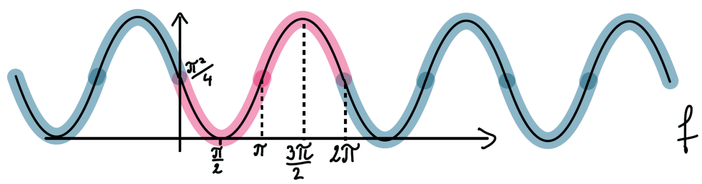
EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 12	December 4 2025

Exercise 1.
We begin by computing the Fourier series of f.



$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\
=& \frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)^2 dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{\pi^2}{2} - \left(x-\frac{3\pi}{2}\right)^2 dx \\
=& \frac{1}{\pi}\left[\frac{1}{3}\left(x-\frac{\pi}{2}\right)^3\right]_{x=0}^{x=\pi} + \frac{\pi^2}{2} - \frac{1}{\pi}\left[\frac{1}{3}\left(x-\frac{3\pi}{2}\right)^3\right]_{x=\pi}^{x=2\pi} \\
=& \frac{1}{3\pi}\left(\frac{\pi^3}{8} - \frac{-\pi^3}{8}\right) + \frac{\pi^2}{2} - \frac{1}{3\pi}\left(\frac{\pi^3}{8} - \frac{-\pi^3}{8}\right) \\
=& \frac{\pi^2}{2}.
\end{aligned}$$

and for $n \geq 1$,

Variant 1: with a small clever change of variable.

$$\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)^2 \cos(nx)dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(\frac{\pi^2}{2} - \left(x-\frac{3\pi}{2}\right)^2\right)\cos(nx)dx \\
\overset{y=x-\pi}{=}& \frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)^2 \cos(nx)dx + \frac{\pi}{2}\int_{\pi}^{2\pi}\cos(nx)dx - \frac{1}{\pi}\int_0^{\pi}\left(y-\frac{\pi}{2}\right)^2 \underbrace{\cos(ny-n\pi)}_{=(-1)^n\cos(ny)} dy \\
=& \frac{1-(-1)^n}{\pi}\underbrace{\int_0^{\pi}\left(x-\frac{\pi}{2}\right)^2\cos(nx)dx}_{:=I_n} + \underbrace{\frac{\pi}{2}\left[\frac{\sin(nx)}{n}\right]_{x=\pi}^{x=2\pi}}_{=0}
\end{aligned}$$

Compute I_n using IBP (IPP)

$$I_n \overset{\text{IPP}}{=} \left[\left(x-\frac{\pi}{2}\right)^2 \frac{1}{n}\sin(nx)\right]_{x=0}^{x=\pi} - \frac{2}{n}\int_0^\pi \left(x-\frac{\pi}{2}\right)\sin(nx)dx \quad \left|\begin{array}{ll} u=\left(x-\frac{\pi}{2}\right)^2 & v=\frac{1}{n}\sin(nx) \\ u'=2\left(x-\frac{\pi}{2}\right) & v'=\cos(nx)\end{array}\right.$$

$$= -\frac{2}{n}\int_0^\pi \left(x-\frac{\pi}{2}\right)\sin(nx)dx$$

$$\overset{\text{IPP}}{=} -\frac{2}{n}\left[\left(x-\frac{\pi}{2}\right)\frac{-1}{n}\cos(nx)\right]_{x=0}^{x=\pi} - \frac{2}{n^2}\int_0^\pi \cos(nx)dx \quad \left|\begin{array}{ll} u=\left(x-\frac{\pi}{2}\right) & v=\frac{-1}{n}\cos(nx) \\ u'=1 & v'=\sin(nx)\end{array}\right.$$

$$= \frac{\pi}{n^2}\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{-\pi}{n^2} - \frac{2}{n^2}\underbrace{\left[\frac{1}{n}\sin(nx)\right]_{x=0}^{x=\pi}}_{=0}$$

$$= \frac{\pi}{n^2}\left(1+(-1)^n\right)$$

To finish,

$$a_n = \frac{\left(1-(-1)^n\right)\left(1+(-1)^n\right)}{n^2} = \frac{1-(-1)^{2n}}{n^2} = 0$$

