

**Exercise 1.**

1. We find after computation

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} e^{(x-\pi)} \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^y \cos(ny + n\pi) dy \\ &= \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} e^y \cos(ny) dy = \frac{2 \sinh \pi}{(1+n^2)\pi}, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} e^{(x-\pi)} \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^y \sin(ny + n\pi) dy \\ &= \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} e^y \sin(ny) dy = \frac{-2n \sinh \pi}{(1+n^2)\pi}. \end{aligned}$$

2. Thanks to Dirichlet's theorem, we have that if

$$Ff(x) = \frac{\sinh \pi}{\pi} + \frac{\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left( \frac{2 \cos(nx)}{1+n^2} - \frac{2n \sin(nx)}{1+n^2} \right),$$

then

$$Ff(x) = f(x) \quad \text{if } x \in (0, 2\pi).$$

3. In particular, if  $x = \pi$  we obtain

$$1 = \frac{\sinh \pi}{\pi} \left( 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} \right),$$

from which

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \sinh \pi} - \frac{1}{2},$$

that is,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \sinh \pi} = \frac{\pi}{e^{\pi} - e^{-\pi}}.$$

**Exercise 2.**

1. We have

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} y^2 dy = \frac{2}{3}\pi^2, \\a_n &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi)^2 \cos(nx) dx = \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} y^2 \cos(ny) dy = \frac{4}{n^2}, \\b_n &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi)^2 \sin(nx) dx = \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} y^2 \sin(ny) dy = 0.\end{aligned}$$

The Fourier series is therefore

$$Ff(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cos(nx)}{n^2}.$$

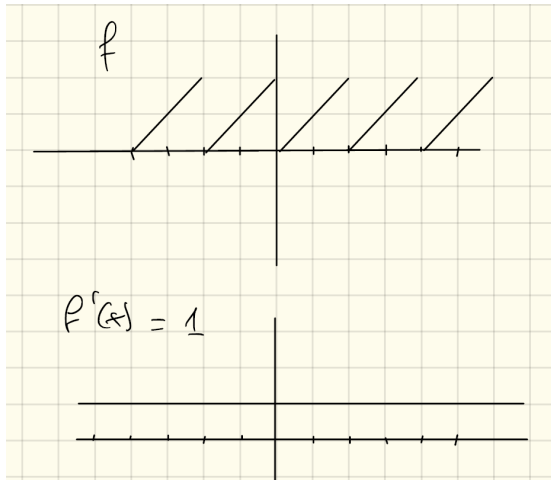
2. Dirichlet's theorem ensures (note that  $f$  is continuous at  $x = 0$  and  $x = 2\pi$ ) that

$$Ff(x) = f(x) \quad \text{for } x \in [0, 2\pi].$$

3. Taking  $x = 0$  (since  $f(0) = \pi^2$ ) and  $x = \pi$  (since  $f(\pi) = 0$ ), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3} &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \\ \sum_{n=1}^{\infty} 4 \frac{(-1)^n}{n^2} = -\frac{\pi^2}{3} &\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.\end{aligned}$$

**Exercise 3.**



Here we have  $T = 2$  and  $f(x) = x$  for  $x \in (0, 2)$ . Thus, for  $n \in \mathbb{Z}^*$ ,

$$\begin{aligned}
 c_0 &= \frac{1}{T} \int_0^T f(x) dx \\
 &= \frac{1}{2} \int_0^2 x dx \\
 &= \frac{1}{2} \left[ \frac{1}{2} x^2 \right]_{x=0}^{x=2} = 1
 \end{aligned}$$

