Solution 1. (a) (1 point) The sample space for this experiment can be written

$$\Omega = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,0)\},$$

where the ordered pairs represent (score from coin, score from die).

(b) (2 points) Let S be the random variate corresponding to the total score of the coin and the die. The probability that S is odd is

$$\begin{aligned}
\Pr(\text{`}S \text{ is odd'}) &= \Pr(\{S=1\}\cup\{S=3\}\cup\{S=5\}) \\
&= \Pr(S=1)+\Pr(S=3)+\Pr(S=5) \\
&= \Pr\{(0,1)\}+\Pr\{(1,0)\}+\Pr\{(0,3)\}+\Pr\{(0,5)\} \\
&= \frac{1}{2}\times\frac{1}{6}+\frac{1}{2}+\frac{1}{2}\times\frac{1}{6}+\frac{1}{2}\times\frac{1}{6}=\frac{3}{4}.
\end{aligned}$$

(c) (2 points) The probability that S equals 3 or less is

$$\begin{aligned}
\Pr(S\le 3) &= \Pr(\{S=1\}\cup\{S=2\}\cup\{S=3\}) \\
&= \Pr(S=1)+\Pr(S=2)+\Pr(S=3) \\
&= \Pr\{(0,1)\}+\Pr\{(1,0)\}+\Pr\{(0,2)\}+\Pr\{(0,3)\} \\
&= \frac{1}{2}\times\frac{1}{6}+\frac{1}{2}+\frac{1}{2}\times\frac{1}{6}+\frac{1}{2}\times\frac{1}{6}=\frac{3}{4}.
\end{aligned}$$

(d) (2 points) Denote H the event that the coin showed heads. The probability that the coin showed heads if the total score equals 3 or less is

$$\Pr(H\mid S\le 3)=\frac{\Pr(H\cap S\le 3)}{\Pr(S\le 3)}=\frac{\Pr\{(1,0)\}}{\Pr(S\le 3)}=\frac{1/2}{3/4}=2/3.$$

Solution 2. (a) (3 points) The probability of winning the lottery on the next draw is

$$\Pr(\text{`win'})=\frac{\binom{5}{5}\binom{2}{2}}{\binom{50}{5}\binom{11}{2}}=1/116'531'800\approx 8.581\times 10^{-09}.$$

(b) (1 points) Denote X the number of winners at the next draw. Since each person has one chance in 116'531'800 to win and the people play independently, $X\sim B(n=80'000'000, p=1/116'531'800)$, and hence the expected number of winners at the next draw is

$$\mathrm{E}(X)=np=\frac{80'000'000}{116'531'800}=0.6865.$$

(c) (3 points) Here the law of small numbers applies (n is very large and p very small), so X follows approximately a Poisson distribution with parameter $\lambda=np=0.6865$. So,

$$\begin{aligned}
\Pr(X=0) &= \frac{0.6865^0}{0!}e^{-0.6865}\approx 0.5033, \\
\Pr(X=1) &= \frac{0.6865^1}{1!}e^{-0.6865}\approx 0.3455,
\end{aligned}$$

and $\Pr(X\ge 2)=1-\Pr(X=0)-\Pr(X=1)\approx 0.1512$.

Alternatively, from the binomial distribution we have

$$\begin{aligned}
\Pr(X=0) &= \binom{80'000'000}{0}\left(\frac{1}{116'531'800}\right)^0\left(1-\frac{1}{116'531'800}\right)^{80'000'000} &&\approx 0.5033, \\
\Pr(X=1) &= \binom{80'000'000}{1}\left(\frac{1}{116'531'800}\right)^1\left(1-\frac{1}{116'531'800}\right)^{79'999'999} &&\approx 0.3455,
\end{aligned}$$

and $\Pr(X\ge 2)=1-\Pr(X=0)-\Pr(X=1)\approx 0.1512$.

(d) (2 points) Let N denote the number of draws until the next winner. The event $\{N = n\}$ happens if there is no winner at the $n-1$ first draws and there is at least one winner at the nth draw. Each draw is a Bernoulli trial with probability of 'success' $\tilde{p} = \Pr(X \geq 1) = 0.4967$. Therefore, the number of draws N until the first win follows a geometric distribution of parameter $\tilde{p}$, and mass function

$$\Pr(N = n) = \tilde{p}(1 - \tilde{p})^{n-1}, \quad n = 1, 2, \ldots.$$

Solution 3. (a) (4 points) Let R and S denote respectively the run times of the first and second jobs. Since they are uniformly distributed from zero to three hours, their cumulative distribution functions are

$$F_R(z) = F_S(z) = \begin{cases} 0, & z \leq 0, \\ \frac{z}{3}, & 0 < z < 3, \\ 1, & z \geq 3. \end{cases}$$

The cumulative distribution function of the longest run time $X = \max(R, S)$ is

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) = \Pr\{\max(R, S) \leq x\} = \Pr(R \leq x, S \leq x) \\ &= \Pr(R \leq x)\Pr(S \leq x) = F_R(x)F_S(x). \end{aligned}$$

