EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 9	November 13 2025

Exercise 1.
(i) Calculation of $\iint_\Sigma \operatorname{curl} F \cdot ds$. We have

$$\operatorname{curl} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & z & x \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ -x^2 \end{pmatrix}$$

and if $A = (0,1) \times (0, 2\pi)$ then

$$\Sigma = \left\{ \sigma(r,\theta) = \left(r^2 \cos\theta, r^2 \sin\theta, r\right) : (r,\theta) \in \bar{A} \right\}$$

The normal vector is given by

$$\sigma_r \wedge \sigma_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2r\cos\theta & 2r\sin\theta & 1 \\ -r^2\sin\theta & r^2\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} -r^2\cos\theta \\ -r^2\sin\theta \\ 2r^3 \end{pmatrix}$$

and consequently

$$\begin{aligned} \iint_\Sigma \operatorname{curl} F \cdot ds &= \int_0^1 \int_0^{2\pi} \left(-1, -1, -r^4\cos^2\theta\right) \cdot \left(-r^2\cos\theta, -r^2\sin\theta, 2r^3\right) drd\theta \\ &= -\int_0^1 \int_0^{2\pi} 2r^7 \cos^2\theta drd\theta = -\frac{\pi}{4} \end{aligned}$$

(ii) Calculation of $\int_{\partial\Sigma} F \cdot dl$. Moreover, we obtain that

$$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where

$$\Gamma_1 = \left\{ \sigma(r,0) = \left(r^2, 0, r\right) : r : 0 \to 1 \right\}$$

$$\begin{aligned} \Gamma_2 &= \{\sigma(1,\theta) = (\cos\theta, \sin\theta, 1) : \theta : 0 \to 2\pi\} \\ \Gamma_3 &= \left\{ \sigma(r, 2\pi) = \left(r^2, 0, r\right) : r : 1 \to 0 \right\} = -\Gamma_1 \\ \Gamma_4 &= \{\sigma(0,\theta) = (0,0,0) : \theta : 2\pi \to 0\} \end{aligned}$$

We therefore deduce that $\partial\Sigma = \Gamma_2$ and that it is positively oriented. We then obtain

$$\begin{aligned}\int_{\partial\Sigma} F \cdot dl &= \int_0^{2\pi} \big(\cos^2\theta\sin\theta, 1, \cos\theta\big) \cdot (-\sin\theta, \cos\theta, 0) d\theta \\ &= -\int_0^{2\pi} \cos^2\theta\sin^2\theta d\theta = -\frac{\pi}{4}\end{aligned}$$

Exercise 2.
(i) Calculation of $\iint_\Sigma \operatorname{curl} F \cdot ds$. We switch to spherical coordinates and write, for $A = \big(0, \frac{\pi}{2}\big) \times \big(\frac{\pi}{6}, \frac{\pi}{3}\big)$,

$$\Sigma = \{\sigma(\theta,\varphi) = (2\cos\theta\sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi) \text{ with } (\theta,\varphi) \in \bar{A}\}$$

We find

$$\sigma_\theta \wedge \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2\sin\theta\sin\varphi & 2\cos\theta\sin\varphi & 0 \\ 2\cos\theta\cos\varphi & 2\sin\theta\cos\varphi & -2\sin\varphi \end{vmatrix} = \begin{pmatrix} -4\cos\theta\sin^2\varphi \\ -4\sin\theta\sin^2\varphi \\ -4\cos\varphi\sin\varphi \end{pmatrix}$$

Since

$$\operatorname{curl} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & 0 \end{vmatrix} = \begin{pmatrix} -2z \\ 0 \\ 0 \end{pmatrix}$$

