MATH 211 EXAM (2024)

(3 hours)

No books, notes, or electronic devices (especially no phones, but also no smart watches, smart pens, headphones etc) are permitted during this exam.

You must show your work to receive credit. **Justify everything**.

Do not unstaple the exam or reorder the pages. All problems must be solved within the space provided (right after the statement of the problem). If you need to use the extra pages at the end, then mention this clearly in the aforementioned space, so your grader knows that they have to also look at the end (they will not check the extra pages unless explicitly told to).

We will provide scratch paper (loose sheets) but do not write solutions on them. **Only the 24 pages of the booklet you're now reading will be graded**.

Please do not leave the room during the first and last 30 minutes of the exam.

Please keep your CAMIPRO face up on the table at all times.

Don't forget to write your SCIPER and sign the exam.

There are 8 problems, worth 100 points in total.

SCIPER: ______________________________

SIGNATURE: ______________________________

PROBLEM 1

(a) Which of the following operations are associative, i.e. $(a*b)*c = a*(b*c)$ (justify: if they are associative include a proof, if they are not associative then provide a counterexample)?

- The operation $*$ on $\mathbb{R}$ defined by $a * b = \dfrac{a+b}{2}$. *(3 points)*

  Solution: This operation is not associative. Indeed:

$$(0 * 1) * 1 = \frac{(0+1)}{2} * 1 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$$

  Yet,

$$0 * (1 * 1) = 0 * \frac{1+1}{2} = 0 * 1 = \frac{1}{2}$$

- The operation $*$ on the set of all finite words on a given alphabet S (i.e. arbitrary sequences $w_1 \dots w_k$ where $w_1, \dots, w_k \in S$ and $k \geq 0$ are arbitrary) given by concatenation, e.g. ginger $*$ bread = gingerbread. *(3 points)*

  Solution: This operation is associative. To see this, let $\alpha = w_1 w_2 \cdots w_k$, $\beta = v_1 v_2 \cdots v_l$ and $\gamma = x_1 \cdots x_r$ be three words in S. In particular:

$$\begin{aligned} \alpha * (\beta * \gamma) &= \alpha * (v_1 \cdots v_l x_1 \cdots x_r) = w_1 \cdots w_k v_1 \cdots v_l x_1 \cdots x_r \\ &= (w_1 \cdots w_k v_1 \cdots v_l) * \gamma = (\alpha * \beta) * \gamma. \quad \square \end{aligned}$$

(b) Consider the usual addition modulo 4 on the set $\mathbb{Z}/4\mathbb{Z}$. Show that this operation makes $\mathbb{Z}/4\mathbb{Z}$ into a group, by checking all the group axioms. *(3 points)*

Solution: We check all axioms, namely, given this binary operation, that (1) existence of neutral element, (2) existence of inverse and (3) existence of inverse are met. We leave the fact this is indeed binary to the reader.

(1) 0 given by all multiples of 4 in $\mathbb{Z}$ is the unit.
(2) Given $[x]_4$, it's inverse can be seen as being $[-x]_4$ where $[.]_4$ is the equivalence class modulo 4. Indeed, given $x + 4k$ and $-x + 4k'$, their sum is a multiple of 4, i.e the zero element.
(3) The operation is associative, since addition is associative in $\mathbb{Z}$:

$$x + 4k + (y + 4k' + z + 4k'') \overset{(\mathbb{Z},+)}{=} (x + 4k + y + 4k') + z + 4k''$$

Thus $[x]_4 + [y + z]_4$ is indeed in the same equivalence class as $[x + y]_4 + [z]_4$ □

(c) However, consider the usual multiplication modulo 4 on the set $\mathbb{Z}/4\mathbb{Z}$. Show that this operation does not make $\mathbb{Z}/4\mathbb{Z}$ into a group, by showing that at least one of the group axioms is violated. *(3 points)*

Solution: Indeed, 2 does not posses an inverse under this new multiplication. If it did, there would exist $k, k', j \in \mathbb{Z}$:

$$(2 + 4k)(x + 4k') = 1 + 4j$$
$$\overset{\exists l \in \mathbb{Z}}{\iff} 2x + 4l = 1 + 4j$$
$$\overset{j-l=:l'}{\iff} 2x = 1 + 2(2l')$$

which would mean an even equals an odd. We have our contradiction. □
Remark: we could have also shown [0] *has no inverse...*

PROBLEM 2

(a) If H_1 and H_2 are subgroups of a group G, prove that $H_1 \cup H_2$ is a subgroup of G if and only if one of the subgroups is contained inside the other (i.e. $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$).