So,

$$F_X(x) = \begin{cases} 0, & z \leq 0, \\ \frac{z^2}{9}, & 0 < z < 3, \\ 1, & z \geq 3. \end{cases}$$

Therefore, the probability density function of X is

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} 2x/9, & 0 < x < 3, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) (2 points) The expected waiting time until both processors are free is

$$\mathrm{E}(X) = \int_0^3 x \frac{2x}{9} \mathrm{d}x = \frac{2}{9} \frac{x^3}{3}\bigg|_0^3 = 2 \text{ hours.}$$

(c) (2 points) The probability that the longest run time was less than one hour given that both jobs were done after two hours is

$$\Pr(X < 1 \mid X < 2) = \frac{\Pr(X < 1, X < 2)}{\Pr(X < 2)} = \frac{\Pr(X < 1)}{\Pr(X < 2)} = \frac{1/9}{4/9} = 1/4.$$

(d) (2 points) The expected running time until both cores are free if the computer is still running after two hours is

$$\begin{aligned} \mathrm{E}(X \mid X > 2) &= \frac{\mathrm{E}\{XI(X > 2)\}}{\Pr(X > 2)} \\ &= \frac{1}{1 - \Pr(X \leq 2)} \int_0^3 xI(x > 2) \frac{2x}{9} \mathrm{d}x \\ &= \frac{9}{5} \int_2^3 x \frac{2x}{9} \mathrm{d}x \\ &= \frac{2}{5} \frac{x^3}{3}\bigg|_2^3 = \frac{38}{15} \approx 2.53. \end{aligned}$$

So, the expected remaining waiting time until both cores are free if the computer is still running after two hours is $\mathrm{E}(X - 2 \mid X > 2) = 38/15 - 2 = 8/15 \approx 0.53$ hour.

Solution 4. (a) (4 points) Let T denote the download time in hours. The probability that the download time exceeds t hours is

$$\Pr(T > t) = \int_t^\infty 10\exp(-10x)\mathrm{d}x = -\exp(-10x)|_t^\infty = \exp(-10t).$$

The conditional probability that the download time exceeds $t + x$ hours, given that it exceeds t hours, is

$$\begin{aligned}\Pr(T > t + x \mid T > t) &= \frac{\Pr(T > t + x, T > t)}{\Pr(T > t)} = \frac{\Pr(T > t + x)}{\Pr(T > t)}\\ &= \frac{\exp(-10t - 10x)}{\exp(-10t)} = \exp(-10x), \quad t, x > 0.\end{aligned}$$

Note that $\Pr(T > t + x \mid T > t) = \Pr(T > x)$. So the download time T exhibits the lack of memory property, which one should expect since T follows an exponential distribution with parameter $\lambda = 10$.

(b) (4 points) First, we find the constant c. To be a valid density, the joint density must integrate to one, so

$$\begin{aligned}1 = \int_0^\infty \int_0^\infty c\exp(-10x - 20y)\mathrm{d}x\mathrm{d}y &= c\int_0^\infty \exp(-10x)\mathrm{d}x \int_0^\infty \exp(-20y)\mathrm{d}y\\ &= c\left[-\frac{1}{10}\exp(-10x)\right]_0^\infty \left[-\frac{1}{20}\exp(-20y)\right]_0^\infty\\ &= c\frac{1}{200},\end{aligned}$$

giving $c = 200$.

The marginal density of Y is

$$\begin{aligned}f_Y(y) &= \int_0^\infty f_{X,Y}(x, y)\mathrm{d}x = 200\exp(-20y)\int_0^\infty \exp(-10x)\mathrm{d}x\\ &= 200\exp(-20y)\left[-\frac{1}{10}\exp(-10x)\right]_0^\infty = 20\exp(-20y), \quad y > 0.\end{aligned}$$

The random variates X and Y are independent since their support is a Cartesian product, $(0, \infty) \times (0, \infty)$, and in the support, the joint density factorises :

$$f_{X,Y}(x, y) = 200\exp(-10x - 20y) = 10\exp(-10x) \times 20\exp(-20y) = f_X(x) \times f_Y(y);$$

clearly $X \sim \exp(\lambda = 10)$ and $Y \sim \exp(\lambda = 20)$.

(c) (4 points) The probability that $Y < X$ is

$$\begin{aligned}\Pr(Y < X) &= \int_{y=0}^\infty \mathrm{d}y \int_{x=y}^\infty \mathrm{d}x f_{X,Y}(x, y)\\ &= \int_{y=0}^\infty \mathrm{d}y\, 20\exp(-20y) \int_{x=y}^\infty \mathrm{d}x\, 10\exp(-10x)\\ &= \int_{d=0}^\infty \mathrm{d}y\, 20\exp(-20y)[-\exp(-10x)]_{x=y}^\infty\\ &= 20\int_0^\infty \exp(-30y)\,\mathrm{d}y = 20\left[-\frac{1}{30}\exp(-30y)\right]_0^\infty = \frac{2}{3}.\end{aligned}$$