Same trick for b_n:

$$b_n = \frac{1}{\pi}\int_0^\pi \left(x-\frac{\pi}{2}\right)^2 \sin(nx)dx + \frac{1}{\pi}\int_\pi^{2\pi}\left(\frac{\pi^2}{2} - \left(x-\frac{3\pi}{2}\right)^2\right)\sin(nx)dx$$

$$\overset{y=x-\pi}{=} \frac{1}{\pi}\int_0^\pi \left(x-\frac{\pi}{2}\right)^2 \sin(nx)dx + \frac{\pi}{2}\int_\pi^{2\pi}\sin(nx)dx - \frac{1}{\pi}\int_0^\pi \left(y-\frac{\pi}{2}\right)^2 \underbrace{\sin(ny-n\pi)}_{=(-1)^n\sin(ny)}dy$$

$$= \frac{1-(-1)^n}{\pi}\underbrace{\int_0^\pi \left(x-\frac{\pi}{2}\right)^2 \sin(nx)dx}_{:=J_n} + \frac{\pi}{2}\left[\frac{-\cos(nx)}{n}\right]_{x=\pi}^{x=2\pi}$$

$$= \frac{1-(-1)^n}{\pi}J_n + \frac{\pi\left((-1)^n-1\right)}{2n} = \left(1-(-1)^n\right)\left(\frac{1}{\pi}J_n - \frac{\pi}{2n}\right)$$

Finally compute J_n with IBP (IPP)

$$
\begin{aligned}
J_n &= \int_0^\pi \left(x-\frac{\pi}{2}\right)^2 \sin(nx)dx \\
&\overset{\text{IPP}}{=} \left[\left(x-\frac{\pi}{2}\right)^2 \frac{-1}{n}\cos(nx)\right]_{x=0}^{x=\pi} + \frac{2}{n}\int_0^\pi \left(x-\frac{\pi}{2}\right)\cos(nx)dx \qquad \left|\begin{array}{ll} u=\left(x-\frac{\pi}{2}\right)^2 & v=\frac{-1}{n}\cos(nx) \\ u'=2\left(x-\frac{\pi}{2}\right) & v'=\sin(nx)\end{array}\right. \\
&= \frac{\pi^2}{4n}\left(1-(-1)^n\right) + \frac{2}{n}\int_0^\pi \left(x-\frac{\pi}{2}\right)\cos(nx)dx \\
&\overset{\text{IPP}}{=} \frac{\pi^2}{4n}\left(1-(-1)^n\right) \\
&\quad + \frac{2}{n}\underbrace{\left[\left(x-\frac{\pi}{2}\right)\frac{1}{n}\sin(nx)\right]_{x=0}^{x=\pi}}_{=0} - \frac{2}{n^2}\int_0^\pi \sin(nx)dx \qquad \left|\begin{array}{ll} u=\left(x-\frac{\pi}{2}\right) & v=\frac{1}{n}\sin(nx) \\ u'=1 & v'=\cos(nx)\end{array}\right. \\
&= \frac{\pi^2}{4n}\left(1-(-1)^n\right) + \frac{2}{n^3}\left[\cos(nx)\right]_{x=0}^{x=\pi} \\
&= \frac{\pi^2}{4n}\left(1-(-1)^n\right) + \frac{2\left((-1)^n-1\right)}{n^3} \\
&= \left(1-(-1)^n\right)\left(\frac{\pi^2}{4n}-\frac{2}{n^3}\right)
\end{aligned}
$$

To conclude,

$$
\begin{aligned}
b_n &= \left(1-(-1)^n\right)\left(\left(1-(-1)^n\right)\left(\frac{\pi}{4n}-\frac{2}{\pi n^3}\right)-\frac{\pi}{2n}\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ 2\left(2\left(\frac{\pi}{4(2k-1)}-\frac{2}{\pi(2k-1)^3}\right)-\frac{\pi}{2(2k-1)}\right) & \text{if } n=2k-1 \text{ with } k\in\mathbb{N}^* \end{cases} \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-8}{\pi(2k-1)^3} & \text{if } n=2k-1 \text{ with } k\in\mathbb{N}^* \end{cases}
\end{aligned}
$$

Variant 2: Brute force.