(8 points)

Solution:

$\Longleftarrow$: If $H_1 \subseteq H_2$ then $H_1 \cup H_2 = H_2$ and so the union is a subgroup, since by hypothesis, $H_2 \leq G$. The other case is dealt with similarly.

$\Longrightarrow$: Suppose $H_1 \cup H_2 \leq G$. We want to show either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. We will suppose both false (*this is a common proof technique. When you wish to assert $P \implies Q$ or $Q \implies P$, it's often easier to proceed by contradiction*).
Indeed if neither is contained in the other (note they at least contain $e \in G$), we may find an $x \in H_1 \setminus H_2$ and a $y \in H_2 \setminus H_1$. In this case $x, y \in H_1 \cup H_2$, so by stability:

$$\boxed{x, y \in H_1 \cup H_2 \overset{H_1 \cup H_2 \leq G}{\Longrightarrow} xy \in H_1 \cup H_2}$$

So assuming first $xy \in H_1$, that means $x^{-1}xy \in H_1$ and so $y \in H_1$, leading to our desired contradiction. The same holds if $xy \in H_2$. $\square$

(b) Let G be the symmetric group S_5, and let $x \in G$ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$, i.e. the permutation taking $1 \to 2 \to 3 \to 4 \to 5 \to 1$. Show that the centralizer $C_G(x)$ (i.e. the set of all elements of G which commute with x) coincides with the subgroup $\langle x \rangle$ generated by x (i.e. the subset of G consisting of all integer powers of x). *(8 points)*

Solution:
First note $\langle x \rangle \subset C_G(x)$ since all powers of x trivially commute with x. We now proceed by showing $|\langle x \rangle| = |C_G(x)|$ since then, they have to be equal (*this is also a common proof writing trick: for example, to avoid showing two inclusions of vector spaces, we show once inclusion and conclude with equal dimensions*).
By the orbit stabilizer theorem ($cl(x)$ is the conjugacy class of x):

$$|cl(x)| = \frac{|S_5|}{|C_G(x)|}$$

We have $|cl(x)| = 4!$, since to conjugate by permutations in S_5 the five cycle $(1\ 2\ 3\ 4\ 5)$, is equivalent to specifying 4 entries where order matters (remember, conjugation conserves cycle type). To see this convince yourself you can always start the new 5 cycle with 1, as shown in the example below:

$$(2\ 3\ 5\ 1\ 4) \overset{\text{shift to the right}}{=} (4\ 2\ 3\ 5\ 1) \overset{\text{again...}}{=} \boxed{(\underline{1}\ 4\ 2\ 3\ 5)}$$

In particular, we readily see that $|C_G(x)| = 5$, and because $|\langle x \rangle| = o(x) = 5$, we can conclude the centralizer of x is indeed the cyclic group generated by x. $\square$

PROBLEM 3

Consider the action of the symmetric group S_3 on the 9-element set

$$X = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

given by

$$\sigma \cdot (i,j) = (\sigma(i), \sigma(j))$$

for all $\sigma \in S_3$ and all $i, j \in \{1,2,3\}$ (you don't need to check that it's a well-defined action).

(a) Explicitly describe the orbits of the action $S_3 \curvearrowright X$ defined above. *(4 points)*

Solution:

Because σ acts on the two entries of (i,j), we consider some (i,i) and some (j,k), $j \neq k$ (i.e diagonal and off-diagonal) and show that the action acts transitively on all diagonal elements (resp. off-diagonal entries). For simplicity, we pick $(1,1)$ as a representative of the diagonal and $(1,2)$ as a representative of the off-diagonal. Let's find $S_3 \cdot (1,1)$ and $S_3 \cdot (1,2)$:

$$\begin{cases} S_3 \cdot (1,1) = \{\big(\sigma(1), \sigma(1)\big) \mid \sigma \in S_3\} = \{(1,1),(2,2),(3,3)\} \\ S_3 \cdot (1,2) = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\} \end{cases}$$

Now, recall orbits form a partition for the set acted upon, and hence we have found all orbits by the above.

