GROUP THEORY 2024 - 25, EXERCISE SHEET 13

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

We saw in class that from a group G one can construct a category BG with one object $\bullet$, whose morphisms is $Mor_{BG}(\bullet, \bullet) = G$, where composition is given by multiplication of the group.

Exercise 2. (easy) Let G, H be groups. What is the data of a functor $BH \to BG$? (Identify the set $Fun(BH, BG)$ of such functors with a known set.)

Exercise 3. (easy) Is there a functor $Z : Gr \to Gr$ such that objects of Gr are assigned to their centre (i.e $G \overset{Z}{\mapsto} Z(G)$)?

Definition 0.1. Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism in $\mathcal{C}$.

- f is a *monomorphism* if for all object Z and diagrams

$$Z \underset{g}{\overset{h}{\rightrightarrows}} X \xrightarrow{f} Y$$

  such that $f \circ g = f \circ h$, then $h = g$.
- Dually f is an *epimorphism* if for all Z and diagrams

$$X \xrightarrow{f} Y \underset{g}{\overset{h}{\rightrightarrows}} Z$$

  such that $g \circ f = h \circ f$, then $h = g$.

Exercise 4. (easy) Let $f : X \to Y$ be a morphism in the category Set of sets and functions. Prove that

(1) f is an epimorphism if and only if f is surjective;
(2) f is a monomorphism if and only if f is injective;

Exercise 5. (medium) *Verifying a Blue Claim from the notes:*
Let $\mathcal{C}$ be a category. Consider the following diagram in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ {\scriptstyle f}\big\downarrow & & \\ Y & & \end{array}$$

Prove that if its pushout exists, then it is unique up to isomorphism.

Remark: Note that even though the pushout of a diagram of the above form is unique if it exists, there are indeed examples of diagrams in certain categories where the pushout doesn't exist.

Exercise 6. (hard) Let G be a group and consider the category **Gset**. An object of **Gset** is a $G-$set, that is a set X along with the information of an action of the group G on X. Let X, Y be $G-$sets, a morphism $f : X \to Y$ in **Gset** is a set function f such that

$$f(g \cdot x) = g \cdot f(x)$$

for all $x \in X$ and $g \in G$.

(1) Let $F : \textbf{Gset} \to Set$ be the forgetful functor to the category of sets. Construct a right adjoint for F. That is, define a functor $H : Set \to \textbf{Gset}$ such that.

$$Mor_{Set}(F(X), Y) \cong Mor_{\textbf{Gset}}(X, H(Y))$$

for all sets Y and $G-$sets X.

(2) Let $Triv : Set \to \textbf{Gset}$ be the functor which assigns to a set X the $G-$set X along with the trivial G action i.e. $g \cdot x = x$ for all $g \in G$ and $x \in X$. Construct a right adjoint for $Triv$.