We start by computing some antiderivatives. For $\alpha \neq 0$,

$$\int^x t\cos(\alpha t)dt \overset{\text{IPP}}{=} \frac{1}{\alpha}x\sin(\alpha x) - \frac{1}{\alpha}\int^x \sin(\alpha t)dt \qquad \left| \begin{array}{ll} u = t & v = \frac{1}{\alpha}\sin\alpha t \\ u' = 1 & v' = \cos\alpha t \end{array} \right.$$

$$= \frac{1}{\alpha}x\sin(\alpha x) + \frac{1}{\alpha^2}\cos(\alpha x)$$

$$\int^x t\sin(\alpha t)dt \overset{\text{IPP}}{=} -\frac{1}{\alpha}x\cos(\alpha x) + \frac{1}{\alpha}\int^x \cos(\alpha t)dt \qquad \left| \begin{array}{ll} u = t & v = -\frac{1}{\alpha}\cos(\alpha t) \\ u' = 1 & v' = \sin(\alpha t) \end{array} \right.$$

$$= -\frac{1}{\alpha}x\cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x)$$

$$\int^x t^2\cos(\alpha t)dt \overset{\text{IPP}}{=} \frac{1}{\alpha}x^2\sin(\alpha x) - \frac{2}{\alpha}\int^x t\sin(\alpha t)dt \qquad \left| \begin{array}{ll} u = t^2 & v = \frac{1}{\alpha}\sin(\alpha t) \\ u' = 2t & v' = \cos(\alpha t) \end{array} \right.$$

$$= \frac{x^2}{\alpha}\sin(\alpha x) + \frac{2x}{\alpha^2}\cos(\alpha x) - \frac{2}{\alpha^3}\sin(\alpha x)$$

$$\int^x t^2\sin(\alpha x)dt \overset{\text{IPP}}{=} -\frac{1}{\alpha}x^2\cos(\alpha t) + \frac{2}{\alpha}\int^x t\cos(\alpha t)dt \qquad \left| \begin{array}{ll} u?t^2 & v = -\frac{1}{\alpha}\cos(\alpha t) \\ u' = 2t & v' = \sin(\alpha t) \end{array} \right.$$

$$= -\frac{x^2}{\alpha}\cos(\alpha x) + \frac{2x}{\alpha^2}\sin(\alpha x) + \frac{2}{\alpha^3}\cos(\alpha x)$$

Thus

$$\int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \cos(nx)dx = \int_0^\pi x^2\cos(nx) - \pi\int_0^\pi x\cos(nx)dx + \frac{\pi^2}{4}\int_0^\pi \cos(nx)dx$$

$$= \left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_{x=0}^{x=\pi}$$

$$- \pi\left[\frac{1}{n}x\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{x=0}^{x=\pi}$$

