

*Aide-memoire: To verify the divergence theorem in  $\mathbb{R}^3$ , proceed as follows:*

1. Sketch the domain  $\Omega$ , then calculate  $\operatorname{div} F(x, y, z)$ .
2. Parameterize the domain  $\Omega$ . Use this parameterization to express

$$\iiint_{\Omega} \operatorname{div} F(x, y, z) \, dx \, dy \, dz$$

*as a triple integral where the bounds and the function to be integrated are explicitly indicated.*

3. Write  $\partial\Omega$  as a union of regular surfaces; for each, give a parameterization and a field of exterior normals. Add this to your sketch.
4. Express

$$\iint_{\partial\Omega} F \cdot \nu \, ds$$

*as a sum of double integrals where the bounds and the functions to be integrated are explicitly indicated.*

5. Check the conclusion of the divergence theorem for  $\Omega$  and  $F$ .

**Exercise 1** (Ex 6.4 page 66).

Verify the divergence theorem for  $F(x, y, z) = (x^2, y^2, z^2)$  and  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : b^2(x^2 + y^2) < a^2z^2 \text{ and } 0 < z < b\}$ .

**Exercise 2** (Ex 6.3 page 66).

Verify the divergence theorem for  $F(x, y, z) = (xy, yz, xz)$  and  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : 0 < z < 1 - x - y, 0 < y < 1 - x, 0 < x < 1\}$ .

**Exercise 3** (Ex 6.6 page 66).

Verify the divergence theorem for  $F(x, y, z) = (x, y, z)$  and  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } x^2 + y^2 < 3z\}$ .

**Exercise 4** (Ex 6.9 page 67).

Verify the divergence theorem for  $F(x, y, z) = (2, 0, xy^2 + z^2)$  and  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x > 0, 0 < z < 2 \text{ and } 4(x^2 + y^2) < (z - 4)^2\}$ .

**Exercise 5** (Ex 6.11 page 67).

Let  $\Omega \subset \mathbb{R}^3$  a regular domain and  $\nu$  a field of unit normals exterior to  $\Omega$ . Let the vector fields  $F$ ,  $G_1$ ,  $G_2$ , and  $G_3$  defined by

$$\begin{aligned} F(x, y, z) &= (x, y, z), & G_1(x, y, z) &= (x, 0, 0), \\ G_2(x, y, z) &= (0, y, 0), & G_3(x, y, z) &= (0, 0, z). \end{aligned}$$

Prove that:

1.  $\text{Volume}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu) \, ds$
2.  $\text{Volume}(\Omega) = \iint_{\partial\Omega} (G_i \cdot \nu) \, ds$  for  $i = 1, 2, 3$ .