(b) Explicitly describe the stabilizers of elements in the orbits you found in part (a), and fill in the dots below with the correct numbers in the orbit-stabilizer theorem.

$$|X| = |\text{first orbit}| + |\text{second orbit}| = \frac{|S_3|}{\dots} + \frac{|S_3|}{\dots}$$

(it's enough to describe the stabilizer of a single element in any given orbit). *(6 points)*

Solution:

Recall, given $G \curvearrowright X$, with G and X finite:

$$\underbrace{|X| = \sum_{x \text{ orbit rep}} |G \cdot x| \overset{\text{O.S Thm}}{=} \sum_{x \text{ orbit rep}} \frac{|G|}{|\text{Stab}_G(x)|}}_{\text{General class equation}}$$

In particular, what the exercise is asking us to do, is find the stabilizers for each orbit representative. Since we have:

$$\text{Stab}_G((1,2)) = \{e\}$$
$$\text{Stab}_G((1,1)) = \{e, (2\ 3)\}$$

This yields:

$$|X| = |S_3 \cdot (1,1)| + |S_3 \cdot (1,2)| = \frac{|S_3|}{1} + \frac{|S_3|}{2} = 9$$

PROBLEM 4

Consider the dihedral group D_{2n} with $n \geq 3$, and the sequence of homomorphisms

$$1 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{f} D_{2n} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 1 \tag{1}$$

where $f(k \bmod n) =$ (rotation by $\frac{2\pi k}{n}$ radians) for all $k \in \mathbb{Z}$, while g(rotations) $= (0 \bmod 2)$ and g(reflections) $= (1 \bmod 2)$. You may assume that f and g are indeed homomorphisms.

(a) Check that (1) is a short exact sequence (i.e. check all the axioms). *(5 points)*

Solution: To show the above is a short exact sequence, we need to show (note we assume them to be homomorphisms):

- f is injective
- g is surjective
- $\mathrm{Im} f = \ker g$

• f is injective if $f(k \bmod n) = e$, the only option is that $k \in n\mathbb{Z}$ i.e $k \equiv 0 \bmod n$.
• g is surjective since both 0 and 1 modulo 2 are hit by an element of D_{2n}.
• We have this equality since the image of f is all rotations, and the kernel of g is also all rotations, because these are the only ones mapped to 0 by definition.

For parts (b) and (c), do not invoke any statement of the sort "D_{2n} is / isn't a (semi)direct product". Prove the statements using only direct computations and the definition of D_{2n}.

(b) Prove that there <u>exists</u> a homomorphism $\psi : \mathbb{Z}/2\mathbb{Z} \to D_{2n}$ such that $g \circ \psi = \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}$.

(4 points)

Solution:
For this let ψ map 0 to the identity, and 1 to the reflection. Since the reflection has order two in D_{2n}, the resulting map must be a homomorphism. Furthermore, to check this is indeed a splitting:

$$\begin{cases} (g \circ \psi)(0) = 0 \\ (g \circ \psi)(1) = g(\text{reflection}) = 1 \end{cases}$$

□

(c) Prove that there <u>does not exist</u> a homomorphism $\phi : D_{2n} \to \mathbb{Z}/n\mathbb{Z}$ such that $\phi \circ f = \text{Id}_{\mathbb{Z}/n\mathbb{Z}}$ (remember that we assumed $n \geq 3$). *(4 points)*

We offer alternative solutions to this question, while using the more compact notation where τ denotes a reflection and σ a rotation.

$\hookrightarrow$ *Solution 1:*
Suppose that such a homomorphism $\phi : D_{2n} \to \mathbb{Z}/n\mathbb{Z}$ exists. Then $\tau\sigma\tau = \sigma^{-1}$ where $\sigma \neq \sigma^{-1}$ since $n \geq 3$. But since $\mathbb{Z}/n\mathbb{Z}$ is Abelian, we have that $\phi(\sigma^{-1}) = \phi(\tau\sigma\tau) = \phi(\sigma)$. But this is a contradiction since ϕ must map distinct rotations to distinct elements of $\mathbb{Z}/n\mathbb{Z}$ since $\phi \circ f = Id_{\mathbb{Z}/n\mathbb{Z}}$.