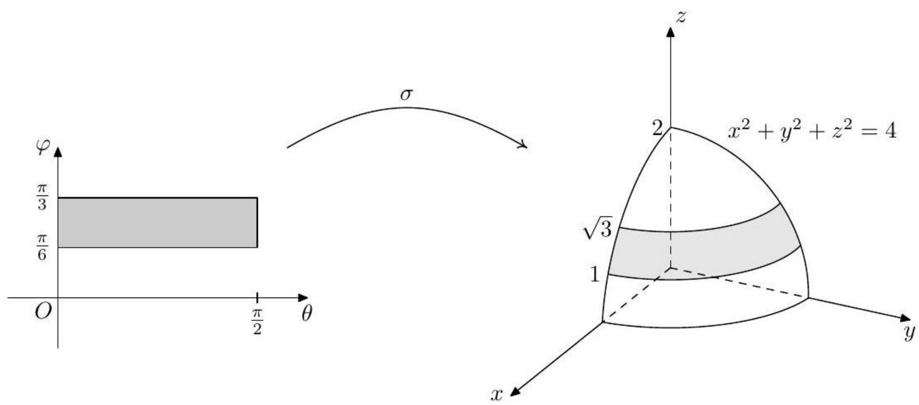
we obtain

$$\begin{aligned}&\operatorname{curl} F \cdot (\sigma_\theta \wedge \sigma_\varphi) \\ &= (-4\cos\varphi, 0, 0) \cdot \big(-4\cos\theta\sin^2\varphi, -4\sin\theta\sin^2\varphi, -4\cos\varphi\sin\varphi\big)\end{aligned}$$

and therefore

$$\iint_\Sigma \operatorname{curl} F \cdot ds = 16 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \cos\theta\cos\varphi\sin^2\varphi \, d\theta \, d\varphi = 2\sqrt{3} - \frac{2}{3}$$

(ii) Calculation of $\int_{\partial\Sigma} F \cdot dl$. We have $\partial\Sigma = \sigma(\partial A) = \bigcup_{i=1}^4 \Gamma_i$, where



$$\Gamma_1 = \left\{ \sigma\left(\theta, \frac{\pi}{6}\right) = (\cos\theta, \sin\theta, \sqrt{3}) : \theta : 0 \to \frac{\pi}{2} \right\}$$
$$\Gamma_2 = \left\{ \sigma\left(\frac{\pi}{2}, \varphi\right) = (0, 2\sin\varphi, 2\cos\varphi) : \varphi : \frac{\pi}{6} \to \frac{\pi}{3} \right\}$$
$$\Gamma_3 = \left\{ \sigma\left(\theta, \frac{\pi}{3}\right) = (\sqrt{3}\cos\theta, \sqrt{3}\sin\theta, 1) : \theta : \frac{\pi}{2} \to 0 \right\}$$
$$\Gamma_4 = \left\{ \sigma(0, \varphi) = (2\sin\varphi, 0, 2\cos\varphi) : \varphi : \frac{\pi}{3} \to \frac{\pi}{6} \right\}$$

A straightforward calculation gives

$$\int_{\Gamma_1} F \cdot dl = \int_0^{\frac{\pi}{2}} (0, 3, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta = \int_0^{\frac{\pi}{2}} 3\cos\theta\, d\theta = 3$$
$$\int_{\Gamma_2} F \cdot dl = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(0, 4\cos^2\varphi, 0\right) \cdot (0, 2\cos\varphi, -2\sin\varphi)\, d\varphi$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 8\cos^3\varphi\, d\varphi = 3\sqrt{3} - \frac{11}{3}$$

$$\int_{\Gamma_3} F \cdot dl = -\int_0^{\frac{\pi}{2}} (0, 1, 0) \cdot (-\sqrt{3}\sin\theta, \sqrt{3}\cos\theta, 0)\, d\theta = -\sqrt{3}$$
$$\int_{\Gamma_4} F \cdot dl = -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(0, 4\cos^2\varphi, 0\right) \cdot (2\cos\varphi, 0, -2\sin\varphi)\, d\varphi = 0$$

Thus, we indeed have

$$\int_{\partial\Sigma} F \cdot dl = \sum_{i=1}^{4} \int_{\Gamma_i} F \cdot dl = 2\sqrt{3} - \frac{2}{3}$$

Exercise 3.