$$\begin{aligned}
 c_n &= \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi n}{T} x} dx \\
 &= \frac{1}{2} \int_0^2 \underbrace{x}_u \underbrace{e^{-in\pi x}}_{v'} dx \quad \left| \begin{array}{l} u = x \quad v = -\frac{1}{in\pi} e^{-in\pi x} \\ u' = 1 \quad v' = e^{-in\pi x} \end{array} \right. \\
 &\stackrel{\text{IBP}}{=} \frac{1}{2} \left[ -\frac{x}{in\pi} e^{-in\pi x} \right]_{x=0}^{x=2} + \frac{1}{2in\pi} \int_0^2 e^{-in\pi x} dx \\
 &= -\frac{1}{in\pi} e^{-in2\pi} + \frac{1}{2in\pi} \left[ \frac{-1}{in\pi} e^{-in\pi x} \right]_{x=0}^{x=2} \\
 &= -\frac{1}{in\pi} - \frac{1}{2i^2 n^2 \pi^2} e^{-in2\pi} + \frac{1}{2i^2 n^2 \pi^2} \\
 &= \frac{i}{n\pi}
 \end{aligned}$$

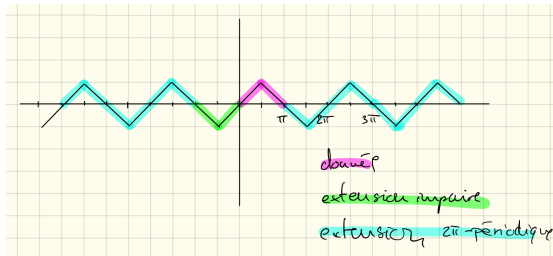
Therefore,

$$Ff(x) = \sum_{n \in \mathbb{Z}} c_n e^{i \frac{2\pi n}{T} x} = 1 + \sum_{n \in \mathbb{Z}^*} \frac{i}{n\pi} e^{in\pi x}$$

**Exercise 4.**

The odd extension of  $f$  to  $]-\pi, \pi]$  is given by

$$f(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq \pi, \\ -f(-x) & \text{if } -\pi < x < 0 \end{cases} = \begin{cases} -x - \pi & \text{if } -\pi < x < \frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$



*Variant 1:* Definition of the complex coefficients.

We have  $T = 2\pi$ , and we know  $f$  on  $]-\pi, \pi]$ . Thus,

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\frac{2\pi n}{T}x} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^{-\pi/2} (-x - \pi) e^{-inx} dx + \int_{-\pi/2}^{\pi/2} x e^{-inx} dx + \int_{\pi/2}^{\pi} (\pi - x) e^{-inx} dx \right) \end{aligned}$$

Note that, by periodicity, one could combine the first and the last integrals into a single one:

$$\int_{-\pi}^{-\pi/2} (-x - \pi) e^{-inx} dx + \int_{\pi/2}^{\pi} (\pi - x) e^{-inx} dx = \int_{\pi/2}^{3\pi/2} (\pi - x) e^{-inx} dx.$$

In short, we replace the green descending branch in the graph with the blue descending branch just to the right of the purple part.

However, since the task waiting for me after writing this solution is particularly unpleasant, I'd rather write a longer solution and compute all three integrals separately.