$\hookrightarrow$ *Solution 2:*
Let ϕ be such a homomorphism. In particular, it fixes all rotations. So to specify such a morphism comes down to finding where it maps τ. First note:

$$\phi(\tau)^2 = \phi(\tau^2) = e$$

thus $\phi(\tau)$ has order 2 in $\mathbb{Z}/n\mathbb{Z}$. If n is odd, we are done, and if not: $\phi(\tau) = n/2$. In particular, since in D_{2n}, $\tau\sigma\tau\sigma = e$, we can obtain the following contradiction:

$$[0] = \phi(e) = \phi(\tau\sigma\tau\sigma) \overset{\phi \text{ hom.}}{=} [n/2] + [1] + [n/2] + [1] = 2$$

where $n \geq 3$.

$\hookrightarrow$ *Solution 3:*
If such a homomorphism existed, $\ker f$ would have to be of cardinality 2. Indeed, this is a consequence of the fact the ϕ above is surjective ($\phi \circ f = \text{Id}_{\mathbb{Z}/n\mathbb{Z}}$) and of the first isomorphism theorem. So we show there is no normal subgroup with cardinality 2 in D_{2n}.
Any such subgroup would be generated by an element of order two, and those are all of the form k rotation or reflection+k rotation, or $\tau\sigma^i$ and σ^i ; $i \in \{0, ..., n-1\}$. Since:

$$\tau\sigma\tau = \sigma^{-1}$$

we may see conjugating by τ the two elements above that:

$$\tau(\sigma^i)\tau = \sigma^{-i} \notin \{e, \sigma^i\}$$
$$\tau(\tau\sigma^i)\tau = \sigma^i\tau \notin \{e, \tau\sigma^i\}$$

because $n \geq 3$. In particular, no normal subgroup of order 2 exists in D_{2n}. $\square$

PROBLEM 5

(a) Prove that if G, G', H are finite abelian groups such that $G \times H$ is isomorphic to $G' \times H$, then G is isomorphic to G'. *(8 points)*

Solution:
The groups involved G, G', H, $G \times H$ and $G' \times H$ are all finite abelian. Hence by the classification theorem of finitely generated abelian groups, it suffices to show that the elementary divisors of G and G' are equal, since they completely characterize finite abelian groups. Recall that the elementary divisors of G are the unique prime powers $p_1^{d_1}, \ldots, p_k^{d_k}$ such that

$$G \cong \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}.$$

Similarly G' has elementary divisors $p_1'^{c_1}, \ldots, p_r'^{c_r}$, and H has elementary divisors $q_1^{a_1}, \ldots, q_s^{a_s}$. Since $G \times H \cong G' \times H$ are finite and abelian, those isomorphic groups are characterized by their elementary divisors. This means that the sets (counted with multiplicities) of elementary divisors of $G \times H$ and $G' \times H$ are equal, i.e.

$$\{p_1^{d_1}, \ \ldots, p_k^{d_k}, q_1^{a_1}, \ldots, q_s^{a_s}\} = \{p_1'^{c_1}, \ldots, p_r'^{c_r}, q_1^{a_1}, \ldots, q_s^{a_s}\}.$$

But this immediately implies that

$$\{p_1^{d_1}, \ \ldots, p_k^{d_k}\} = \{p_1'^{c_1}, \ldots, p_r'^{c_r}\}.$$

This means that the elementary divisors of G and G' coincide, which implies that $G \cong G'$.$\square$

(b) Find a counterexample to part (a) if we drop the "finite abelian" assumption, i.e. find groups G, G', H such that $G \ncong G'$ but $G \times H \cong G' \times H$. Prove your assertions.

(4 points, hard)

Solution:
We have seen that two free abelian groups are isomorphic if and only if their generating set have the same cardinality (are in bijections). Let G be free abelian on $S = \{*\}$, G' be free abelian on $T = \{*_1, *_2\}$, and H be free abelian on $R = \mathbb{N}$. Then $G \times H \cong G \times H'$ are isomorphic since they are free abelian on $S \cup R \cong T \cup R \cong \mathbb{N}$. However G and G' are not isomorphic since they are free abelian on $S \neq T$. $\square$

PROBLEM 6

(a) If G_1, G_2, H_1, H_2 are simple groups such that there exists an isomorphism

$$G_1 \times G_2 \cong H_1 \times H_2$$

then prove that either

- $G_1 \cong H_1$ and $G_2 \cong H_2$, or
- $G_1 \cong H_2$ and $G_2 \cong H_1$.

