Solution 1. Two interpretations of the wording are possible, namely, either (i) the gain is \$$2^r \times 1000$ for $r > 0$ (and \$0 if $r = 0$), or (ii) the gain is \$$2^r \times 1000$ for $r \geq 0$. In the following, we provide the solutions and the number of points under both (i) and (ii).

(a) i) and ii) (3 points) The sample space for this experiment can be written

$$\Omega = \{(1,0),(1,1),(2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(3,3)\},$$

where the ordered pair (n,k) represents the event "the number n was written on the card" and "you answered k question(s) correctly".
Then,

$$\Pr\{(n,k)\} = \frac{1}{3} \times \binom{n}{k}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{n-k} = \binom{n}{k}\frac{1}{3 \times 2^n}, \quad 1 \leq k \leq n \leq 3,$$

since each card can be chosen with probability $1/3$, and the number of correct answers follows a binomial distribution with parameters n and $1/2$. Hence, the probabilities are

n	k	$\Pr\{(n,k)\}$
1	0	$1/6 \approx 0.1667$
1	1	$1/6 \approx 0.1667$
2	0	$1/12 \approx 0.0833$
2	1	$1/6 \approx 0.1667$
2	2	$1/12 \approx 0.0833$
3	0	$1/24 \approx 0.0417$
3	1	$1/8 \approx 0.125$
3	2	$1/8 \approx 0.125$
3	3	$1/24 \approx 0.0417$

and

$$\sum_{n=1}^{3}\sum_{k=1}^{n}\Pr\{(n,k)\} = 3 \times \frac{1}{6} + 2 \times \frac{1}{8} + 2 \times \frac{1}{12} + 2 \times \frac{1}{24} = 1.$$

(b) i) (1.5 points) Let the random variable X denote the amount that you win. Then,

$$\begin{aligned}\Pr(X=0) &= \Pr\{(1,0)\cup(2,0)\cup(3,0)\} = \Pr\{(1,0)\} + \Pr\{(2,0)\} + \Pr\{(3,0)\}\\ &= \frac{1}{6} + \frac{1}{12} + \frac{1}{24} = \frac{7}{24} \approx 0.2917\end{aligned}$$

since the events are mutually exclusive.

ii) (1 point) Let the random variable X denote the amount that you win. Then,

$$\Pr(X=0) = 0$$

since $X \in \{1000, 2000, 4000, 8000\}$.

(c) i) and ii) (1.5 points) Similarly,

$$\begin{aligned}\Pr(X=2000) &= \Pr\{(1,1)\cup(2,1)\cup(3,1)\} = \Pr\{(1,1)\} + \Pr\{(2,1)\} + \Pr\{(3,1)\}\\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{8} = \frac{11}{24} \approx 0.4583.\end{aligned}$$

(d) i) (2 points) Let the random variable K denote the number of correct answers. Then,

$$\begin{aligned}\Pr(X \geq 4000 \mid X \geq 1000) &= \Pr(K \geq 2 \mid K > 0) \\ &= \frac{\Pr(K \geq 2, K > 0)}{\Pr(K > 0)} = \frac{\Pr(K \geq 2)}{\Pr(K \geq 1)} \\ &= \frac{\Pr\{(2,2) \cup (3,2) \cup (3,3)\}}{\Pr\{(1,1) \cup (2,1) \cup (2,2) \cup (3,1) \cup (3,2) \cup (3,3)\}} \\ &= \frac{\frac{1}{12} + \frac{1}{8} + \frac{1}{24}}{2 \times \frac{1}{6} + \frac{1}{12} + 2 \times \frac{1}{8} + \frac{1}{24}} = \frac{6}{17} \approx 0.3529.\end{aligned}$$

ii) (1 point) Let the random variable K denote the number of correct answers. Then, noting that $K \in \{0, 1, 2, 3\}$,

$$\begin{aligned}\Pr(X \geq 4000 \mid X \geq 1000) &= \Pr(K \geq 2 \mid K \geq 0) \\ &= \Pr(K \geq 2) \\ &= \Pr\{(2,2) \cup (3,2) \cup (3,3)\} \\ &= \frac{1}{12} + \frac{1}{8} + \frac{1}{24} = \frac{1}{4} = 0.25.\end{aligned}$$

Solution 2. (a) (2 points) Denote D the event "the download is complete/successful" and I the event "the downloaded file is not corrupted". From the wording we have $\Pr(D) = p$ and $\Pr(I \mid D) = q$.