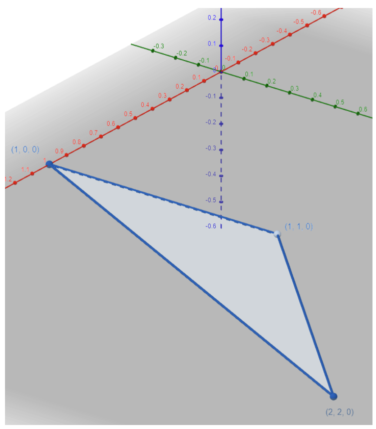
(i) *Computation of* $\iint_\Sigma \operatorname{curl} F\, ds$.

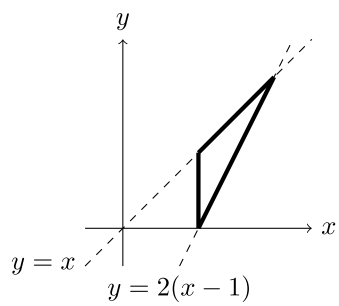
We have

$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x \end{pmatrix}$$

Sketch of Σ*:*

We observe that Σ lies in the plane $z = 0$:





Notice that the domain in $\mathbb{R}^2$ is y-simple. It could also be split into two x-simple domains, but in that case one must take care to keep the same orientation of the normal vector for both parts of the integral. We therefore present only the y-simple variant. *Computation:*

We parameterize Σ by

$$\sigma\colon \left\{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\ 2(x-1) \leq y \leq x\right\} \to \mathbb{R}^3, \qquad \sigma(x,y) = (x,y,0)$$

We have

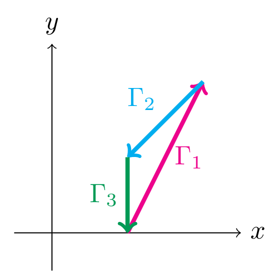
$$\operatorname{curl} F(\sigma(x,y)) = (0,0,2x)$$

$$\sigma_x \wedge \sigma_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
\iint_\Sigma \operatorname{curl} F \, ds &= \int_1^2 \int_{2x-2}^{x} \langle (0,0,2x), (0,0,1) \rangle \; dy \, dx \\
&= \int_1^2 2x \, (x - (2x-2)) \, dx \\
&= \int_1^2 (4x - 2x^2) dx \\
&= \left[2x^2 - \frac{2}{3} x^3 \right]_{x=1}^{x=2} \\
&= 8 - \frac{16}{3} - 2 + \frac{2}{3} \\
&= \frac{24 - 16 - 6 + 2}{3} = \frac{4}{3}
\end{aligned}
$$

(ii) *Computation of* $\int_{\partial\Sigma} F \, dl$.

The boundary of $A = \left\{ (x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, \; 2(x-1) \leq y \leq x \right\}$ is divided into three parts:



$\boxed{\Gamma_1}$ $\oplus$

$$
\begin{aligned}
&\gamma_1 \colon [1,2] \to \mathbb{R}^2 \\
&\gamma_1(t) = (t, 2t-2) \\
&\sigma \circ \gamma_1(t) = (t, 2t-2, 0) \\
&(\sigma \circ \gamma_1)'(t) = (1, 2, 0)
\end{aligned}
$$

$\boxed{\Gamma_2}$

$$\gamma_2 \colon [1,2] \to \mathbb{R}^2 \qquad \ominus$$
$$\gamma_2(t) = (t,t)$$
$$\sigma \circ \gamma_2(t) = (t,t,0)$$
$$(\sigma \circ \gamma_2)'(t) = (1,1,0)$$