$$\begin{aligned}
I_1 &= \int_{-\pi}^{-\pi/2} (-x - \pi)e^{-inx} dx \\
&\stackrel{\text{IBP}}{=} \left[ (x + \pi) \frac{1}{in} e^{-inx} \right]_{-\pi}^{-\pi/2} - \frac{1}{in} \int_{-\pi}^{-\pi/2} e^{-inx} dx \quad \left| \begin{array}{l} u = -x - \pi \quad v = \frac{-1}{in} e^{-inx} \\ u' = -1 \quad v' = e^{-inx} \end{array} \right. \\
&= \frac{\pi}{2} \frac{1}{in} e^{in\frac{\pi}{2}} - \frac{1}{n^2} [e^{-inx}]_{-\pi}^{-\pi/2} \\
i^{-1} &\stackrel{=}{=} -i - \frac{i\pi}{2n} e^{in\frac{\pi}{2}} - \frac{1}{n^2} e^{in\frac{\pi}{2}} + \frac{1}{n^2} (-1)^n.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{-\pi/2}^{\pi/2} x e^{-inx} dx \\
&= \frac{-1}{in} [x e^{-inx}]_{-\pi/2}^{\pi/2} + \frac{1}{in} \int_{-\pi/2}^{\pi/2} e^{-inx} dx \quad \left| \begin{array}{l} u = x \quad v = \frac{-1}{in} e^{-inx} \\ u' = 1 \quad v' = e^{-inx} \end{array} \right. \\
&= -\frac{\pi}{2in} e^{-in\frac{\pi}{2}} - \frac{\pi}{2in} e^{in\frac{\pi}{2}} + \frac{1}{n^2} [e^{-inx}]_{-\pi/2}^{\pi/2} \\
i^{-1} &\stackrel{=}{=} -\frac{i\pi}{2n} e^{-in\frac{\pi}{2}} + \frac{i\pi}{2n} e^{in\frac{\pi}{2}} + \frac{1}{n^2} e^{-in\frac{\pi}{2}} - \frac{1}{n^2} e^{in\frac{\pi}{2}}.
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{\pi/2}^{\pi} (\pi - x) e^{-inx} dx \\
&\stackrel{\text{IBP}}{=} \frac{i}{n} [(\pi - x) e^{-inx}]_{\pi/2}^{\pi} + \frac{i}{n} \int_{\pi/2}^{\pi} e^{-inx} dx \quad \left| \begin{array}{l} u = \pi - x \quad v = \frac{i}{n} e^{-inx} \\ u' = -1 \quad v' = e^{-inx} \end{array} \right. \\
&= -\frac{i\pi}{2n} e^{-in\frac{\pi}{2}} - \frac{1}{n^2} [e^{-inx}]_{\pi/2}^{\pi} \\
&= -\frac{i\pi}{2n} e^{-in\frac{\pi}{2}} + \frac{1}{n^2} e^{-in\frac{\pi}{2}} - \frac{1}{n^2} (-1)^n.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1 + I_2 + I_3 &= -\frac{2}{n^2} e^{in\frac{\pi}{2}} + \frac{2}{n^2} e^{-in\frac{\pi}{2}} = -\frac{4i}{n^2} \sin\left(n\frac{\pi}{2}\right) \\
&= \begin{cases} 0 & \text{if } n = 2k, k \in \mathbb{Z}, \\ -\frac{4i}{(2k-1)^2} \sin\left(k\pi - \frac{\pi}{2}\right) & \text{if } n = 2k-1, k \in \mathbb{Z} \end{cases} \\
&= \begin{cases} 0 & \text{if } n = 2k, k \in \mathbb{Z}, \\ \frac{4i(-1)^k}{(2k-1)^2} & \text{if } n = 2k-1, k \in \mathbb{Z}. \end{cases}
\end{aligned}$$

This gives

$$c_n = \frac{1}{2\pi}(I_1 + I_2 + I_3) = \begin{cases} 0 & \text{if } n = 2k, k \in \mathbb{Z}, \\ \frac{2i(-1)^k}{\pi(2k-1)^2} & \text{if } n = 2k-1, k \in \mathbb{Z}. \end{cases}$$

*Variante 2:* Compute the real coefficients and then use the formulas to obtain the complex ones.

Since  $f$  is odd, all the cosine Fourier coefficients vanish. Moreover, for  $n \geq 1$ ,

$$\begin{aligned} b_n &= \frac{4}{T} \int_0^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi n}{T}x\right) dx \\ &= \frac{2}{\pi} \left( \int_0^{\frac{\pi}{2}} x \sin(nx) dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin(nx) dx \right) \end{aligned}$$

We compute an antiderivative of  $x \sin(ax)$  for  $a \in \mathbb{R}$ :

$$\begin{aligned} \int^x t \sin(at) dt &\stackrel{\text{IBP}}{=} \left[ -\frac{t}{a} \cos(at) \right]^{t=x} + \int^x \frac{1}{a} \cos(at) dt \quad \left| \begin{array}{l} u = t \quad v = -\frac{1}{a} \cos(at) \\ u' = 1 \quad v' = \sin(at) \end{array} \right. \\ &= -\frac{x}{a} \cos(ax) + \frac{1}{a^2} \sin(ax) \end{aligned}$$