The probability of downloading a non-corrupted file at a first attempt is

$$\Pr(D \cap I) = \Pr(I \mid D)\Pr(D) = pq.$$

(b) (2 points) Let the random variable X denote the waiting time (number of download attempts) to the first successful download. Then,

$$\Pr(X = x) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots,$$

since one must have $x - 1$ failures before the successful download. So, the waiting time to the first successful download follows a geometric distribution with parameter p, i.e., $X \sim \mathrm{Geom}(p)$.

Then, the expected number of downloads is $\mathrm{E}(X) = 1/p$.

(c) (1 points) By the independence of the outcome of successive downloads, the probability that the first successful download is of a corrupt file is

$$\Pr(I^c \mid D) = 1 - q.$$

Solution 3. (a) (5 points) Define the event B 'the bus is late' and write T for the time taken by the bus to reach the station; $\Pr(B) = 0.2$. Conditional on B^c, $T = 10$ minutes with probability one, and conditional on B, $T \sim N(10 + 3, 1)$. Then

$$\begin{aligned}\Pr(T \leq 14) &= \Pr(T \leq 14 \mid B)\Pr(B) + \Pr(T \leq 14 \mid B^c)\Pr(B^c) \\ &= \Phi\{(14 - 13)/1\} \times 0.2 + 1 \times 0.8 \\ &= 0.8413 \times 0.2 + 0.8 \\ &\approx 0.9683.\end{aligned}$$

Alternatively: let the random variable $L = 1$ if the bus is late, and $L = 0$ otherwise. So, $L \sim \text{Bern}(p = 0.2)$, and X and L are independent. Let the random variable Y denote the delay of tomorrows' bus journey. Then $Y = LX$.

The probability that I will catch the train tomorrow is

$$\begin{aligned}
\Pr(Y \le 4) &= \Pr(LX \le 4) \\
&= \Pr(LX \le 4 \mid L = 1)\Pr(L = 1) + \Pr(LX \le 4 \mid L = 0)\Pr(L = 0) \\
&= \Pr(X \le 4) \times p + 1 \times (1 - p)
\end{aligned}$$

by the independence of X and L. Now,

$$\Pr(X \le 4) = \Pr\left(\frac{X-3}{1} \le 1\right) = \Pr(Z \le 1) = \Phi(1) \approx 0.8413.$$

since Z is a standard normal random variable. Thus,

$$\Pr(Y \le 4) = 0.2 \times \Phi(1) + 1 \times 0.8 \approx 0.9683.$$

(b) (3 points) We seek

$$\begin{aligned}
\Pr(T > 14 \mid T > 13) &= \frac{\Pr(T > 14, T > 13)}{\Pr(T > 13)} \\
&= \frac{\Pr(T > 14 \mid B)\Pr(B) + \Pr(T > 14 \mid B^c)\Pr(B^c)}{\Pr(T > 13 \mid B)\Pr(B) + \Pr(T > 13 \mid B^c)\Pr(B^c)} \\
&= \frac{\Pr(T > 14 \mid B)\Pr(B)}{\Pr(T > 13 \mid B)\Pr(B)} \\
&= \frac{\Pr(T > 14 \mid B)}{\Pr(T > 13 \mid B)} \\
&= \frac{1 - \Phi(14 - 13)}{1 - \Phi(13 - 13)} \\
&= 2\{1 - \Phi(1)\} \approx 0.3174.
\end{aligned}$$