$\boxed{\Gamma_3}$

$$\gamma_3 \colon [0,1] \to \mathbb{R}^2 \qquad \ominus$$
$$\gamma_3(t) = (1,t)$$
$$\sigma \circ \gamma_3(t) = (1,t,0)$$
$$(\sigma \circ \gamma_3)'(t) = (0,1,0)$$

Thus,

$$\int_{\sigma(\Gamma_1)} F \cdot dl = \int_1^2 \big\langle (0,t^2,0),(1,2,0) \big\rangle \, dt = \int_1^2 2t^2 \, dt = \left[\frac{2}{3} t^3 \right]_{t=1}^{t=2}$$
$$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

$$\int_{\sigma(\Gamma_2)} F \cdot dl = \int_1^2 \big\langle (0,t^2,0),(1,1,0) \big\rangle \, dt = \int_1^2 t^2 \, dt = \left[\frac{1}{3} t^3 \right]_{t=1}^{t=2}$$
$$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_{\sigma(\Gamma_3)} F \cdot dl = \int_0^1 \langle (0,1,0),(0,1,0) \rangle \, dt = \int_0^1 1 \, dt = 1$$

$$\int_{\partial\Sigma} F \cdot dl = + \int_{\sigma(\Gamma_1)} F \cdot dl - \int_{\sigma(\Gamma_2)} F \cdot dl - \int_{\sigma(\Gamma_3)} F \cdot dl$$
$$= \frac{14}{3} - \frac{7}{3} - 1 = \frac{4}{3},$$

which is the expected result.

Exercise 4.

(i) *Computation of* $\iint_\Sigma \operatorname{curl} F \, ds$.

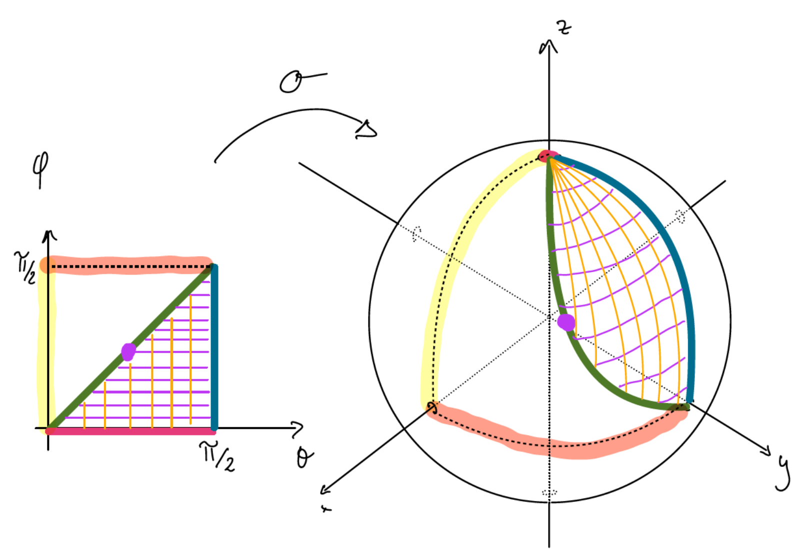
We have

$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y+z^2 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Sketch of Σ*:*

We switch to spherical coordinates $(x,y,z) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$ with $r \geq 0$, $\theta \in [-\pi,\pi]$, and $\varphi \in [0,\pi]$. Our conditions become

$$x^2+y^2+z^2 = 4 \Leftrightarrow r = 2$$
$$x, y \geq 0 \Leftrightarrow \theta \in \left[0, \frac{\pi}{2}\right]$$
$$z \geq 0 \Leftrightarrow \varphi \in \left[0, \frac{\pi}{2}\right]$$
$$0 \leq \arccos\frac{z}{2} \leq \arctan\frac{y}{x} \leq \frac{\pi}{2} \Leftrightarrow 0 \leq \varphi \leq \theta \leq \frac{\pi}{2}$$



Calculation:

We parametrize Σ with

$$\sigma\colon \left\{(\theta,\varphi) \in \mathbb{R}^2 : 0 \leq \varphi \leq \theta \leq \frac{\pi}{2}\right\} \to \mathbb{R}^3$$

defined by

$$\sigma(\theta,\varphi) = (2\cos\theta\sin\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\varphi)$$

We have

$$\operatorname{curl} F(\sigma(\theta,\varphi)) = (1,0,0)$$

$$\sigma_\theta \wedge \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2\sin\theta\sin\varphi & 2\cos\theta\sin\varphi & 0 \\ 2\cos\theta\cos\varphi & 2\sin\theta\cos\varphi & -2\sin\varphi \end{vmatrix} = \begin{pmatrix} -4\cos\theta\sin^2\varphi \\ -4\sin\theta\sin^2\varphi \\ -4\sin\varphi\cos\varphi \end{pmatrix}$$

Variant 1: $d\varphi\, d\theta$

$$\begin{aligned}
\int_\Sigma \operatorname{curl} F\, ds &= \int_0^{\frac{\pi}{2}} \int_0^{\theta} \left\langle (1,0,0), (-4\cos\theta\sin^2\varphi, -4\sin\theta\sin^2\varphi, -4\sin\varphi\cos\varphi) \right\rangle \, d\varphi\, d\theta \\
&= -4 \int_0^{\frac{\pi}{2}} \cos\theta \int_0^{\theta} \sin^2\varphi \, d\varphi\, d\theta \\
&= -4 \int_0^{\frac{\pi}{2}} \cos\theta \left[\frac{1}{2}(\varphi - \sin\varphi\cos\varphi) \right]_{\varphi=0}^{\varphi=\theta} d\theta \\
&= -4 \int_0^{\frac{\pi}{2}} \cos\theta \frac{1}{2}(\theta - \sin\theta\cos\theta)\, d\theta \\
&= -2 \int_0^{\frac{\pi}{2}} \theta\cos\theta - \sin\theta\cos^2\theta \, d\theta \\
&= -2 \int_0^{\frac{\pi}{2}} \theta\cos\theta\, d\theta - \left[\frac{2}{3}\cos^3\theta \right]_0^{\frac{\pi}{2}} \\
&= -2 \int_0^{\frac{\pi}{2}} \theta\cos\theta\, d\theta + \frac{2}{3} \qquad \left| \begin{array}{ll} u = \theta & v = \sin\theta \\ u' = 1 & v' = \cos\theta \end{array} \right. \\
&\overset{\text{IBP}}{=} -2\left([\theta\sin\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \right) + \frac{2}{3} \\
&= -\pi - 2[\cos\theta]_0^{\frac{\pi}{2}} + \frac{2}{3} \\
&= -\pi + 2 + \frac{2}{3} \\
&= \frac{8}{3} - \pi
\end{aligned}$$

Variant 2: $d\theta\, d\varphi$

$$
\begin{aligned}
\int_\Sigma \operatorname{curl} F\, ds &= \int_0^{\frac{\pi}{2}} \int_\varphi^{\frac{\pi}{2}} \big\langle (1,0,0), (-4\cos\theta\sin^2\varphi, -4\sin\theta\sin^2\varphi, -4\sin\varphi\cos\varphi) \big\rangle\, d\theta\, d\varphi \\
&= -4\int_0^{\frac{\pi}{2}} \sin^2\varphi \int_\varphi^{\frac{\pi}{2}} \cos\theta\, d\theta\, d\varphi \\
&= -4\int_0^{\frac{\pi}{2}} \sin^2\varphi [\sin\theta]_{\theta=\varphi}^{\theta=\frac{\pi}{2}}\, d\varphi \\
&= -4\int_0^{\frac{\pi}{2}} \sin^2\varphi - \sin^3\varphi\, d\varphi \\
&= -4\int_0^{\frac{\pi}{2}} \sin^2\varphi - \sin\varphi(1-\cos^2\varphi)\, d\varphi \\
&= 4\int_0^{\frac{\pi}{2}} -\sin^2\varphi + \sin\varphi - \cos^2\varphi\sin\varphi\, d\varphi \\
&= 4\left[-\frac{1}{2}\varphi + \frac{1}{2}\cos\varphi\sin\varphi - \cos\varphi + \frac{1}{3}\cos^3\varphi\right]_0^{\frac{\pi}{2}} \\
&= -\pi + 4 - \frac{4}{3} = \frac{8}{3} - \pi
\end{aligned}
$$