$$+ \frac{\pi^2}{4}\left[\frac{1}{n^2}\sin(nx)\right]_{x=0}^{x=\pi}$$

$$= \frac{2\pi}{n^2}\cos(n\pi) - \pi\frac{1}{n^2}(\cos(n\pi) - 1)$$

$$= \frac{2\pi}{n^2}(-1)^n - \frac{\pi}{n^2}(-1)^n + \frac{\pi}{n^2}$$

$$= \frac{\pi}{n^2}((-1)^n + 1)$$

and

$$\int_{\pi}^{2\pi} \left(\frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2\right)\cos(nx)dx = -\int_{\pi}^{2\pi} x^2\cos(nx) + 3\pi\int_{\pi}^{2\pi} x\cos(nx)dx - \frac{7\pi^2}{4}\int_{\pi}^{2\pi}\cos(nx)dx$$
$$= -\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_{x=\pi}^{x=2\pi}$$
$$+ 3\pi\left[\frac{1}{n}x\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{x=\pi}^{x=2\pi}$$
$$- \frac{7\pi^2}{4}\left[\frac{1}{n^2}\sin(nx)\right]_{x=\pi}^{2\pi}$$
$$= -\frac{2\pi}{n^2}(2 - \cos(n\pi)) + \frac{3\pi}{n^2}(1 - \cos(n\pi))$$
$$= -\frac{\pi}{n^2} - \frac{\pi}{n^2}(-1)^n$$
$$= -\frac{\pi}{n^2}(1 + (-1)^n)$$

And so,

$$a_n = \frac{1}{\pi}\int_0^{\pi}\left(x - \frac{\pi}{2}\right)^2\cos(nx)dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(\frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2\right)\cos(nx)dx$$
$$= \frac{1}{n^2}(1 + (-1)^n) - \frac{1}{n^2}(1 + (-1)^n) = 0$$

Next,

$$\int_0^{\pi}\left(x - \frac{\pi}{2}\right)^2\sin(nx)dx = \int_0^{\pi} x^2\sin(nx) - \pi\int_0^{\pi} x\sin(nx)dx + \frac{\pi^2}{4}\int_0^{\pi}\sin(nx)dx$$
$$= \left[-\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_{x=0}^{x=\pi}$$
$$- \pi\left[-\frac{1}{n}x\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{x=0}^{x=\pi}$$
$$+ \frac{\pi^2}{4}\left[-\frac{1}{n}\cos(nx)\right]_{x=0}^{x=\pi}$$
$$= -\frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n^3}(\cos(n\pi) - 1) + \frac{\pi^2}{n}\cos(n\pi) - \frac{\pi^2}{4n}(\cos(n\pi) - 1)$$
$$= \frac{2}{n^3}((-1)^n - 1) + \frac{\pi^2}{4n}(1 - (-1)^n)$$

and

$$
\begin{aligned}
\int_{\pi}^{2\pi}\left(\frac{\pi^2}{2}-\left(x-\frac{3\pi}{2}\right)^2\right)\sin(nx)dx =& -\int_{\pi}^{2\pi} x^2\sin(nx)+3\pi\int_{\pi}^{2\pi} x\sin(nx)dx-\frac{7\pi^2}{4}\int_{\pi}^{2\pi}\sin(nx)dx\\
=& -\left[-\frac{x^2}{n}\cos(nx)+\frac{2x}{n^2}\sin(nx)+\frac{2}{n^3}\cos(nx)\right]_{x=\pi}^{x=2\pi}\\
&+3\pi\left[-\frac{1}{n}x\cos(nx)+\frac{1}{n^2}\sin(nx)\right]_{x=\pi}^{x=2\pi}\\
&-\frac{7\pi^2}{4}\left[-\frac{1}{n}\cos(nx)\right]_{x=\pi}^{x=2\pi}\\
=&\frac{\pi^2}{n}\left(4-\cos(n\pi)\right)-\frac{2}{n^3}\left(1-\cos(n\pi)\right)-\frac{3\pi^2}{n}\left(2-\cos(n\pi)\right)\\
&+\frac{7\pi^2}{4n}\left(1-\cos(n\pi)\right)\\
=&\frac{\pi^2}{n}\left(\frac{7}{4}+4-6\right)+\frac{\pi^2}{n}(-1)^n\left(-1+3-\frac{7}{4}\right)+\frac{2}{n^3}\left((-1)^n-1\right)\\
=&\frac{\pi^2}{4n}\left((-1)^n-1\right)+\frac{2}{n^3}\left((-1)^n-1\right)
\end{aligned}
$$