Alternatively: the requested probability is

$$\begin{aligned}
\Pr(Y > 4 \mid Y \ge 3) &= \Pr(LX > 4 \mid LX > 3) = \Pr(LX > 4 \mid L = 1, X > 3) \\
&= \Pr(X > 4 \mid L = 1, X > 3) = \frac{\Pr(L > 1, X > 4)}{\Pr(L > 1, X > 3)} \\
&= \frac{\Pr(L > 1)\Pr(X > 4)}{\Pr(L > 1)\Pr(X > 3)} = \frac{1 - \Pr(X \le 4)}{1 - \Pr(X \le 3)} \\
&= \frac{1 - \Phi(1)}{1 - \Phi(0)} \approx 2(1 - 0.8413) = 0.3174.
\end{aligned}$$

(c) (3 points) We use Bayes' theorem and the result from (a) to obtain

$$\begin{aligned}
\Pr(B \mid T \le 14) &= \frac{\Pr(T \le 14 \mid B)\Pr(B)}{\Pr(T \le 14)} \\
&= \frac{\Phi(14 - 13) \times 0.2}{0.9683} \\
&\approx 0.1738.
\end{aligned}$$

Alternatively: using Bayes' theorem and the law of total probability, the requested probability is

$$\begin{aligned}\Pr(L=1 \mid Y \le 4) &= \frac{\Pr(Y \le 4 \mid L=1)\Pr(L=1)}{\Pr(Y \le 4)} \\ &= \frac{\Pr(LX \le 4 \mid L=1) \times p}{1-p\{1-\Phi(1)\}} \\ &= \frac{\Pr(X \le 4) \times p}{1-p\{1-\Phi(1)\}} \\ &= \frac{0.2 \times \Phi(1)}{1-0.2\{1-\Phi(1)\}} \\ &\approx 0.1738.\end{aligned}$$

Alternatively: the requested probability is

$$\begin{aligned}\Pr(L=1 \mid Y \le 4) = \Pr(L=1 \mid LX \le 4) &= \Pr(L=1 \mid \{L=0\} \cup \{X \le 4\}) \\ &= \frac{\Pr(\{L=1\} \cap [\{L=0\} \cup \{X \le 4\}])}{\Pr(\{L=0\} \cup \{X \le 4\})} \\ &= \frac{\Pr(\{L=1\} \cap \{X \le 4\})}{\Pr(L=0) + \Pr(X \le 4) - \Pr(\{L=0\} \cap \{X \le 4\})} \\ &= \frac{\Pr(L=1)\Pr(X \le 4)}{\Pr(L=0) + \Pr(X \le 4) - \Pr(L=0)\Pr(X \le 4)} \\ &= \frac{0.2 \times \Phi(1)}{0.8 + \Phi(1) - 0.8 \times \Phi(1)} \approx 0.1738.\end{aligned}$$

(d) (2 points)

The expected bus journey duration is

$$\begin{aligned}\mathrm{E}(T) &= \mathrm{E}(T \mid B)\Pr(B) + \mathrm{E}(T \mid B^c)\Pr(B^c) \\ &= 13 \times 0.2 + 10 \times 0.8 \\ &= 10 + 3 \times 0.2 = 10.6 \text{ minute,}\end{aligned}$$

and the expected arrival time at the station is 7.55 and 36 seconds.

Alternatively: from the wording, the joint density function of $L \sim \mathrm{Bern}(p = 0.2)$ and $X \sim N(\mu = 3, \sigma^2 = 1)$ is

$$f_{L,X}(l, x) = \{pI(l=1) + (1-p)I(l=0)\} \times \phi_{\mu,\sigma^2}(x), \quad l \in \{0,1\}, x \in \mathbb{R},$$

where $I(\cdot)$ is the indicator function, and $\phi_{\mu,\sigma^2}(x)$ denotes the density function of the normal distribution with mean μ and variance σ^2. So, the expected delay is

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(LX) \\ &= \int_{-\infty}^{\infty} \sum_{l=0}^{1} lx \times \left[\{pI(l=1) + (1-p)I(l=0)\} \times \phi_{\mu,\sigma^2}(x)\right] \mathrm{d}x \\ &= p \int_{-\infty}^{\infty} x\phi_{\mu,\sigma^2}(x)\, \mathrm{d}x = p\mathrm{E}(X) = 0.6 \text{ minute,}\end{aligned}$$

and the expected arrival time at the station is 7.55 and 36 seconds.

