EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 13	December 11 2025

Exercise 1.
The condition $y(x+2\pi) = y(x)$ tells us that we are looking for a 2π-periodic solution, so we seek it in the form of a Fourier series:

$$\begin{aligned} y(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \\ y'(x) &= \sum_{n=1}^{\infty} (nb_n \cos(nx) - na_n \sin(nx)) \\ y''(x) &= \sum_{n=1}^{\infty} \left(-n^2 a_n \cos(nx) - n^2 b_n \sin(nx)\right) \\ y'(x-\pi) &= \sum_{n=1}^{\infty} (nb_n \cos(n(x-\pi)) - na_n \sin(n(x-\pi))) \\ &= \sum_{n=1}^{\infty} ((-1)^n nb_n \cos(nx) - (-1)^n na_n \sin(nx)) \end{aligned}$$

Therefore,

$$\begin{aligned} y''(x)&+5y'(x-\pi) - y(x) \\ &= -\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(-a_n + 5(-1)^n nb_n - n^2 a_n\right)\cos(nx) + \left(-b_n - 5(-1)^n na_n - n^2 b_n\right)\sin(nx)\right]. \end{aligned}$$

Now, the right-hand side of the equation is given as a Fourier series (in fact a finite sum), and by equating coefficients term by term, we obtain the systems:

$$\boxed{n=0} \quad -\frac{a_0}{2} = 2$$

$$\boxed{n=1} \quad \begin{cases} -a_1 - 5b_1 - a_1 = 1 \\ -b_1 + 5a_1 - b_1 = 0 \end{cases}$$

$$\boxed{n=2} \quad \begin{cases} -a_2 + 10b_2 - 4a_2 = 0 \\ -b_2 - 10a_2 - 4b_2 = -3 \end{cases}$$

$$\boxed{n \geq 3} \quad \begin{cases} -a_n + 5(-1)^n nb_n - n^2 a_n = 0 \\ -b_n - 5(-1)^n na_n - n^2 b_n = 0 \end{cases}$$

whose solutions are

$$\begin{aligned} a_0 =& -4 \\ a_1 =& -\frac{2}{29} \\ b_1 =& -\frac{5}{29} \\ a_2 =& \frac{6}{25} \\ b_2 =& \frac{3}{25} \\ a_n =& 0 \qquad \forall n \geq 3 \\ b_n =& 0 \qquad \forall n \geq 3. \end{aligned}$$

Thus,

$$y(x) = -2 - \frac{2}{29}\cos(x) - \frac{5}{29}\sin(x) + \frac{6}{25}\cos(2x) + \frac{3}{25}\sin(2x)$$

Exercise 2.
Applying the Fourier transform inversion theorem for the function f defined by $f(x) = xe^{-\omega|x|}$ (i.e., it is continuous) we obtain:

$$\begin{aligned} xe^{-\omega|x|} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathfrak{F}(f)(\alpha) e^{i\alpha x} \mathrm{d}\alpha = \frac{4\omega}{2\pi i} \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} e^{i\alpha x} \mathrm{d}\alpha \\ &= \frac{2\omega}{\pi i} \Big[\int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \cos(\alpha x) \mathrm{d}\alpha + i \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha \Big]. \end{aligned}$$

As $f(x) = xe^{-\omega|x|} \in \mathbb{R}$, we get:

$$\int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \cos(\alpha x) \mathrm{d}\alpha = 0,$$

$$\int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha = \frac{\pi}{2\omega} xe^{-\omega|x|}.$$

The function g defined by $g(\alpha) = \dfrac{\alpha}{(\alpha^2+\omega^2)^2}\sin(\alpha x)$ is even, then we obtain:

$$\int_{0}^{+\infty} \frac{\alpha}{(\alpha^2+\omega^2)^2} \sin(\alpha x) \mathrm{d}\alpha = \frac{\pi}{4\omega} xe^{-\omega|x|}.$$

Evaluating it at $\alpha = t$ and chosing $\omega = 2$ and $x = \dfrac{1}{2}$ we obtain:

$$\int_{0}^{+\infty} \frac{t}{(t^2+4)^2} \sin\left(\frac{t}{2}\right) \mathrm{d}t = \frac{\pi}{16e}.$$

Exercise 3.
Using the convolution product notation, our equation is written as

$$y + 3y * f = f$$

Taking the Fourier transform:

$$\begin{aligned} \hat{y} + 3\widehat{y * f} &= \hat{f} \\ \hat{y} + 3\sqrt{2\pi}\, \hat{y}\, \hat{f} &= \hat{f} \\ \hat{y}\left(1 + 3\sqrt{2\pi}\, \hat{f}\right) &= \hat{f} \end{aligned}$$

Using the hint with $\omega = 1$, we find the Fourier transform of f:

$$\begin{aligned} \hat{y} &= \frac{\hat{f}}{1 + 3\sqrt{2\pi}\, \hat{f}} \\ &= \frac{\sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2+1}}{1 + 3\sqrt{2\pi}\sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2+1}} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{1 + \alpha^2 + 6} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + 7} \end{aligned}$$

Thus, using the hint with $\omega = \sqrt{7}$, we obtain $y(x) = \frac{1}{\sqrt{7}} e^{-\sqrt{7}|x|}$.

Exercise 4.
Writing the equation using the convolution product, we get

$$3y + (y'' - y) * f = g$$

Taking the Fourier transform, we arrive at

$$\begin{aligned} 3\hat{y} + \widehat{(y'' - y) * f} &= \hat{g} \\ 3\hat{y} + \sqrt{2\pi}\, \widehat{(y'' - y)}\, \hat{f} &= \hat{g} \\ 3\hat{y} + \sqrt{2\pi}\big((i\alpha)^2\hat{y} - \hat{y}\big)\sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + 1} &= \hat{g} \\ 3\hat{y} - 2(\alpha^2 + 1)\hat{y}\frac{1}{\alpha^2 + 1} &= \hat{g} \\ 3\hat{y} - 2\hat{y} &= \hat{g} \\ \hat{y} &= \hat{g} \end{aligned}$$

and therefore $y(x) = g(x) = xe^{-x^2}$