \cap A]}{\Pr(A)} = \frac{1}{6},$$

and so forth, giving $\Pr(X = 3 \mid A) = 2/6$, $\Pr(X = 4 \mid A) = 3/6$, and finally

$$\mathrm{E}(X \mid A) = 2 \times \tfrac{1}{6} + 3 \times \tfrac{2}{6} + 4 \times \tfrac{3}{6} = \frac{2+6+12}{6} = \frac{10}{3}.$$

Ex. 2 **a)** [*2 marks*] Here is one possible formulation (we accept other equivalent ones). Let T_s, T_p denote sword fight and pistol fight respectively. When they fight with swords, let A and P denote that Aramis and Porthos win. When they fight with pistols, let A, A^c, P, P^c denote Aramis hits, misses, and Porthos hits, misses respectively. The sample space Ω can be written as

$$\{(T_s,A),(T_s,P),(T_p,A),(T_p,A^cP)\} \cup \{(T_p,(A^cP^c)^kA),(T_p,(A^cP^c)^kA^cP) : k \in \mathbb{N}\} \cup \{(T_p,(A^cP^c)^\infty)\},$$

where $(A^cP^c)^k$ denotes k successive repetitions of A^cP^c. (Bonus if they include the last part, no loss of points if they don't.)

b) [*2 marks*] Let $p = \Pr(P \mid T_p)$ denote the probability that Porthos will win if they fight with pistols, and note that p can be obtained recursively from

$$p = \Pr(A^c \mid T_p)\ \Pr(P) + \Pr(A^c \mid T_p)\ \Pr(P^c \mid T_p)\ p.$$

Solving for p yields

$$p = \frac{(1-0.4)\times 0.6}{1-(1-0.4)\times(1-0.6)} \approx 0.47.$$

Alternatively, we can write

$$p = \sum_{k=0}^{\infty} \Pr\{(A^cP^c)^kA^cP\} = \sum_{k=0}^{\infty}(0.6\times 0.4)^k 0.6 \times 0.6 = \frac{0.36}{1-0.24} = 0.47,$$

using geometric series summation.

c) [*2 marks*] Using an obvious notation, p from **b)** and the fact that $\Pr(T_s) = \Pr(T_p) = 0.5$, we have

$$\Pr(T_s \mid A) = \frac{\Pr(A \mid T_s)\Pr(T_s)}{\Pr(A \mid T_s)\Pr(T_s) + \Pr(A \mid T_p)\Pr(T_p)} = \frac{1-0.4}{(1-0.4)+1-p} \approx 0.53.$$

d) [*4 marks*] Let N denote the number of shots if they fight with pistols, and let N_A, N_P denote the numbers of shots when Aramis wins and when Porthos wins respectively, so $\mathrm{E}(N) = \mathrm{E}(N_A) + \mathrm{E}(N_P)$. Then

$$\mathrm{E}(N_A) = \sum_{n=0}^{\infty}(2n+1)\Pr\{(A^cP^c)^n A\} = \sum_{n=0}^{\infty}(2n+1) \times (0.6 \times 0.4)^n \times 0.4.$$

Similarly, we have

$$\mathrm{E}(N_P) = \sum_{n=0}^{\infty}(2n+2)\Pr\{(A^cP^c)^n A^cP\} = \sum_{n=0}^{\infty}(2n+2) \times (0.6 \times 0.4)^n \times 0.6 \times 0.6.$$

Using the hint, we get

$$\mathrm{E}(N) = 0.4\left[2\sum_{n=0}^{\infty} n\ (0.6 \times 0.4)^n + \sum_{n=0}^{\infty}(0.6 \times 0.4)^n\right] + 0.6^2 \times 2\sum_{n=0}^{\infty}(n+1)(0.6 \times 0.4)^n$$

and since the question only asked for an expression for $\mathrm{E}(N)$, it would be OK to stop here (with full marks).

For completeness we compute the expected number of shots using formulae for geometric series:

$$\begin{aligned}
\mathrm{E}(N) &= 0.4\left[2\sum_{n=0}^{\infty} n\ (0.6 \times 0.4)^n + \sum_{n=0}^{\infty}(0.6 \times 0.4)^n\right] + 0.6^2 \times 2\sum_{n=0}^{\infty}(n+1)(0.6 \times 0.4)^n \\
&= 0.4 \times 2 \times 0.6 \times 0.4\sum_{n=1}^{\infty} n(0.6 \times 0.4)^{n-1} + 0.4\tfrac{1}{1-0.6\times0.4} + 0.6^2 \times 2\sum_{n=1}^{\infty} n(0.6 \times 0.4)^{n-1} \\
&= \tfrac{0.4\times2\times0.6\times0.4+0.6^2\times2}{(1-0.6\times0.4)^2} + \tfrac{0.4}{1-0.6\times0.4} \approx 2.1.
\end{aligned}$$