(the two options are not exclusive, i.e. they are allowed to hold at the same time) (5 points)

Solution:
To show this, we will use the uniqueness part of the Jordan-Hölder theorem. For this, let's exhibit two distinct composition series (we leave it to you to check these do indeed constitute composition series):

$$1 \triangleleft G_1 \times \{1\} \triangleleft G_1 \times G_2$$

and:

$$1 \triangleleft H_1 \times \{1\} \triangleleft H_1 \times H_2$$

In particular, the simple blocks resulting from the series are $\{G_1 \times G_2 / G_1 \times \{1\},\ G_1 \times \{1\}/1\}$ which is up to isomorphism simply $\{G_2, G_1\}$, and same goes for the composition series of $H_1 \times H_2$: $\{H_2, H_1\}$.

Now, because the blocks are isomorphic up to permutation, we see either $G_1 \cong H_1$ and $G_2 \cong H_2$ or $G_1 \cong H_2$ and $G_2 \cong H_1$ $\square$

(b) Show that the symmetric group S_4 is solvable, by exhibiting subgroups

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = S_4$$

where G_{i-1} is normal inside G_i and the quotient G_i/G_{i-1} is a <u>cyclic</u> group, for all $i \in \{1, \dots, k\}$. The choice of number k is up to you. *(10 points)*

Sketch:
The main hurdle in this exercise is to find a normal subgroup for A_4, which is abelian, since then the tail of the composition series won't be hard to find (recall, you've done this in Sheet 7, exercise 6).

$$1 \triangleleft \text{easy...} \triangleleft \underbrace{?}_{\text{abelian \& finite}} \triangleleft A_4 \triangleleft S_4$$

s.t that "?" is maximal in A_4 (in other words, $A_4/?$ is simple). To get a feel for what the mystery subgroup is, recall the elements of A_4 (don't worry, there are only twelve of them):

$$\{\underbrace{e, (12)(34), (13)(24), (14)(23)}_{=:S}, (123), (132), (234), (243), (134), (143), (124), (142)\}$$

If S formed a subgroup, it would certainly be normal, since conjugation preserves cycle type (important fact to always remember!). And indeed, S is a group with the operation of A_4. We leave it to you to check this.

With this in mind, let's see what you are expected to write on the exam:

Solution:
We will show the following chain of subgroups constitutes a composition series:

$$1 \subset \{e, (12)(34)\} \subset \{e, (12)(34), (13)(24), (14)(23)\} \subset A_4 \subset S_4$$

We have A_4 is maximally normal in S_4, since their quotient are $\mathbb{Z}/2\mathbb{Z}$ (recall, $\ker \text{sign} = A_n$). We have $K := \{e, (12)(34), (13)(24), (14)(23)\}$ is normal in A_4, since it contains all permutations of the form $(ij)(kl)$ and it's a subgroup. Furthermore because it has 4 elements, it's necessarily maximal, because $A_4/K \cong \mathbb{Z}/3\mathbb{Z}$ is simple. Finally, because K is abelian (any group of order 4 is abelian) and $(12)(34)$ has order two, we have the last chain is maximal and normal $\square$

PROBLEM 7

(a) Show that any group of order 56 has a normal Sylow p-subgroup for some prime $p \in \{2, 7\}$.

Partial credit will be given for useful remarks concerning the number of Sylow p-subgroups, or when such a subgroup is normal, which apply to the problem at hand. *(8 points)*

Solution:

First observe that $56 = 2^3 \cdot 7$. Since we know that Sylow p-subgroups of G are conjugate to each other, it suffices to show that $n_p = 1$ for $p = 2$ or 7, where n_p is the number n_p of Sylow p-subgroups. By the third Sylow theorem, we know that $n_2|7$ and $n_2 = 1 \mod 2$, while $n_7|8$ and $n_7 = 1 \mod 7$. This implies that $n_2 \in \{1, 7\}$ and $n_7 \in \{1, 8\}$.

Suppose that $n_7 = 8$. Since Sylow 7-subgroups are of order 7, they are cyclic and every non-trivial element is a generator (of order 7). Moreover this implies that two distinct Sylow 7-subgroups intersect trivially. It follows that there is $8 \cdot 6 = 48$ elements of order 7 . Those elements of order 7 can't be in a Sylow 2-subgroups. Since the order of a Sylow 2-subgroup is 8, and $56 - 48 = 8$, there is only room for a single Sylow 2-subgroup, because for two distinct primes, sylow subgroups intersect trivially (this is a consequence of Lagrange's theorem) $\square$

(b) Find, with proof, which well-known group is isomorphic to a Sylow 2-subgroup of S_5.