And therefore,

$$
\begin{aligned}
b_n =&\frac{1}{\pi}\int_0^{\pi}\left(x-\frac{\pi}{2}\right)^2\sin(nx)dx+\frac{1}{\pi}\int_{\pi}^{2\pi}\left(\frac{\pi^2}{2}-\left(x-\frac{3\pi}{2}\right)^2\right)\sin(nx)dx\\
=&\frac{4}{\pi n^3}\left((-1)^n-1\right)\\
=&\begin{cases} 0 & \text{if } n \text{ is even}\\ \frac{-8}{\pi(2k-1)^3} & \text{if } n=2k-1 \text{ with } k\in\mathbb{N}^* \end{cases}
\end{aligned}
$$

Now that we know the Fourier coefficients of f, we look for a solution of the differential equation in the form of a Fourier series:

$$
\begin{aligned}
y(x) =&\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(A_n\cos(nx)+B_n\sin(nx)\right)\\
y'(x) =&\sum_{n=1}^{\infty}\left(nB_n\cos(nx)-nA_n\sin(nx)\right)\\
y'(x)+y(x) =&\frac{A_0}{2}+\sum_{n=1}^{\infty}\left\{(A_n+nB_n)\cos(nx)+(B_n-nA_n)\sin(nx)\right\}\\
=&f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)
\end{aligned}
$$

We thus obtain the family of systems

$$\boxed{n=0} \quad A_0 = a_0 = \frac{\pi^2}{2}$$

$$\boxed{n\geq 1} \quad \begin{cases} A_n + nB_n = a_n = 0 \\ B_n - nA_n = b_n \end{cases} \quad \Rightarrow \quad \begin{cases} A_n = \dfrac{-n}{n^2+1} b_n \\ \\ B_n = \dfrac{1}{n^2+1} b_n \end{cases}$$

and therefore

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \\ \frac{8}{\pi(2k-1)^2((2k-1)^2+1)} & \text{if } n = 2k-1 \text{ with } k \in \mathbb{N}^* \end{cases}$$

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \\ \frac{-8}{\pi(2k-1)^3((2k-1)^2+1)} & \text{if } n = 2k-1 \text{ with } k \in \mathbb{N}^* \end{cases}$$

To finish,

$$y(x) = \frac{\pi^2}{4} + \sum_{k=1}^{\infty} \left(\frac{8\cos((2k-1)x)}{\pi(2k-1)^2\left((2k-1)^2+1\right)} - \frac{8\sin((2k-1)x)}{\pi(2k-1)^3\left((2k-1)^2+1\right)} \right)$$

Remark: The given equation is a first-order linear differential equation with constant coefficients. In Analysis II we saw that, for example by the integrating factor method, one can find the general solution as follows:

$$\begin{aligned} & y'(x) + y(x) = f(x) \\ & y'(x)e^x + y(x)e^x = f(x)e^x \\ & \frac{d}{dx}\left[y(x)e^x\right] = f(x)e^x \\ & y(x)e^x = \int_0^x f(t)e^t dt + C \\ & y(x) = e^{-x}\int_0^x f(t)e^t dt + Ce^{-x} \end{aligned}$$

To answer the exercise question, one must determine whether there exists a constant C such that the y given above is 2π-periodic, knowing that f is.

It turns out this is possible!

We have

$$
\begin{aligned}
y(x+2\pi) =& e^{-x-2\pi}\int_0^{x+2\pi} f(t)e^t dt + Ce^{-x-2\pi} \\
=& e^{-x-2\pi}\left(\int_0^{2\pi} f(t)e^t dt + \int_{2\pi}^{x+2\pi} f(t)e^t dt\right) + Ce^{-x-2\pi} \\
\overset{t=s+2\pi}{=}& e^{-x-2\pi}\left(\int_0^{2\pi} f(t)e^t dt + \int_0^{x} f(s+2\pi)e^{s+2\pi} ds\right) + Ce^{-x-2\pi} \\
=& e^{-x-2\pi}\left(\int_0^{2\pi} f(t)e^t dt + e^{2\pi}\int_0^{x} f(s)e^{s} ds\right) + Ce^{-x-2\pi} \\
=& e^{-x}\int_0^x f(s)e^s ds + Ce^{-x} \\
& - Ce^{-x} + e^{-x-2\pi}\int_0^{2\pi} f(t)e^t dt + Ce^{-x-2\pi} \\
=& y(x) + e^{-x}\left(C\left(e^{-2\pi}-1\right) + e^{-2\pi}\int_0^{2\pi} f(t)e^t dt\right)
\end{aligned}
$$