Thus,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin(nx) dx - \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} x \sin(nx) dx + 2 \int_{\frac{\pi}{2}}^{\pi} \sin(nx) dx \\ &= \frac{2}{\pi} \left[ \frac{1}{n^2} \sin(nx) - \frac{x}{n} \cos(nx) \right]_{x=0}^{x=\frac{\pi}{2}} - \frac{2}{\pi} \left[ \frac{1}{n^2} \sin(nx) - \frac{x}{n} \cos(nx) \right]_{x=\frac{\pi}{2}}^{x=\pi} + 2 \left[ \frac{-1}{n} \cos(nx) \right]_{x=\frac{\pi}{2}}^{x=\pi} \\ &= \frac{2}{n^2\pi} \sin\left(n\frac{\pi}{2}\right) - \frac{1}{n} \cos\left(n\frac{\pi}{2}\right) - \frac{2}{n^2\pi} \underbrace{\sin(n\pi)}_{=0} + \frac{2}{n} \cos(n\pi) + \frac{2}{n^2\pi} \sin\left(n\frac{\pi}{2}\right) \\ &\quad - \frac{1}{n} \cos\left(n\frac{\pi}{2}\right) - \frac{2}{n} \cos(n\pi) + \frac{2}{n} \cos\left(n\frac{\pi}{2}\right) \\ &= \frac{4}{n^2\pi} \sin\left(n\frac{\pi}{2}\right) \end{aligned}$$

Finally,

$$\begin{aligned} \text{for } n \geq 1, \quad c_n &= \frac{a_n - ib_n}{2} = \frac{-i}{2} b_n = \frac{-2i}{n^2\pi} \sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k, \\ \frac{2i(-1)^k}{(2k-1)^2\pi} & \text{if } n = 2k-1, \end{cases} \\ \text{for } n \leq -1, \quad c_n &= \frac{a_{-n} + ib_{-n}}{2} = \frac{i}{2} b_{-n} = \frac{2i}{n^2\pi} \sin\left(-n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k, \\ \frac{2i(-1)^k}{(2k-1)^2\pi} & \text{if } n = 2k-1. \end{cases} \end{aligned}$$

We conclude that

$$Ff(x) = \sum_{k \in \mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} e^{i(2k-1)x}.$$

From the graph, we observe that  $f$  is piecewise  $C^1$  and continuous. Hence, for all  $x$ ,

$$Ff(x) \stackrel{(\text{Dirichlet})}{=} \frac{f(x-0) + f(x+0)}{2} \stackrel{f \in C^0}{=} f(x).$$

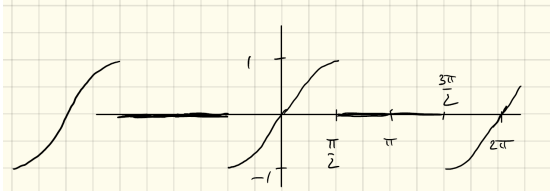
To compute  $\sum_{k \in \mathbb{Z}} \frac{1}{(2k-1)^2}$  from  $\sum_{k \in \mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} e^{i(2k-1)x}$ , we want to choose  $x$  such that  $e^{i(2k-1)x} \approx i(-1)^k$ . Here,  $x = \pm \frac{\pi}{2}$  works. Then,

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = Ff\left(\frac{\pi}{2}\right) = \sum_{k \in \mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} \underbrace{e^{ik\pi}}_{(-1)^k} \underbrace{e^{-i\frac{\pi}{2}}}_{-i} = \frac{2}{\pi} \sum_{k \in \mathbb{Z}} \frac{1}{(2k-1)^2},$$

and therefore

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}.$$

### Exercise 5.



Since  $f$  is odd, we have  $a_n = 0$  for all  $n$ , and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx = \frac{1}{2\pi} \left( \int_0^{\pi/2} \sin(x) \sin(nx) dx + \int_{3\pi/2}^{2\pi} \sin(x) \sin(nx) dx \right).$$