(4 points, hard)

Solution:
To do this, we recall that S_5 has $2^3 \cdot 3 \cdot 5$ elements, while S_4 has only $2^3 \cdot 3$ elements. Crucially, these have equal powers of two, meaning given some embedding $\iota : S_4 \hookrightarrow S_5$ of your liking, and P a sylow 2 subgroup of S_4, it's also a Sylow 2 subgroup of S_5! Thus, let's reduce the problem to finding a subgroup of S_4 of order 8. Since S_4 acts on the square by permutation of it's vertices, we see a subgroup of those permutations consist of flipping and rotating the square, in particular $D_{2\cdot 4}$ in S_4. Then, $\iota(D_{2\cdot 4}) \cong D_{2\cdot 4}$ and so, the dihedral group with 8 elements is a sought after Sylow-2 subgroup of S_5. But by Sylow's second theorem, all Sylow-p subgroups are isomorphic (because they are conjugates of each other). Hence:

The dihedral group with 8 elements is the sought after Sylow-2 subgroup of S_5 $\square$

PROBLEM 8

Show that a finite group G is nilpotent if and only if

$$xy = yx$$

for all $x, y \in G$ whose orders are coprime. *You may use the fact that G is nilpotent $\Leftrightarrow$ it is the direct product of its Sylow p-subgroups $\Leftrightarrow$ all the Sylow p-subgroups of G are normal.*

only if implication, i.e. G nilpotent implies "$xy = yx$, $\forall x, y$ of coprime orders". *(6 points)*

Solution:
For this first implication, recall nilpotent groups are isomorphic to a product of Sylow p-groups:

$$G \cong P_1 \times ... \times P_n$$

In particular, if x corresponds to $(x_1, ..., x_n)$ via this isomorphism (resp. y with $(y_1, ..., y_n)$) because their orders are coprime, we must have $x_i \neq e \implies y_i = e$. Indeed, if (without loss of generality) $x_1, y_1 \neq e$, then they would both have order a non trivial power of p, and since:

$$o(x_1, ..., x_n) = \text{lcm}(o(x_1), ..., o(x_n)) \quad (\#)$$

We obtain $p^n = o(x_1) \mid o(x)$ and $p^m = o(y_1) \mid o(y)$, yielding the contradiction (since they wouldn't be coprime anymore). Finally, let's quickly justify (#).Suppose the order strictly divides the lcm. Because the lcm is the product of the maximum of prime powers pick q^l a power of some prime dividing the lcm but not the order of x. Pick the element x_i who's order divides q^l, and note: $x_i^{o(x_1,...,x_n)} \neq e$. Thus the order is at least the lcm of the orders, and this as one can check indeed annihilates x. $\square$

(see next page)

if implication, i.e. "$xy = yx, \forall x, y$ of coprime orders" implies G nilpotent. *(4 points, hard)*

Solution:
For the second implication, let G be a finite group such that $xy = yx$ whenever $x \in G$ and $y \in G$ have co-prime orders. Let $p_1, ..., p_n$ be all the distinct prime divisors of $|G|$ and let P_i denote a Sylow p_i- subgroup of G. We claim that the following map is an injective group homomorphism:

$$\psi : P_1 \times ... \times P_n \to G, \ (a_1, ..., a_n) \mapsto a_1...a_n.$$

Indeed the assumption on commutation of elements with co-prime orders implies that $a_1b_1...a_nb_n = a_1...a_nb_1...b_n$ for all $a_i, b_i \in P_i$ and $i \in \{1, ..., n\}$. Hence $\psi((a_1, ..., a_n) \cdot (b_1, ..., b_n)) = \psi(a_1b_1, ..., a_nb_n)$ which implies that ψ is a group homomorphism.

Now we show that ψ is injective. If $\psi(a_1, ..., a_n) = 1$ then $a_1...a_n = 1$. Since the a_i's commute and have co-prime order, this forces that $a_i = 1$ for all $i \in 1, ..., n$. This is because if some $a_i \neq 0$ then the order of $a_1...a_l$ would be divisible by p_i.

Since ψ is an injective group homomorphism between groups of equal order it is an isomorphism. This shows that G is nilpotent since it is the product of it's Sylow p-subgroups.
$\square$

EXTRA SHEET

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