Alternatively: by the law of iterated expectations, the expected delay is

$$\mathrm{E}(Y) = \mathrm{E}(LX) = \mathrm{E}_L\{\mathrm{E}(LX \mid L)\}.$$

Now, by the independence of X and L,

$$\mathrm{E}(LX \mid L = k) = \mathrm{E}(kX \mid L = k) = k\mathrm{E}(X \mid L = k) = k\mathrm{E}(X) = 3k,$$

and therefore $\mathrm{E}(IX \mid L) = 3I$. Thus,

$$\mathrm{E}(Y) = \mathrm{E}_L(3L) = 3p = 0.6 \text{ minute},$$

and the expected arrival time at the station is 7.55 and 36 seconds.

Solution 4. (a) (3 points) The joint probability density function of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}(1-e^{-x})(1-e^{-y}) = \frac{\partial}{\partial y} e^{-x}(1-e^{-y}) \\ &= e^{-x-y}, \quad x, y \geq 0. \end{aligned}$$

Clearly, the joint density factorizes and the support is a Cartesian product, so the flows X and Y are independent.

Alternatively: computing the marginal densities (see (b)) yields $f_X(x) = e^{-x}$ for $x > 0$, and $f_Y(y) = e^{-y}$ for $y > 0$. So the flows X and Y are independent since

$$f_{X,Y}(x,y) = e^{-x-y} = e^{-x} \times e^{-y} = f_X(x) \times f_Y(y), \quad x, y > 0.$$

(b) (4 points) The marginal density of X is

$$f_x(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^{\infty} e^{-x-y}\,\mathrm{d}y = e^{-x}\left[-e^{-y}\right]_0^{\infty} = e^{-x}, \quad x > 0.$$

Then,

$$\Pr(X > 3) = \int_3^{\infty} f_X(x)\,\mathrm{d}x = \int_3^{\infty} e^{-x}\,\mathrm{d}x = -e^{-x}|_3^{\infty} = e^{-3} \approx 0.0498,$$

and

$$\begin{aligned} \Pr(X > 3, Y > 3) &= \int_3^{\infty}\int_3^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y = \int_3^{\infty}\int_3^{\infty} e^{-x}e^{-y}\,\mathrm{d}x\mathrm{d}y \\ &= \int_3^{\infty} e^{-x}\,\mathrm{d}x \int_3^{\infty} e^{-y}\,\mathrm{d}y = \left(e^{-3}\right)^2 = e^{-6} \approx 0.0025. \end{aligned}$$

(c) (3 points) Let $\mathcal{A} = \{(x,y) : x+y \leq 6,\ x, y \geq 0\}$. The probability that no alert is sounded today is

$$\begin{aligned} \Pr(X + Y \leq 6) &= \int\int_{(x,y)\in\mathcal{A}} f_{X,Y}(x,y)\,\mathrm{d}y\mathrm{d}x = \int_0^6\int_0^{6-x} e^{-x}e^{-y}\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^6 e^{-x}\left[-e^{-y}\right]_0^{6-x}\,\mathrm{d}x = \int_0^6 (e^{-x} - e^{-6})\,\mathrm{d}x \\ &= \left[-e^{-x} - xe^{-6}\right]_0^6 = 1 - 7e^{-6} \approx 0.9826. \end{aligned}$$

Noting that $\{\{X > 3\} \cap \{Y > 3\}\} \subset \{X + Y > 6\}$, we have

$$\Pr(X + Y \leq 6) = 1 - \Pr(X + Y > 6) \leq 1 - \Pr(X > 3, Y > 3).$$