One could also find the antiderivatives of trigonometric functions using Euler's formulas.

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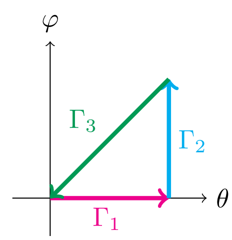
(ii) *Computation of* $\int_{\partial\Sigma} F\, dl$.

The boundary of $A = \left\{(\theta,\varphi) \in \mathbb{R}^2 : 0 \le \varphi \le \theta \le \frac{\pi}{2}\right\}$ is divided into three parts:



$\boxed{\Gamma_1}$

$\gamma_1 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$

$\gamma_1(t) = (t, 0)$

$\sigma \circ \gamma_1(t) = (0,0,2)$

is a single point, so it contributes nothing.

$\boxed{\Gamma_2}$ $\gamma_2 \colon [0, \frac{\pi}{2}] \to \mathbb{R}^2$ $\oplus$

$$\gamma_2(t) = (\tfrac{\pi}{2}, t)$$
$$\sigma \circ \gamma_2(t) = (0, 2\sin t, 2\cos t)$$
$$(\sigma \circ \gamma_2)'(t) = (0, 2\cos t, -2\sin t)$$

$\boxed{\Gamma_3}$ $\gamma_3 \colon [0, \frac{\pi}{2}]$ $\ominus$

$$\gamma_3(t) = (t, t)$$
$$\sigma \circ \gamma_3(t) = (2\cos t \sin t, 2\sin^2 t, 2\cos t)$$
$$(\sigma \circ \gamma_3)'(t) = (-2\sin^2 t + 2\cos^2 t, 4\sin t \cos t, -2\sin t)$$

Thus,

$$\begin{aligned}
\int_{\sigma(\Gamma_2)} F \cdot dl &= \int_0^{\frac{\pi}{2}} \big\langle (0, 0, 2\sin t + 4\cos^2 t), (0, 2\cos t, -2\sin t) \big\rangle \, dt \\
&= \int_0^{\frac{\pi}{2}} -4\sin^2 t - 8\sin t \cos^2 t \, dt \\
&= \left[-2(t - \sin t \cos t) + \frac{8}{3}\cos^3 t \right]_0^{\frac{\pi}{2}} \\
&= -\pi - \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
\int_{\sigma(\Gamma_3)} F \cdot dl &= \int_0^{\frac{\pi}{2}} \big\langle (0, 0, 2\sin^2 t + 4\cos^2 t), (-2\sin^2 t + 2\cos^2 t, 4\sin t \cos t, -2\sin t) \big\rangle \, dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin^3 t + 2\sin t \cos^2 t \, dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin t - \sin t \cos^2 t + 2\sin t \cos^2 t \, dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin t + \sin t \cos^2 t \, dt \\
&= -4 \left[-\cos t - \frac{1}{3}\cos^3 t \right]_0^{\frac{\pi}{2}} \\
&= -4 - \frac{4}{3} = -\frac{16}{3}
\end{aligned}$$

$$\int_\Sigma F \cdot dl = + \int_{\sigma(\Gamma_2)} F \cdot dl - \int_{\sigma(\Gamma_3)} F \cdot dl = \frac{8}{3} - \pi$$

which indeed matches the expected result.