Ex. 3 **a)** [*4 marks*] Let X, S, D denote the power that will be generated tomorrow, on a sunny day and on a dull day respectively, and let p_S, p_D denote the probability of having a sunny or a dull day respectively. Then the law of total probability gives

$$\begin{aligned}
\Pr(25 < X < 35) &= p_S\Pr(25 < S \le 35) + p_D\Pr(25 < D \le 35) \\
&= p_S\Pr\left(\tfrac{25-30}{3} < \tfrac{S-30}{3} \le \tfrac{35-30}{3}\right) + p_D\Pr\left(\tfrac{25-20}{4} < \tfrac{D-20}{4} \le \tfrac{35-20}{4}\right) \\
&= p_S\{\Phi(5/3) - \Phi(-5/3)\} + p_D\{\Phi(15/4) - \Phi(5/4)\} \\
&= 0.3\{2\Phi(5/3) - 1\} + 0.7\{\Phi(15/4) - \Phi(5/4)\} \\
&\approx 0.345.
\end{aligned}$$

b) [*3 marks*] Let Y denote the power generated this week. Then $Y = S_1 + \cdots + S_5 + D_6 + D_7$, and since the power generated on different days is independent, Y has a Gaussian distribution with $\mathrm{E}(Y) = 5 \times 30 + 2 \times 20 = 190$, and $\mathrm{var}(Y) = 5 \times 3^2 + 2 \times 4^2 = 77$. Thus,

$$\Pr(Y \geqslant 200) = \Pr\left(\tfrac{Y-190}{\sqrt{77}} \geqslant \tfrac{200-190}{\sqrt{77}}\right) = 1 - \Phi(10/\sqrt{77}) \approx 0.13.$$

c) [*3 marks*] Let Z denote the power generated this week in this new situation. By an argument similar to that above, $Z = 130 + S + \sum_{i=1}^{2} D_i$ has a Gaussian distribution with $\mathrm{E}(Z) = 130 + 30 + 2 \times 20 = 200$, and $\mathrm{var}(Z) = 3^2 + 2 \times 4^2 = 41$. Thus,

$$\Pr(Z \geqslant 220) = \Pr\left(\tfrac{Z-200}{\sqrt{41}} \geqslant \tfrac{220-200}{\sqrt{41}}\right) = 1 - \Phi(20/\sqrt{41}) \approx 0.0009.$$

d) [*3 marks*] Let $X = 1_S S + 1_D D$, using independence of S and D, we have

$$E(X) = p_S E(S) + p_D E(D) = 0.3 \times 30 + 0.7 \times 20 = 23 \text{ kWh},$$

and

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E^2(X) \\
&= p_S E(S^2) + p_D E(D^2) - E^2(X) \\
&= p_S\{\mathrm{Var}(S) + E^2(S)\} + p_D\{\mathrm{Var}(D) + E^2(D)\} - E^2(X) \\
&= 0.3(3^2 + 30^2) + 0.7(4^2 + 20^2) - 23^2 = 34.9 \text{ kWh}^2.
\end{aligned}$$

Note that $\mathrm{Var}(X) \neq p_S^2 \mathrm{Var}(S) + p_D^2 \mathrm{Var}(D)$ as 1_S and 1_D are not deterministic but random variables.

Ex. 4 **a)** [*3 marks*] As $X_{1,2}, X_{1,3}, X_{1,4}, X_{2,4}$ are four independent Bernoulli variables, there are $2^4 = 16$ different configurations, and clearly $S \sim B(4, \frac{1}{2})$, so the numbers of ways to have $S = 0, 1, 2, 3, 4$ are respectively $1, 4, 6, 4, 1$.

It is then easy to check (e.g., by drawing a tree) that the complete table is

s	0	1	1	2	2	3	3	3	4
t	0	0	1	0	1	0	1	2	4
Number of configurations N_{st}	1	3	1	1	5	0	0	4	1

Clearly, $\Pr(S = s, T = t) = N_{st}/16$.

b) [*3 marks*] As S has a binomial distribution with denominator $n = 4$ and probability $\frac{1}{2}$, the probabilities that $S = 0, 1, 2, 3, 4$ are $1/16, 4/16, 6/16, 4/16, 1/16$.

Clearly

$$\begin{aligned}
\Pr(T = 0) &= (1 + 3 + 1)/16 = 5/16, \\
\Pr(T = 1) &= (1 + 5)/16 = 6/16, \\
\Pr(T = 2) &= 4/16, \\
\Pr(T = 4) &= 1/16,
\end{aligned}$$

where we note that the total probability is $(5 + 6 + 4 + 1)/16 = 1$.

The variables S and T are clearly not independent; for example, $S = 0$ implies $T = 0$.

c) [*2 marks*] The conditional distribution of S given that $T = 1$ is

$$\begin{aligned}
\Pr(S = 1 \mid T = 1) &= \frac{\Pr(S = 1, T = 1)}{\Pr(T = 1)} = \frac{1/16}{6/16} = 1/6, \\
\Pr(S = 2 \mid T = 1) &= \frac{\Pr(S = 2, T = 1)}{\Pr(T = 1)} = \frac{5/16}{6/16} = 5/6,
\end{aligned}$$

so

$$E(S \mid T = 1) = \sum_s s \Pr(S = s \mid T = 1) = 1 \times \tfrac{1}{6} + 2 \times \tfrac{5}{6} = 11/6.$$

(We also accept $2 + 11/6 = 23/6$, since the question does not specify whether these are additional links or just links.)