Thus choosing

$$
C = \frac{-e^{-2\pi}}{e^{-2\pi}-1}\int_0^{2\pi} f(t)e^t dt = \frac{1}{e^{2\pi}-1}\int_0^{2\pi} f(t)e^t dt
$$

which is independent of x, we obtain that y is periodic.

Note further that the work done with Fourier series is the analogue of the method of undetermined coefficients[1] with a parameter n running around.

Exercise 2.

1. Since f is even, we have $b_n = 0$ for all n. Moreover,

$$
a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[-\frac{1}{3}\cos(3x)\right]_{x=0}^{x=\pi} = \frac{4}{3\pi}.
$$

For $n \geq 1$,

$$
a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(3x)\cos(nx)\,dx.
$$

[1] Sometimes called the "I guess the shape of the solution" method. See `https://sma.epfl.ch/~struett/analyse2/main_proofs_at_the_end.pdf#theorem.1.26`

Using Euler's formulas, we obtain

$$\sin(3x)\cos(nx) = \frac{e^{i3x} - e^{-i3x}}{2i}\,\frac{e^{inx} + e^{-inx}}{2} = \frac{1}{2}\big(\sin((n+3)x) - \sin((n-3)x)\big).$$

Thus, for $n = 3$,

$$a_3 = \frac{1}{\pi}\int_0^\pi \sin(6x)\,dx = \frac{1}{\pi}\left[-\frac{1}{6}\cos(6x)\right]_0^\pi = \frac{1}{\pi}\left(-\frac{1}{6} + \frac{1}{6}\right) = 0.$$

For $n \neq 3$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[-\frac{1}{n+3}\cos((n+3)x) + \frac{1}{n-3}\cos((n-3)x)\right]_0^\pi \\ &= \frac{1}{\pi}\left(-\frac{1}{n+3}(-1)^{n+3} + \frac{1}{n-3}(-1)^{n-3} + \frac{1}{n+3} - \frac{1}{n-3}\right) \\ &= \frac{1}{\pi}\left(\frac{1+(-1)^n}{n+3} - \frac{1+(-1)^n}{n-3}\right) \\ &= \frac{1+(-1)^n}{\pi}\,\frac{n-3-n-3}{n^2-9} = -\frac{6}{\pi}\,\frac{1+(-1)^n}{n^2-9}. \end{aligned}$$

Hence,

$$a_n = \begin{cases} -\dfrac{12}{\pi(4k^2-9)} & \text{if } n = 2k,\ k \geq 1, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The Fourier series of f is therefore

$$Ff(x) = \frac{2}{3\pi} - \frac{12}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-9}.$$

2. Since we are looking for an even solution, we seek it in the form

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(nx).$$

We have

$$y(x-\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(nx - n\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(-1)^n A_n\cos(nx),$$

and thus,

$$y(x) - 2y(x-\pi) = -\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\big(1 - 2(-1)^n\big)\cos(nx).$$

The coefficients A_n must therefore satisfy:

$$\boxed{n=0} \qquad -\frac{A_0}{2} = -\frac{2}{3\pi},$$

$$\boxed{n=2k} \qquad -A_n = -\frac{12}{\pi(4k^2-9)},$$

$$\boxed{n=2k+1} \qquad 3A_n = 0.$$

Finally,

$$y(x) = \frac{-2}{3\pi} + \frac{12}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-9}.$$

