

Remark.

The theorems introduced in this Series were studied in Analysis II.

Definition 1.

Let $\Omega \subset \mathbb{R}^m$ be an open set, $\mathbf{F} = (F_1, \dots, F_n) : \Omega \rightarrow \mathbb{R}^n$, such that $\mathbf{F} \in C^1(\Omega, \mathbb{R}^n)$ and $1 \leq i \leq m$. We define:

$$\frac{\partial \mathbf{F}}{\partial x_i} = \left(\frac{\partial F_1}{\partial x_i}, \dots, \frac{\partial F_n}{\partial x_i} \right).$$

Exercise 1.

Let $\Omega \subset \mathbb{R}^m$ be an open set, $\mathbf{F}, \mathbf{G} \in C^1(\Omega, \mathbb{R}^n)$ and $1 \leq i \leq m$. Show that:

1.

$$\frac{\partial}{\partial x_i} [\langle \mathbf{F}, \mathbf{G} \rangle] = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}, \mathbf{G} \right\rangle + \left\langle \mathbf{F}, \frac{\partial \mathbf{G}}{\partial x_i} \right\rangle,$$

where for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the dot product of these two Euclidean vectors \mathbf{a} and \mathbf{b} , and is defined by:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n a_i b_i,$$

2. In three-dimensional space ($n = 3$),

$$\frac{\partial}{\partial x_i} [\mathbf{F} \wedge \mathbf{G}] = \frac{\partial \mathbf{F}}{\partial x_i} \wedge \mathbf{G} + \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x_i},$$

where for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{a} \wedge \mathbf{b} \in \mathbb{R}^3$ denotes the cross product of \mathbf{a} and \mathbf{b} , and is defined by:

$$\mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

Definition 2 (Jacobian matrix and determinant).

Let $\mathbf{u} = (u_1, \dots, u_n) : \Omega \subset \mathbb{R}^n \rightarrow \Omega' \subset \mathbb{R}^n$, $\mathbf{u} = \mathbf{u}(\mathbf{x})$ such that:

- $\mathbf{u} \in C^\infty(\Omega; \Omega')$,

- \mathbf{u} is invertible and $\mathbf{u}^{-1} \in C^\infty(\Omega'; \Omega)$.

The Jacobian matrix of the vector-valued function \mathbf{u} , represented by $\nabla \mathbf{u}$, is a square matrix whose entries are:

$$(\nabla \mathbf{u})_{i,j} = \frac{\partial u_i}{\partial x_j}.$$

The Jacobian determinant, or simply the Jacobian, denoted by $\text{Jac } \mathbf{u}(\mathbf{x})$ is defined as:

$$\text{Jac } \mathbf{u}(\mathbf{x}) = \det \nabla \mathbf{u}(\mathbf{x}).$$

Exercise 2 (§8.5 pages 119-121).

Compute the Jacobian for the following mapping functions:

1. Polar coordinates:

$$\mathbf{u}(r, \theta) = (r \cos \theta, r \sin \theta),$$

2. Spherical coordinates:

$$\mathbf{u}(r, \theta, \varphi) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi),$$

3. Cylindrical coordinates:

$$\mathbf{u}(r, \theta, z) = (r \cos \theta, r \sin \theta, z),$$

4. Cartesian coordinates:

$$\mathbf{u}(x, y, z) = (x, y, z).$$

Theorem 3 (§8.5 page 119).

Let $f : \Omega' \rightarrow \mathbb{R}$ be a continuous function, $\mathbf{u} : \Omega \rightarrow \Omega'$ be a function as the one involved in Definition 2, and $A \subset \Omega'$ be a closed and bounded set. Then:

$$\int_A f(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{u}^{-1}(A)} f(\mathbf{u}(\tilde{\mathbf{x}})) |\text{Jac } \mathbf{u}(\tilde{\mathbf{x}})| d\tilde{\mathbf{x}}.$$

Exercise 3.

Sketch the set A , and compute the integral $\int_A f(\mathbf{x}) d\mathbf{x}$ in the following cases:

1. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$, and $f(x, y) = (x^2 + y^2)^{-\frac{1}{2}}$,
2. $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, x - 3 \leq y \leq 3 - x\}$,
and $f(x, y) = x^2 + \sin^3(y)$,
3. $A = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2}\}$,
and $f(x, y) = y(1 + x^2)$,
4. $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 9\}$, and $f(x, y, z) = \sqrt{x^2 + y^2}$,
5. $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4\}$,
and $f(x, y, z) = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$.

Theorem 4.

Let $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$, $f \in C^1(\Omega')$ and $g = (g_1, \dots, g_m) \in C^1(\Omega, \mathbb{R}^m)$ such that $g(\Omega) \subset \Omega'$. Then, the chain rule expresses the derivatives of the composition function $f \circ g \in C^1(\Omega)$ as:

$$\frac{\partial f \circ g}{\partial x_i}(x) = \sum_{j=1}^m \frac{\partial f}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x), \quad \text{for } i = 1 \dots n.$$

Exercise 4.

1. Let $p \geq 1$ and the function:

$$h_p(x) := |x|^p = \left(\sqrt{x_1^2 + \dots + x_n^2} \right)^p, \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}.$$

Compute $\nabla h_p(x)$.

2. Let $\mathbf{G} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be a map that never reaches zero, and f_p the function defined by:

$$f_p(t) := \frac{1}{p} |\mathbf{G}(tx)|^p, \quad \forall t \in \mathbb{R}.$$

Compute $\frac{d}{dt} f_p(t)$.