

Exercise 1.

Let $\Sigma \subset \mathbb{R}^3$ be a regular orientable surface with a field of unit normal vectors ν . Let $F : \Sigma \mapsto \mathbb{R}^3$ be a continuous vector field. Prove that the flux of F through the surface Σ in the direction ν is equal to the integral of the scalar field $F \cdot \nu$ on Σ .

Exercise 2 (Ex 5.1 page 56).

Let $f(x, y, z) = xy + z^2$ and

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\}.$$

Compute $\iint_{\Sigma} f ds$.

Exercise 3 (Ex 5.3 page 57).

Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0 \text{ and } 0 \leq z \leq 1\}$. Compute the mass of the surface Σ knowing that the density is $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

Exercise 4 (Ex 5.5 page 57).

Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 1\}$. Compute the area of $\partial\Omega$.

Exercise 5 (Ex 5.2 page 57).

Let $F(x, y, z) = (x^2, y^2, z^2)$ and

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } 0 \leq z \leq 1\}.$$

Compute the flux through Σ in the upward direction (i.e., in the direction of $z > 0$).

Exercise 6 (Ex 5.4 page 57).

Let $F(x, y, z) = (0, z, z)$ and

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 6 - 3x - 2y \text{ and } x \geq 0, y \geq 0, z \geq 0\}.$$

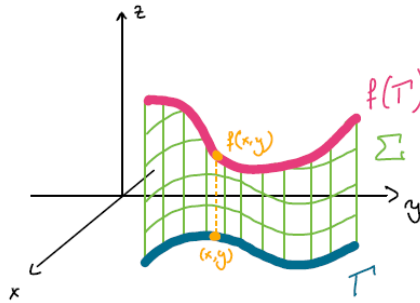
Compute the flux that passes through the surface and away from the origin.

Exercise 7 (Ex 5.7 page 57).

Let $\Omega \subset \mathbb{R}^2$ be an open set and $f : \bar{\Omega} \rightarrow \mathbb{R}$ be a continuous scalar field such that $f(x, y) > 0$ for all $(x, y) \in \bar{\Omega}$. Consider a simple and regular curve $\Gamma \subset \Omega$ and the surface $\Sigma \subset \mathbb{R}^3$ defined by

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Gamma \text{ and } 0 \leq z \leq f(x, y)\}.$$

Prove that $\text{Area}(\Sigma) = \int_{\Gamma} f dl$.



Remark: This result can be interpreted as follows: The line integral of a scalar field in \mathbb{R}^2 gives the area of the surface between Γ and the graph of f .

Exercise 8.

Let $\alpha, \beta > 0$ and $F_{\alpha, \beta} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

$$F_{\alpha, \beta}(x, y, z) = \left(\frac{x}{(y^2 + z^2)^\alpha}, y, \frac{z}{|x|^\beta} \right).$$

Identify for which values of α and β , the following integral is well-defined:

$$\left| \iint_S F_{\alpha, \beta} \cdot dS \right| < +\infty,$$

being S the unit sphere centered at the origin, *i.e.*:

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$