Exercise 3.
We look for a solution of the form

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right).$$

We then compute

$$\begin{aligned} y(x-\pi) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx-n\pi) + b_n\sin(nx-n\pi)\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n(-1)^n\cos(nx) + b_n(-1)^n\sin(nx)\right), \end{aligned}$$

and therefore

$$\begin{aligned} y(x) + 2y(x-\pi) &= \frac{3}{2}a_0 + \sum_{n=1}^{\infty}\left\{(1+2(-1)^n)\,a_n\cos(nx) + (1+2(-1)^n)\,b_n\sin(nx)\right\} \\ &= \cos(x) + 3\sin(2x) + 4\cos(5x). \end{aligned}$$

Moreover, if A_n and B_n denote the Fourier coefficients of $\cos(x) + 3\sin(2x) + 4\cos(5x)$, then $A_1 = 1$, $A_5 = 4$, $B_2 = 3$, and all other coefficients are zero.

Thus, we obtain the systems

$$\boxed{n=0} \qquad \frac{3}{2}a_0 = 0,$$

$$\boxed{n=1} \qquad \begin{cases} -1\,a_1 = 1, \\ -1\,b_1 = 0, \end{cases}$$

$$\boxed{n=2} \qquad \begin{cases} 3a_2 = 0, \\ 3b_2 = 3, \end{cases}$$

$$\boxed{n=5} \qquad \begin{cases} -1\,a_5 = 4, \\ -1\,b_5 = 0, \end{cases}$$

$$\boxed{n \in \mathbb{N} \setminus \{0,1,2,5\}} \qquad \begin{cases} (1+(-1)^n)a_n = 0, \\ (1+(-1)^n)b_n = 0. \end{cases}$$

Therefore, we conclude that

$$y(x) = -\cos(x) + \sin(2x) - 4\cos(5x).$$

Exercise 4.
We have

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x} e^{-i\alpha x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \int_{0}^{N} e^{-(1+i\alpha)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \left[\frac{-1}{1+i\alpha} e^{-(1+i\alpha)x} \right]_0^N \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\alpha} - \frac{1}{1+i\alpha} \lim_{N\to\infty} e^{-(1+i\alpha)N} \right). \end{aligned}$$

Now,

$$\lim_{N\to\infty} \left| e^{-(1+i\alpha)N} \right| = \lim_{N\to\infty} e^{\mathrm{Re}(-(1+i\alpha)N)} = \lim_{N\to\infty} e^{-N} = 0,$$

and therefore $\lim_{N\to\infty} e^{-(1+i\alpha)N} = 0$.

Alternatively,

$$\lim_{N\to\infty} \mathrm{Re}\left(e^{-(1+i\alpha)N}\right) = \lim_{N\to\infty} e^{-N}\cos(\alpha N) = 0,$$

$$\lim_{N\to\infty} \mathrm{Im}\left(e^{-(1+i\alpha)N}\right) = \lim_{N\to\infty} e^{-N}\sin(\alpha N) = 0.$$

Finally,

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\alpha}.$$

Exercise 5.
The function f is piecewise-defined and we can verify that:

$$\int_{-\infty}^{+\infty} |f(x)|\, dx = \int_{-1}^{1} 1\, dx = 2 < +\infty.$$

Thus, the Fourier transform is well-defined. We compute:

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\alpha x}\, dx = \int_{-1}^{1} e^{-i\alpha x}\, dx \\ &= -\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-i\alpha x}}{i\alpha}\right]_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\frac{e^{i\alpha}-e^{-i\alpha}}{i\alpha} = \sqrt{\frac{2}{\pi}}\frac{\sin\alpha}{\alpha}.\end{aligned}$$

Exercise 6.

