

Exercise 1.

1. Let $w, \xi > 0$, and let $f, \varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the functions defined by

$$f(x) = x^2 e^{-w^2 x^2}, \quad \varphi(x) = \frac{2w^2 + \xi^2 - 4w^4 x^2}{4w^4} e^{-w^2 x^2}.$$

Show (using Fourier transform tables and standard Fourier transform properties) that

$$\hat{f}(\alpha) = \frac{2w^2 - \alpha^2}{4\sqrt{2}w^5} e^{-\frac{\alpha^2}{4w^2}}, \quad \hat{\varphi}(\alpha) = \frac{\xi^2 + \alpha^2}{4\sqrt{2}w^5} e^{-\frac{\alpha^2}{4w^2}}.$$

2. Find a function $u: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$u(x) + \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} u''(t) e^{-\sqrt{2}|x-t|} dt = x^2 e^{-x^2}.$$

Exercise 2.

We check whether the following are distributions or not.

1. Show that T is a distribution, where

$$T(\phi) = \int_{-1}^1 \phi(x) dx. \quad (1)$$

2. Show that T is a distribution, where

$$T(\phi) = \int_{-\infty}^{\infty} \phi(x) dx. \quad (2)$$

3. Show that S is **not** a distribution, where

$$S(\phi) = \int_0^1 |\phi(x)| dx. \quad (3)$$

Exercise 3.

Find a function $u: \mathbb{R} \rightarrow \mathbb{R}$, 2π -periodic, satisfying

$$u''(x) - 2u(x + \pi) = 3 + \sin\left(\frac{3}{2}x\right) \sin\left(\frac{x}{2}\right).$$

Hint: You may use the identities

$$2 \sin(a) \sin(b) = \cos(a - b) - \cos(a + b),$$
$$\sin(n(x \pm \pi)) = (-1)^n \sin(nx), \quad \cos(n(x \pm \pi)) = (-1)^n \cos(nx).$$

Exercise 4.

Find the distributional derivative of the function

$$f(x) = |x|. \tag{4}$$

Exercise 5.

Solve for $u = u(x, t)$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0, \pi), t > 0 \\ u(0, t) = u(\pi, t) = 0 & t > 0 \\ u(x, 0) = \cos x - \cos(3x) & x \in (0, \pi) \end{cases}$$

Exercise 6.

Find $u = u(x, y)$ satisfying

$$\begin{cases} \Delta u = 0 & x, y \in (0, \pi), \\ \frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, \pi) = 0 & x \in (0, \pi), \\ u(0, y) = \cos(2y), \quad u(\pi, y) = 0 & y \in (0, \pi). \end{cases}$$