We can compute the primitives of the integrands using Euler's formulas:

$$\begin{aligned} \sin(x) \sin(nx) &= \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{inx} - e^{-inx}}{2i} \\ &= -\frac{1}{4} \left( e^{i(n+1)x} - e^{-i(n-1)x} - e^{i(n-1)x} + e^{-i(n+1)x} \right) \\ &= \frac{1}{2} \left( \cos((n-1)x) - \cos((n+1)x) \right), \end{aligned}$$

and thus

$$\int \sin(x) \sin(nx) dx = \begin{cases} \frac{1}{2} \left( \frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1} \right), & \text{if } n \neq 1, \\ \frac{1}{2} \left( x - \frac{\sin(2x)}{2} \right), & \text{if } n = 1. \end{cases}$$

We now compute the integrals using integration by parts for  $n \neq 1$ .

Case  $n = 1$ :

$$I_1 = \int_0^{\pi/2} \sin^2(x) dx = \frac{1}{2} [x - \cos(x) \sin(x)]_0^{\pi/2} = \frac{\pi}{4},$$
$$I_2 = \int_{3\pi/2}^{2\pi} \sin^2(x) dx = \frac{1}{2} [x - \cos(x) \sin(x)]_{3\pi/2}^{2\pi} = \pi - \frac{3\pi}{4} = \frac{\pi}{4}.$$

Hence,

$$b_1 = \frac{1}{\pi} (I_1 + I_2) = \frac{1}{2}.$$

Case  $n > 1$ :

$$\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin(x) \sin(nx) dx \\
&\stackrel{\text{IBP}}{=} [-\cos(x) \sin(nx)]_0^{\pi/2} + n \int_0^{\pi/2} \cos(x) \cos(nx) dx & \left| \begin{array}{ll} u = \sin(nx) & v = -\cos(x) \\ u' = n \cos(nx) & v' = \sin(x) \end{array} \right. \\
&\stackrel{\text{IBP}}{=} n [\sin(x) \cos(nx)]_0^{\pi/2} + n^2 \int_0^{\pi/2} \sin(x) \sin(nx) dx & \left| \begin{array}{ll} u = \cos(nx) & v = \sin(x) \\ u' = -n \sin(nx) & v' = \cos(x) \end{array} \right. \\
&= n \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \cos\left(n\frac{\pi}{2}\right) + n^2 I_1, \\
\Rightarrow I_1 &= n \cos\left(n\frac{\pi}{2}\right) + n^2 I_1.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{3\pi/2}^{2\pi} \sin(x) \sin(nx) dx \\
&\stackrel{\text{IBP}}{=} [-\cos(x) \sin(nx)]_{3\pi/2}^{2\pi} + n \int_{3\pi/2}^{2\pi} \cos(x) \cos(nx) dx & \left| \begin{array}{ll} u = \sin(nx) & v = -\cos(x) \\ u' = n \cos(nx) & v' = \sin(x) \end{array} \right. \\
&\stackrel{\text{IBP}}{=} n [\sin(x) \cos(nx)]_{3\pi/2}^{2\pi} + n^2 \int_{3\pi/2}^{2\pi} \sin(x) \sin(nx) dx & \left| \begin{array}{ll} u = \cos(nx) & v = \sin(x) \\ u' = -n \sin(nx) & v' = \cos(x) \end{array} \right. \\
&= -n \underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1} \cos\left(n\frac{3\pi}{2}\right) + n^2 I_2 \\
&= n \cos\left(n\frac{3\pi}{2}\right) + n^2 I_2.
\end{aligned}$$

If  $n$  is even ( $n = 2k$ ):

$$\cos\left(n\frac{\pi}{2}\right) = \cos(k\pi) = (-1)^k, \quad \cos\left(n\frac{3\pi}{2}\right) = \cos(3k\pi) = (-1)^k,$$

so

$$\begin{aligned}
I_1 &= 2k(-1)^k + 4k^2 I_1 \Rightarrow I_1 = \frac{2k(-1)^{k+1}}{4k^2 - 1}, \\
I_2 &= 2k(-1)^k + 4k^2 I_2 \Rightarrow I_2 = \frac{2k(-1)^{k+1}}{4k^2 - 1}, \\
I_1 + I_2 &= \frac{4k(-1)^{k+1}}{4k^2 - 1}.
\end{aligned}$$