a) The function f is piecewise-defined and integrable over $\mathbb{R}$. Its Fourier transform is

$$\begin{aligned}\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sqrt{\frac{\pi}{2}}e^{-i\alpha x} dx \\ &= \frac{1}{2}\left[\frac{1}{-i\alpha}e^{-i\alpha x}\right]_{-1}^{1} = \frac{1}{2i\alpha}\left(e^{i\alpha}-e^{-i\alpha}\right) = \frac{\sin(\alpha)}{\alpha}.\end{aligned}$$

b) The function g is continuous and integrable over $\mathbb{R}$. Its Fourier transform is

$$
\begin{aligned}
\hat{g}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^{0} (\pi + \frac{\pi}{2} x) e^{-i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{2} (\pi - \frac{\pi}{2} x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\alpha} e^{-i\alpha x} (\pi + \frac{\pi}{2} x) \right]_{-2}^{0} - \frac{1}{\sqrt{2\pi}} \int_{-2}^{0} \frac{\pi}{-2i\alpha} e^{-i\alpha x} dx \\
&\quad + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-i\alpha} e^{-i\alpha x} (\pi - \frac{\pi}{2} x) \right]_{0}^{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{2} \frac{\pi}{2i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{\pi}{2} \left(\left[\frac{e^{-i\alpha x}}{-\alpha^2} \right]_{0}^{2} - \left[\frac{e^{-i\alpha x}}{-\alpha^2} \right]_{-2}^{0} \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}} \left(\frac{e^{2i\alpha} + e^{-2i\alpha}}{2} \frac{2}{-\alpha^2} - \frac{2}{-\alpha^2} \right) \\
&= \frac{\sqrt{\pi}}{-\alpha^2 \sqrt{2}} (\cos(2\alpha) - 1) = \sqrt{2\pi} \, \frac{\sin^2(\alpha)}{\alpha^2}.
\end{aligned}
$$

We observe that $\hat{g}(\alpha) = \sqrt{2\pi} \hat{f}(\alpha)^2$. So $g = f * f$.

Exercise 7.

1. We first compute

$$
\mathfrak{F}_c(f)(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(\alpha x) \, dx.
$$

Since $f(x) = e^{-x} \cos x$, we obtain

$$
\begin{aligned}
\mathfrak{F}_c(f)(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-x} \left[\frac{\cos((\alpha+1)x) + \cos((\alpha-1)x)}{2} \right] dx \\
&= \sqrt{\frac{2}{\pi}} \left[-\frac{e^{-x} [\cos((\alpha-1)x) - (\alpha-1)\sin((\alpha-1)x)]}{2\,(1 + (\alpha-1)^2)} \right. \\
&\qquad \left. + \frac{e^{-x} [(\alpha+1)\sin((\alpha+1)x) - \cos((\alpha+1)x)]}{2\,(1 + (\alpha+1)^2)} \right]_0^{+\infty}.
\end{aligned}
$$

Therefore,

$$
\mathfrak{F}_c(f)(\alpha) = \sqrt{\frac{2}{\pi}} \, \frac{2 + \alpha^2}{(2 - 2\alpha + \alpha^2)\,(2 + 2\alpha + \alpha^2)}.
$$

2. We now compute

$$\mathfrak{F}_s(f)(\alpha) = -i\sqrt{\frac{2}{\pi}}\int_0^{+\infty} f(x)\sin(\alpha x)\,dx.$$

Again using $f(x) = e^{-x}\cos x$, we obtain

$$\begin{aligned}\mathfrak{F}_s(f)(\alpha) &= -i\sqrt{\frac{2}{\pi}}\int_0^{+\infty} e^{-x}\left[\frac{\sin((\alpha+1)x)+\sin((\alpha-1)x)}{2}\right]dx\\ &= -i\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}[-(\alpha-1)\cos((\alpha-1)x)-\sin((\alpha-1)x)]}{2\,(1+(\alpha-1)^2)}\right.\\ &\qquad\left.-\frac{e^{-x}[(\alpha+1)\cos((\alpha+1)x)+\sin((\alpha+1)x)]}{2\,(1+(\alpha+1)^2)}\right]_0^{+\infty}.\end{aligned}$$

Thus,

$$\mathfrak{F}_s(f)(\alpha) = -i\sqrt{\frac{2}{\pi}}\,\frac{\alpha^3}{(2-2\alpha+\alpha^2)\,(2+2\alpha+\alpha^2)}.$$