If  $n$  is odd ( $n = 2k - 1$ ):

$$\cos\left(n\frac{\pi}{2}\right) = \cos\left(k\pi - \frac{\pi}{2}\right) = 0, \quad \cos\left(n\frac{3\pi}{2}\right) = \cos\left(3k\pi - \frac{3\pi}{2}\right) = 0,$$

and thus

$$I_1 = I_2 = 0 \quad \Rightarrow \quad I_1 + I_2 = 0.$$

Final result:

$$b_n = \frac{1}{\pi}(I_1 + I_2) = \begin{cases} \frac{1}{2}, & \text{if } n = 1, \\ \frac{4k(-1)^{k+1}}{\pi(4k^2 - 1)}, & \text{if } n = 2k \geq 2, \\ 0, & \text{if } n = 2k - 1 \geq 3. \end{cases}$$

and therefore

$$Ff(x) = \frac{1}{2}\sin(x) + \sum_{k=1}^{\infty} \frac{4k(-1)^{k+1}}{\pi(4k^2 - 1)} \sin(2kx).$$

**Exercise 6.** 1. Since  $f$  is even, we have  $b_n = 0$  for all  $n$ . Moreover,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} |\cos(x)| dx = \frac{1}{\pi} \left( \int_0^{\pi/2} \cos(x) dx - \int_{\pi/2}^{3\pi/2} \cos(x) dx + \int_{3\pi/2}^{2\pi} \cos(x) dx \right) = \frac{4}{\pi} \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} |\cos(x)| \cos(nx) dx \\ &= \frac{1}{\pi} \left( \int_0^{\pi/2} \cos(x) \cos(nx) dx - \int_{\pi/2}^{3\pi/2} \cos(x) \cos(nx) dx + \int_{3\pi/2}^{2\pi} \cos(x) \cos(nx) dx \right) \end{aligned}$$

As in the previous exercise, we could compute these integrals by performing successive integrations by parts, leading to an equation involving the same integral. This time, however, we use Euler's formulas to directly obtain a primitive.

We have

$$\begin{aligned} \cos(x) \cos(nx) &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{inx} + e^{-inx}}{2} \\ &= \frac{1}{4} \left( e^{i(n+1)x} + e^{-i(n-1)x} + e^{i(n-1)x} + e^{-i(n+1)x} \right) \\ &= \frac{1}{2} \left( \cos((n+1)x) + \cos((n-1)x) \right). \end{aligned}$$

Hence,

$$\int \cos(x) \cos(nx) dx = \begin{cases} \frac{x}{2} + \frac{1}{4} \sin(2x), & \text{if } n = 1, \\ \frac{\sin((n+1)x)}{2(n+1)} + \frac{\sin((n-1)x)}{2(n-1)}, & \text{if } n \geq 2. \end{cases}$$

We then obtain, for  $n = 1$ ,

$$\begin{aligned} \int_0^{2\pi} |\cos(x)| \cos(nx) dx &= \frac{1}{4} \sin(\pi) + \frac{\pi}{4} - 0 - 0 - \frac{1}{4} \sin(3\pi) - \frac{3\pi}{4} + \frac{1}{4} \sin(\pi) + \frac{\pi}{4} + \frac{1}{4} \sin(4\pi) \\ &+ \pi - \frac{1}{4} \sin(3\pi) - \frac{3\pi}{2} = 0 \end{aligned}$$

and for  $n \geq 2$ ,

$$\begin{aligned} \int_0^{2\pi} |\cos(x)| \cos(nx) dx &= \frac{1}{2(n+1)} \sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{2(n-1)} \sin\left((n-1)\frac{\pi}{2}\right) \\ &- \frac{1}{2(n+1)} \sin((n+1)0) + \frac{1}{2(n-1)} \sin((n-1)0) \\ &- \frac{1}{2(n+1)} \sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{2(n-1)} \sin\left((n-1)\frac{3\pi}{2}\right) \\ &+ \frac{1}{2(n+1)} \sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{2(n-1)} \sin\left((n-1)\frac{\pi}{2}\right) \\ &+ \frac{1}{2(n+1)} \sin((n+1)2\pi) + \frac{1}{2(n-1)} \sin((n-1)2\pi) \\ &- \frac{1}{2(n+1)} \sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{2(n-1)} \sin\left((n-1)\frac{3\pi}{2}\right) \\ &= \frac{1}{n+1} \sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1} \sin\left((n-1)\frac{\pi}{2}\right) \\ &- \frac{1}{n+1} \sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{n-1} \sin\left((n-1)\frac{3\pi}{2}\right) \end{aligned}$$

If  $n$  is odd, each term is zero. If  $n$  is even ( $n = 2k$ ), then

$$\begin{aligned} \sin\left((n+1)\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k, \\ \sin\left((n-1)\frac{\pi}{2}\right) &= \sin\left(-\frac{\pi}{2} + k\pi\right) = (-1)^{k+1}, \\ \sin\left((n+1)\frac{3\pi}{2}\right) &= \sin\left(\frac{3\pi}{2} + 3k\pi\right) = (-1)^{k+1}, \\ \sin\left((n-1)\frac{3\pi}{2}\right) &= \sin\left(-\frac{3\pi}{2} + 3k\pi\right) = (-1)^k, \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^{2\pi} |\cos(x)| \cos(nx) dx &= \frac{(-1)^k}{2k+1} + \frac{(-1)^{k+1}}{2k-1} - \frac{(-1)^{k+1}}{2k+1} - \frac{(-1)^k}{2k-1} \\ &= \frac{2(-1)^k(2k-1-2k-1)}{4k^2-1} = \frac{4(-1)^{k+1}}{4k^2-1}. \end{aligned}$$

Finally,

$$Ff(x) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)} \cos(2kx)$$

2. Observing the graph, we note that  $f$  is piecewise  $C^1$  and continuous. Therefore, for all  $x$ ,

$$Ff(x) \stackrel{\text{Dirichlet}}{=} \frac{f(x-0) + f(x+0)}{2} \stackrel{f \in C^0}{=} f(x)$$

To compute  $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$  from  $\sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)} \cos(2kx)$ , we must choose  $x$  such that  $\cos(2kx) \approx (-1)^k$ . Choosing  $x = \frac{\pi}{2}$  works:

$$0 = |\cos(\frac{\pi}{2})| = f(\frac{\pi}{2}) = Ff(\frac{\pi}{2}) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)} (-1)^k = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

and hence

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}.$$

**Exercise 7.** 1. Since  $f$  is even, the sine coefficients vanish. For the cosine coefficients, we have  $T = 2\pi$ ,  $f(x) = \cos(\alpha x)$  for  $x \in (-\pi, \pi]$ , and therefore

$$a_0 = \frac{4}{T} \int_0^{\frac{T}{2}} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \cos(\alpha x) dx = \frac{2}{\alpha\pi} [\sin(\alpha x)]_{x=0}^{x=\pi} = \frac{2}{\alpha\pi} \sin(\alpha\pi)$$

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} \cos(\alpha x) \cos(nx) dx$$

Once again, we can choose between Euler's formulas or integration by parts to compute the integral:

*Option 1: Using Euler's formulas.*

$$\begin{aligned}
\cos(\alpha x) \cos(nx) &= \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} \cdot \frac{e^{inx} + e^{-inx}}{2} \\
&= \frac{1}{4} \left( e^{i(\alpha+n)x} + e^{i(\alpha-n)x} + e^{-i(\alpha-n)x} + e^{-i(\alpha+n)x} \right) \\
&= \frac{1}{2} \cos((\alpha+n)x) + \frac{1}{2} \cos((\alpha-n)x)
\end{aligned}$$

$$\int_0^x \cos(\alpha x) \cos(nx) dx = \frac{1}{2(\alpha+n)} \sin((\alpha+n)x) + \frac{1}{2(\alpha-n)} \sin((\alpha-n)x).$$

Thus,

$$\begin{aligned}
a_n &= \frac{2}{\pi} \left[ \frac{1}{2(\alpha+n)} \sin((\alpha+n)x) + \frac{1}{2(\alpha-n)} \sin((\alpha-n)x) \right]_{x=0}^{x=\pi} \\
&= \frac{1}{\pi(\alpha+n)} \sin(\alpha\pi + n\pi) + \frac{1}{\pi(\alpha-n)} \sin(\alpha\pi - n\pi) \\
&= \frac{(-1)^n}{\pi(\alpha+n)} \sin(\alpha\pi) + \frac{(-1)^n}{\pi(\alpha-n)} \sin(\alpha\pi) \\
&= \frac{(-1)^n}{\pi} \frac{\alpha-n + \alpha+n}{\alpha^2 - n^2} \sin(\alpha\pi) \\
&= \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi)
\end{aligned}$$

*Option 2: Using Integration by Parts (IBP).*

$$\begin{aligned}
I &= \int_0^\pi \underbrace{\cos(\alpha x)}_{=u'} \underbrace{\cos(nx)}_v dx && \left| \begin{array}{ll} u = \frac{1}{\alpha} \sin(\alpha x) & v = \cos(nx) \\ u' = \cos(\alpha x) & v' = -n \sin(nx) \end{array} \right. \\
\stackrel{\text{IBP}}{=} & \left[ \frac{1}{\alpha} \sin(\alpha x) \cos(nx) \right]_0^\pi + \frac{n}{\alpha} \int_0^\pi \sin(\alpha x) \sin(nx) dx \\
&= \frac{(-1)^n}{\alpha} \sin(\alpha\pi) + \frac{n}{\alpha} \int_0^\pi \underbrace{\sin(\alpha x)}_{=u'} \underbrace{\sin(nx)}_v dx && \left| \begin{array}{ll} u = \frac{-1}{\alpha} \cos(\alpha x) & v = \sin(nx) \\ u' = \sin(\alpha x) & v' = n \cos(nx) \end{array} \right. \\
\stackrel{\text{IBP}}{=} & \frac{(-1)^n}{\alpha} \sin(\alpha\pi) - \frac{n}{\alpha} \left[ \frac{1}{\alpha} \cos(\alpha x) \sin(nx) \right]_0^\pi + \frac{n^2}{\alpha^2} \int_0^\pi \cos(\alpha x) \cos(nx) dx \\
&= \frac{(-1)^n}{\alpha} \sin(\alpha\pi) + \frac{n^2}{\alpha^2} I
\end{aligned}$$

Hence,

$$I = \frac{\frac{(-1)^n}{\alpha} \sin(\alpha\pi)}{1 - \frac{n^2}{\alpha^2}} = \frac{\alpha(-1)^n}{\alpha^2 - n^2} \sin(\alpha\pi),$$

and therefore,

$$a_n = \frac{2}{\pi} I = \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi).$$

Finally, the Fourier series of  $f$  is

$$\begin{aligned} Ff(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}x\right) \\ &= \frac{1}{\alpha\pi} \sin(\alpha\pi) + \sum_{n=1}^{\infty} \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi) \cos(nx). \end{aligned}$$

2. To find the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2}$$

from

$$\sum_{n=1}^{\infty} \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi) \cos(nx),$$

we must choose  $x$  such that  $(-1)^n \cos(nx) \approx 1$ . Choosing  $x = \pi$  works, since  $\cos(nx) = \cos(n\pi) = (-1)^n$ , giving

$$Ff(\pi) = \frac{1}{\alpha\pi} \sin(\alpha\pi) - \frac{2\alpha \sin(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2}.$$

Moreover, since  $f$  is continuous, we have  $Ff(\pi) = f(\pi) = \cos(\alpha\pi)$ .

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \sin(\alpha\pi)} Ff(\pi) = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)},$$

which is the desired result.